SIZE BOUNDS AND RAMSEY-TYPE NUMBERS

by

John Weigand

B.A. Mathematics, University of Colorado at Boulder, 1971

M.S. Computer Science, University of Colorado at Denver, 1996

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John Weigand

has been approved

by

Michael Jacobson

William Cherowitzo

Jan Mandel

Michael Ferrara

Thomas Altman

Date

Weigand, John (Ph.D., Applied Mathematics)

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ABSTRACT

Staton introduced the notion of Ramsey-type numbers in his 1977 dissertation. He defined the Ramsey-type number $R_D(m,n)$ to be the smallest integer for which, any graph of order $R_D(m,n)$ and maximum degree at most D, must contain a clique of size m or an independent set of size n. Independence ratios, which are define by $\mu(G) = \frac{\alpha(G)}{\nu(G)}$, play an important role in this theory. If we let $\mathcal{G}_D(m)$ denote the class of K_m-free graphs of maximum degree at most D, then a lower bound for the independence ratios of such graphs yields an upper bound for the Ramsey-type number $R_D(m,n)$. Lower bounds for independence ratios follow from size bounds of the form $\mathrm{e}(G) \geq a\nu(G) + b\alpha(G)$.

This dissertation is concerned with the problem of calculating size and independence ratio bounds for various classes of graphs, and then using them to establish upper bounds for Ramsey-type numbers. Expressions that yield coefficients for multiple size bounds or multiple independence ratio bounds are of particular importance. For example, if we let $m \geq 3$, and define

$$a_m = 2m - 3 \quad \text{and} \quad b_m = -\frac{7}{2}m^2 + \frac{3}{2}m - 2,$$

then, with one exception for the case $m = 5$, any connected $G \in \mathcal{G}_m(m)$ satisfies

$$\mathrm{e}(G) \geq a_m \nu(G) + b_m \alpha(G).$$

Given this size bound, we immediately obtain the independence ratio bound

$$\mu(G) \geq \frac{3m-6}{3m^2-7m+4}.$$

These results are sharpened for members of $\mathcal{G}_6(6)$ and non-regular members of $\mathcal{G}_7(7)$.

An infinite class of size bounds that hold for all graphs is introduced, and the set of independence ratios for which each is tight is established. The more difficult problem of defining tight size bounds for classes of graphs that satisfy a forbidden clique constraint, but no maximum degree constraint is also studied. These bounds are shown to be closely related to an independence ratio bound introduced by Fajtlowicz.

This abstract accurately represents the content of the candidate's thesis. I recommend its publication.

Signed ______________________________
Michael Jacobson

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1. Introduction

This dissertation is concerned with the development of techniques for calculating bounds for Ramsey-type numbers, but, before addressing this issue, it is necessary to present some background information.

This chapter is organized as follows: Notational conventions and basic graph-theoretic definitions are introduced in Section 1.1. Historical information pertaining to research in the area of Ramsey-type numbers is presented in Section 1.2. This material includes the definition of Ramsey-type numbers, citations of relevant research and an informal introduction to established proof techniques. Section 1.3 contains a chapter by chapter summary of the contents of this dissertation.

1.1 Basic Graph-Theoretic Definitions

A collection of basic graph theoretic definitions is presented in this section. Most of this material can be found in an introductory graph theory text such as West [23], however several notational conventions that will be used throughout this paper, and a few specialized concepts are also introduced. In particular, the use of ordinal numbers as indexing sets is somewhat unusual. These numbers arise frequently in the following chapters, and in some of the definitions presented in this section, we begin with a brief explanation of their use.

We let $\mathbb{N}$ denote the set of non-negative integers, and we assume that, in addition to being a number, each $n \in \mathbb{N}$ is also a set. Specifically, we let $n = \{0, 1, \ldots, n-1\}$ (In particular, notice that $0 = \emptyset$). Given this convention, we are using $\mathbb{N}$ to denote the set of finite ordinal numbers. Since members of $\mathbb{N}$ are both numbers and sets, peculiar-looking notation such as the following may arise:

$$\cup_{i\in 5} A_i, \quad \exists\, i \in n, \quad \forall\, j \in 2, \quad \{p_i\}^{i\in n} \quad \text{and} \quad \{q_j\}^{j\in k+2}. \tag{1.1}$$

No confusion should result as long as the reader remains aware that both arithmetic and set-theoretic operations are applicable to ordinal numbers.

Sizes of finite sets play an important role in Ramsey theory. A set S is defined to be finite if there is a bijection between it and some $n \in \mathbb{N}$. In this case, n is unique. Furthermore, we say that S has size n, and we write $|S| = n$. Since $|S|$ is a set, statements such as $Q = \{q_j\}^{j\in|Q|}$ are permitted.

We now present some basic graph-theoretic definitions. All of this material will be used in the following chapters.

Definition 1.1

1. *A graph is defined to be an ordered pair* $G = (\mathrm{V}(G), \mathrm{E}(G))$*, where* $\mathrm{V}(G)$ *is a finite set and* $\mathrm{E}(G)$ *is a collection of two element subsets of* $\mathrm{V}(G)$*.*

2. $\mathrm{V}(G)$ *is said to be the vertex set of* G*.*

3. $\mathrm{E}(G)$ *is said to be the edge set of* G*.*

4. $\mathcal{G}$ *denotes the class of all graphs.*

If $\{u, v\} \in \mathrm{E}(G)$, then we simply write $uv \in \mathrm{E}(G)$, and we say that u and v are adjacent. In this case, we may also write $u \leftrightarrow v$; furthermore the edge uv is said to be incident with u and v, and u and v are said to be its endpoints. If u and v are not adjacent, then we write $u \not\leftrightarrow v$. It is possible for the vertex set of a graph to be empty, in this case, we refer to the resulting graph as the empty graph.

Sizes of vertex and edge sets of graphs are of great importance in graph theory. Notation for these two graph parameters is introduced in the following:

Definition 1.2 *Let* $G = (\mathrm{V}(G), \mathrm{E}(G)) \in \mathcal{G}$*.*

1. *Define* $\nu(G) = |\mathrm{V}(G)|$*.* $\nu(G)$ *is said to be the order of* G*.*

2. *Define* $\mathrm{e}(G) = |\mathrm{E}(G)|$*.* $\mathrm{e}(G)$ *is said to be the size of* G*.*

If a pair of graphs have distinct vertex sets, then they are technically different objects, but it is convenient to blur this distinction and consider two graphs to be identical if they have the same structure. The notion of isomorphism allows us to make this statement precise.

Definition 1.3 *Let* $G_0, G_1 \in \mathcal{G}$*. Assume that there exists a bijection* $\Phi : \mathrm{V}(G_0) \to \mathrm{V}(G_1)$ *such that* $\Phi(u) \leftrightarrow \Phi(v) \iff u \leftrightarrow v \ \forall\, u, v \in \mathrm{V}(G_0)$*. Then* G_0 *and* G_1 *are said to be isomorphic and* Φ *is said to be an isomorphism from* G_0 *to* G_1*.*

In this dissertation, we do not distinguish between isomorphic graphs, so that we are actually considering structures of "isomorphism classes" of graphs.

Several of the proofs in the following chapters proceed by induction on order, with the empty graph providing the base case. The induction step involves splitting the graph into two pieces, and then considering the structures of the resulting pair of graphs. The notions of subgraphs and induced subgraphs, which we introduce next, are therefore of considerable importance.

Definition 1.4 *Let G and H be graphs.*

1. *H is defined to be a subgraph of G, if $\mathrm{V}(H) \subseteq \mathrm{V}(G)$ and $\mathrm{E}(H) \subseteq \mathrm{E}(G)$. In this case we write $H \leq G$.*

2. *Assume that $H \leq G$, and $H \neq G$. Then H is said to be a proper subgraph of G, and we write $H < G$.*

3. *Assume that $H \leq G$ such that $uv \in \mathrm{E}(H)$ if and only if $u, v \in \mathrm{V}(H)$ and $uv \in \mathrm{E}(G)$. Then H is said to be an induced subgraph of G.*

4. *If $S \subseteq \mathrm{V}(G)$, then we define $\langle S \rangle$ to be the graph with vertex set $\mathrm{V}(\langle S \rangle) = S$ and edge set $\mathrm{E}(\langle S \rangle) = \{uv \mid u, v \in S \text{ and } uv \in \mathrm{E}(G)\}$. $\langle S \rangle$ is said to be the subgraph of G induced by S.*

5. *Define the graph $\overline{G} = \left(\mathrm{V}\left(\overline{G}\right), \mathrm{E}(\overline{G})\right)$ by $\mathrm{V}\left(\overline{G}\right) = \mathrm{V}(G)$ and $uv \in \mathrm{E}(\overline{G})$ exactly when $uv \notin \mathrm{E}(G)$. $\overline{G}$ is said to be the complement of G.*

Definition 1.5 *Let $u \in \mathrm{V}(G)$ and $U, W \subseteq \mathrm{V}(G)$.*

1. *The open neighborhood of u is defined by $\mathrm{N}(u) = \{v \in \mathrm{V}(G) \mid v \leftrightarrow u\}$.*

2. *$\mathrm{N}(u) \cup \{u\}$ is defined to be the closed neighborhood of u. We denote this set by $\mathrm{N}[u]$.*

3. *The closed neighborhood of U is defined by $\mathrm{N}[U] = \cup_{u \in U} \mathrm{N}[u]$.*

4. *Define $\mathrm{N}_W(u) = \mathrm{N}(u) \cap W$. $\mathrm{N}_W(u)$ is said to be the open neighborhood of u relative to W.*

5. *The set* $\mathrm{M}[W;U] = \{v \in \mathrm{N}[W] \mid v \notin \mathrm{N}[U]\}$ *is said to be the neighborhood difference of* W *and* U*.*

The first three items of Definition 1.5 introduce concepts that arise frequently in graph-theoretic literature, while the last two are required for the proofs presented in this paper. The structure introduced in (1.5.5) is always used within the context where $U \subseteq \mathrm{N}[W]$. Now that open neighborhoods have been defined, we may introduce the concept of degree.

Definition 1.6 *Let* G *be a graph,* $u \in \mathrm{V}(G)$ *and* $W \subseteq \mathrm{V}(G)$*.*

1. *The degree of* u *is defined by* $\mathrm{d}(u) = |\mathrm{N}(u)|$*.*

2. *Define* $\mathrm{d}(W) = \sum_{u \in W} \mathrm{d}(u)$*.* $\mathrm{d}(W)$ *is said to be the degree sum of* W*.*

3. *Define* $\delta(G) = \min\{\mathrm{d}(u) \mid u \in \mathrm{V}(G)\}$*.* $\delta(G)$ *is said to be the minimum degree of* G*.*

4. *Define* $\Delta(G) = \max\{\mathrm{d}(u) \mid u \in \mathrm{V}(G)\}$*.* $\Delta(G)$ *is said to be the maximum degree of* G*.*

5. *If* $\delta(G) = \Delta(G) = m$*, then* G *is said to be* m*-regular.*

6. *Define* $\mathrm{d}_W(u) = |\mathrm{N}_W(u)|$*.* $\mathrm{d}_W(u)$ *is said to be the degree of* u *relative to* W*.*

7. *Define* $\mathrm{MinDeg}(G) = \{v \in \mathrm{V}(G) \mid \mathrm{d}(v) = \delta(G)\}$*.* $\mathrm{MinDeg}(G)$ *is said to be the minimum degree vertex set for* G*.*

8. *Let* $D \geq 1$*, and define* $\mathcal{G}_D = \{G \in \mathcal{G} \mid \Delta(G) \leq D\}$*.*

The notions of independent sets and cliques are fundamental concepts in Ramsey theory, and independence ratios, as defined in (1.7.4), play a particularly important role in the study of Ramsey-type numbers.

Definition 1.7 *Let $U \subseteq \mathrm{V}(G)$.*

1. *U is defined to be independent if $u, v \in U \implies u \not\leftrightarrow v$.*
2. *The set of independent subsets if $\mathrm{V}(G)$ is denoted by $\mathrm{Indep}(G)$.*
3. *The independence number of G is defined to be the size of a maximum independent subset of $\mathrm{V}(G)$. We denote this number by $\alpha(G)$.*
4. *The independence ratio of G is defined by $\mu(G) = \dfrac{\alpha(G)}{\nu(G)}$.*

Definition 1.8 *Let $K \subseteq \mathrm{V}(G)$.*

1. *K is defined to be a clique if $u, v \in K \implies u \leftrightarrow v$.*
2. *The set of cliques of $\mathrm{V}(G)$ is denoted by $\mathrm{Clique}(G)$.*
3. *The clique number of G is defined to be the size of a maximum clique in $\mathrm{V}(G)$. We denote this number by $\omega(G)$.*
4. *Let $m > 0$. A graph G is said to be K_m-free if $\omega(G) < m$.*

Cliques, as defined in 1.8.1, are sets of vertices. Given such a set K, we may also refer to $\langle K \rangle$ as a clique. When considering graphs to be cliques, it is clear that two such graphs are isomorphic exactly when they have the same order. Therefore no ambiguity arises if we let K_m denote the clique of order m. Additionally, if a given graph is K_3-free, we may refer to it as being triangle-free.

Definition 1.9 *Let G be a graph.*

1. *If it is possible to partition $\mathrm{V}(G)$ into a pair of disjoint, non-empty sets S and T such that S and T are independent, then G is said to be bipartite and S and T are referred to as partite sets.*

2. *Let G be a bipartite graph, and let $S, T \subseteq \mathrm{V}(G)$ as in (1), Assume that $|S| = m$, and $|T| = n$. If $s \in S$ and $t \in T$ imply $s \leftrightarrow t$, then G is said to be a complete bipartite graph. In this case we write $G = \mathrm{K}_{m,n}$.*

3. *If one of the partite sets of a complete bipartite graph G has size one, then G is said to be a star. The member of that partite set is said to be the center of the star, and the members of the other partite sets are said to be its points. When both partite sets contain only one vertex, the star is said to be degenerate; in this case, either vertex may be designated the center.*

If G is an incomplete bipartite graph, then the partite sets need not be unique, but they are always unique if G is complete. The next definition generalizes (1.9.1) and (1.9.2).

Definition 1.10 *Let G be a graph and let $r \geq 2$.*

1. *If it is possible to partition $\mathrm{V}(G)$ into r pairwise-disjoint, non-empty sets $\{S_i\}^{i \in r}$ such that each S_i is independent, then G is said to be r-partite. In this case the sets S_i are said to be partite sets.*

2. *If G is an r-partite graph with partite sets $\{S_i\}^{i \in r}$ such that $s^i \leftrightarrow s^j$ whenever $i, j \in r$, $i \neq j$, $s^i \in S_i$ and $s^j \in S_j$, then G is said to be a complete r-partite graph.*

3. *Let G by an r-partite graph satisfying the property that any two partite sets differ in size by at most one, then G is said to be a balanced r-partite graph.*

4. *If the r-partite graph G is both balanced and complete, then G is said to be a Turan graph. In the case that $\nu(G) = n$, we write $G = \mathrm{T}_{n,r}$.*

In the next chapter, it will be shown that complements of Turan graphs play an important role in the theory of Ramsey-type numbers.

Several of the theorems in the following chapters are applicable to graphs that are K_m-free for a particular value of m. In addition, we may also require graphs to satisfy upper bounds on their maximum degree. The final definition of this section provides notation for these two classes of graphs.

Definition 1.11 *Let $D \geq 1$, and let $m \geq 2$.*

1. *Let $\mathcal{G}(m)$ denote the class of K_m-free graphs.*

2. *Define $\mathcal{G}_D(m) = \{G \in \mathcal{G}(m) \mid \Delta(G) \leq D\}$.*

The notation introduced in this definition will be used frequently in the next section when discussing results obtained by various authors, but it should be noted that none of them used this particular notation in their publications.

Additional graph-theoretic definitions that pertain specifically to the proof techniques used in this dissertation are introduced in Chapter 3.

1.2 History of Ramsey-Type Numbers

This section contains a synopsis of the published work in the field of Ramsey-type numbers. There are several classes of numbers that are called Ramsey-type numbers; in this paper, we only consider Ramsey-type numbers as defined by Staton [21] in his 1977 dissertation. He introduced these numbers in order to gain some insight into the problem of determining Ramsey numbers.

It is appropriate to begin with a brief discussion of Ramsey's original work. In 1930, he introduced a class of numbers that have come to bear his name. While he was able to establish the existence of these numbers via an elegant double induction argument, the calculation of their values has proved to be extremely difficult. There is a subclass of the Ramsey numbers, known as the classical, two-color Ramsey numbers, which can be defined in graph-theoretic terms. These numbers constitute the simplest class of Ramsey numbers, yet only minimal progress has been made in this area. This class of Ramsey numbers can be informally defined in terms of edge colorings of graphs as follows:

Let K be a complete graph, and assign to each edge of K either the color blue or the color red. We say that a subset W of $\mathrm{V}(K)$ induces a blue (red) clique if each edge joining members of W is assigned the color blue (red). In this context, Ramsey's Theorem states that, given any pair of positive integers m and n, there exists a smallest integer $R(m,n)$ satisfying the property that, given any edge coloring of the complete graph K of order $R(m,n)$, there exists either a subset of $\mathrm{V}(K)$ of size m that induces a blue clique or a subset of size n that induces a red clique. The number $R(m,n)$ is said to be a Ramsey number.

There is an alternative formulation of this problem that is phrased in terms of clique and independence numbers: Given an edge coloring of the complete graph K, we define a subgraph G by letting $\mathrm{V}(G) = \mathrm{V}(K)$ and taking the edge set of G to consist exactly of those edges of K that are assigned the color blue. Now, if $W \subset \mathrm{V}(G)$, then W is a clique in G exactly when each edge that has both endpoints in W is assigned the color blue, and it is independent in G exactly when each edge that has both endpoints in W is assigned the color red. This observation allows us to formally define the classical, two-color Ramsey numbers.

Definition 1.12 *(Classical, Two-Color Ramsey Numbers) Let $m, n \geq 2$.*

1. *Define $R(m,n)$ to be the smallest integer satisfying the property that, given a graph G of order $R(m,n)$, either $\omega(G) \geq m$ or $\alpha(G) \geq n$. The numbers $R(m,n)$ are said to be the classical, two-color Ramsey numbers.*

2. *Let G be a graph and assume that $\nu(G) = R(m,n) - 1$, $\omega(G) < m$ and $\alpha(G) < n$. Then G is said to be (m,n)-critical.*

Tight lower bounds for Ramsey numbers are generally established by exhibiting critical graphs.

Since any graph either does or does not contain an edge, the numbers $R(2,n)$ follow immediately, but numbers of the form $R(3,n)$ are much more difficult to obtain. The problem of determining $R(3,3)$ appeared on a Putnam exam several years ago. In 1955 Greenwood and Gleason [10] calculated $R(3,4) = 9$, $R(3,5) = 14$, and $R(4,4) = 18$ and Kalbfleisch included the calculation of $R(3,7) = 23$ and $R(3,9) = 36$ in his 1966 dissertation. The Ramsey number $R(4,5) = 25$ was obtained by MacKay and Radziszowski [17] in 1995, and no new numbers have been computed since. Known bounds for several Ramsey numbers are included in Radziszowski's Dynamic Survey in the Electronic Journal of Combinatorics [18]. The large differences between known lower and upper bounds for relatively small values of m and n, as reported in that survey, attest to the difficulty of this problem.

The classical, two-color problem has proved to be so difficult that graph theorists have proposed several simplifications of it. In this dissertation, we utilize Staton's notion of Ramsey-type Numbers. He suggested restricting the problem size by imposing a maximum degree constraint, as specified in the following:

Definition 1.13 *(Ramsey-Type Numbers) Let $D \geq 1$, and let $m, n \geq 2$.*

1. *Define $R_D(m,n)$ to be the smallest integer for which, given a graph G, $\nu(G) = R_D(m,n)$ and $\Delta(G) \leq D$ imply that either $\omega(G) \geq m$ or $\alpha(G) \geq n$. $R_D(m,n)$ is said to be a Ramsey-type number.*

2. *If G is a graph for which $\nu(G) = R(m,n) - 1$, $\Delta(G) \leq D$, $\omega(G) < m$ and $\alpha(G) < n$, then G is said to be (D, m, n)-critical.*

Staton additionally calculated the Ramsey-type numbers $R_3(3,n)$, for all values of n. The existence of Ramsey numbers clearly ensures the existence of Ramsey-type numbers. In fact we have

$$R_D(m,n) \leq R(m,n), \tag{1.2}$$

for any choices of D, m and n. While Definition 1.13 yields a simplification of the original problem, only a few classes of these numbers have been determined.

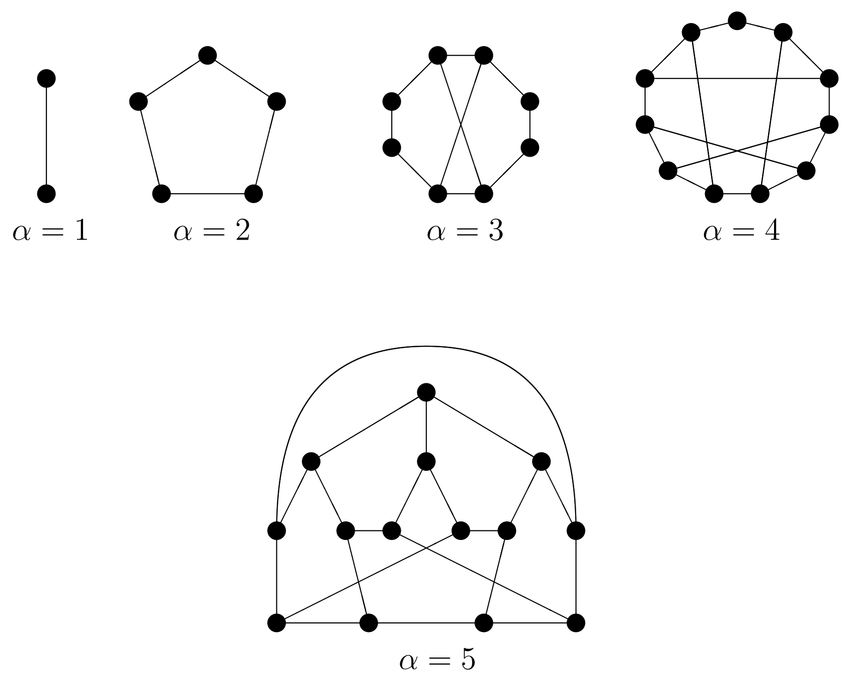
Independence ratios, which were introduced in Definition 1.7.4, played an important role in Staton's work. He showed that, for each $G \in \mathcal{G}_3(3)$,

$$\mu(G) \geq \frac{5}{14} \tag{1.3}$$

If we let $n \geq 2$, then (1.3) immediately implies

$$R_3(3, n) \leq \left\lfloor \frac{14}{5}(n-1) \right\rfloor + 1. \tag{1.4}$$

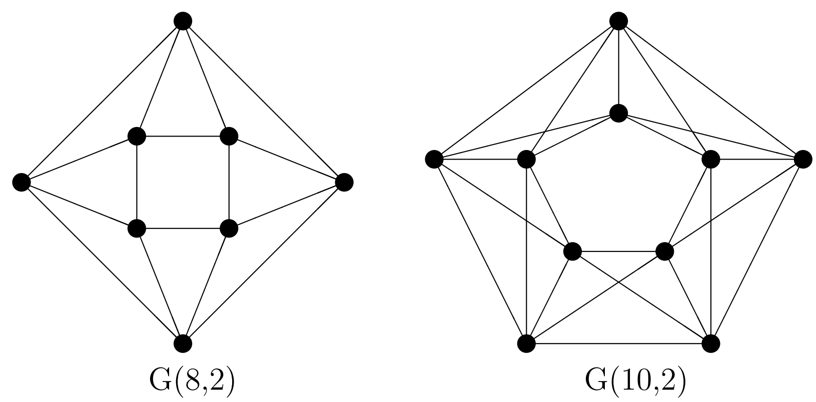
Fajtlowicz, Staton's advisor, had previously constructed the order-14 graph depicted in Figure 1.1. Since this graph belongs to $\mathcal{G}_3(3)$ and has independence five, the tightness of Staton's independence ratio bound follows. Sums of these graphs yield critical graphs for numbers of the form $R_3(3, 5k+1)$, and sums of these graphs plus one of the other graphs depicted in Figure 1.1 yield critical graphs for the remaining independence numbers.



These graphs are used in the construction of critical graphs for the Ramsey-type numbers $R_3(3, n)$ as follows: If $n - 1 = 5k$, then we form a sum consisting of k copies of the order-14-graph. Otherwise we form a sum consisting of copies of that graph and one copy of a graph of smaller independence.

Figure 1.1: BUILDING BLOCKS FOR $R_3(3, n)$-CRITICAL GRAPHS

An interesting result pertaining to independence ratios was published in 1976 by Albertson, Bollobas and Tucker [1]. They showed that, for each $m \geq 3$, any connected, m-regular, K_m-free graph satisfies the bound $\mu(G) \geq \frac{1}{m}$. They also showed that, with the exception of the two graphs depicted in Figure 1.2, this bound is strict. We shall refer to these authors as ABT.



These two graphs are the unique connected, K_m-free graphs for which $\Delta = m$ and $\mu = \frac{1}{m}$ when $m \geq 3$. They were constructed by Albertson, Bollobas and Tucker in 1976.

Figure 1.2: THE EXCEPTIONAL GRAPHS OF ABT

This theorem does not yield upper bounds for the Ramsey-type numbers $R_m(m,n)$, because its hypothesis requires graphs to be regular. This problem was remedied in 1977 by Fajtliwicz [4], when he showed that any K_m-free graph G of maximum degree Δ satisfies the bound

$$\mu(G) \geq \frac{2}{m+\Delta}. \tag{1.5}$$

He established the tightness of this bound for of graphs of independence two in the case that

$$3m - 2\Delta = 5, \tag{1.6}$$

and showed that it is not tight if

$$3m - 2\Delta > 5. \tag{1.7}$$

He returned to this problem in 1984 [5], and showed that, given (1.6), independence two is necessary in order to attain tightness, and, in that case, it is yielded by a unique graph. He called these graphs butterflies because they have large chromatic numbers and possess a great deal of symmetry. In his review of this article, Albertson noted that Catlin used butterflies as counterexamples to Hayos's conjecture.

In his 1977 article, Fajtlowicz stated that the independence ratio bounds $\mu(G) \geq \frac{1}{m}$ are not tight when $m \geq 7$, and, with the exception of the two graphs exhibited by ABT, the bounds for the cases $m = 4, 5$ also fail to be tight.

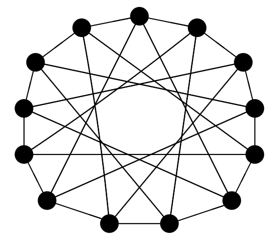
In her 1982 dissertation, Fraughnaugh [6], another of Fajtlowicz's students, followed Staton by calculating all Ramsey-type numbers $R_4(3, n)$. She accomplished this by showing that the independence ratio of any member of $\mathcal{G}_4\left(3\right)$ must be at least $\frac{4}{13}$. The graph illustrated in Figure 1.3 establishes tightness.

She derived this independence ratio bound by showing that any $G \in \mathcal{G}_4\left(3\right)$ satisfies the size bound

$$\mathrm{e}\left(G\right) \geq 6\nu(G) - 13\alpha(G). \tag{1.8}$$

Since each $G \in \mathcal{G}_4\left(3\right)$ also satisfies

$$2\nu(G) \geq \mathrm{e}\left(G\right), \tag{1.9}$$



$\alpha = 4$

This graph has order 13, independence 4 and is 4-regular. Given that independence ratios of triangle-free graphs with $\Delta \leq 4$ must be at least $\frac{4}{13}$, we see that this graph is $(4,3,5)$-critical. Since $R(3,5) = 14$, this graph is also $(3,5)$-critical.

Figure 1.3: THE $(4,3,5)$-CRITICAL GRAPH

the desired result follows by juxtaposing the preceding two inequalities and solving for $\mu(G)$. Kreher and Radziszowski [19], unaware of Fraughnaugh's work, showed that (1.8) holds for all K_3-free graphs regardless of maximum degree. The technique of deriving independence ratio bounds from size bounds has become the primary tool for computing upper bounds for Ramsey-type numbers.

Staton did not use a size bound to obtain the $\frac{5}{14}$ independence ratio bound, but Fraughnaugh [15] did use this technique in an alternate proof by showing that

$$\mathrm{e}\,(G) \geq \frac{13}{2}\nu(G) - 14\alpha(G) \tag{1.10}$$

for each member of $\mathcal{G}_3\,(3)$, and noting that

$$\frac{3}{2}\nu(G) \geq \mathrm{e}\,(G)\,. \tag{1.11}$$

Additional independence ratio bounds were obtained by Fraughnaugh in collaboration with Locke. In 1994 [7], they proved that each $G \in \mathcal{G}_3\,(3)$ satisfies the bound

$$\mu(G) \geq \frac{11}{30} - \frac{2}{15\nu(G)}, \tag{1.12}$$

thereby showing that Staton's $\frac{5}{14}$ bound is not tight when $\nu(G) > 14$. They used a variation of the size bound technique to establish (1.12) by defining

$$\gamma(G) = \mathrm{e}\,(G) - 7\nu(G) + 15\alpha(G) \tag{1.13}$$

and showing that

$$\gamma(G) \geq -2. \tag{1.14}$$

They also constructed graphs of arbitrarily large order that satisfy the bound

$$\mu(G) \leq \frac{11}{30} - \frac{1}{15\nu(G)}, \tag{1.15}$$

and thereby obtained a tightness result.

In 1997, [9] they published the following three size bounds for K_4-free graphs:

$$\mathrm{e}\,(G) \geq \nu(G) - \alpha(G), \tag{1.16}$$

$$\mathrm{e}\,(G) \geq 2\nu(G) - 3\alpha(G) \tag{1.17}$$

and

$$\mathrm{e}\,(G) \geq 5\nu(G) - 12\alpha(G). \tag{1.18}$$

By imposing the additional constraint that graphs have maximum degree at most 4, they were able to use (1.18) to obtain an alternative proof that the independence ratio of any member of $\mathcal{G}_4\,(4)$ must be at least $\frac{1}{4}$. They then used this bound to compute the Ramsey-type numbers $R_4(4, n)$.

They also obtained some interesting tightness results for these size bounds. They showed that (1.16) is tight for independence ratios between $\frac{1}{2}$ and 1, (1.17) is tight for independence ratios between $\frac{1}{3}$ and $\frac{1}{2}$ and (1.18) is tight for independence ratios between $\frac{1}{4}$ and $\frac{1}{3}$. This is the only publication in which the problem of multiple size bounds for a given class of graphs has been addressed.

Locke and Lou [16] considered the problem of finding independence ratio bounds for 4-regular, K_4-free graphs. They showed that

$$\alpha(G) \geq \frac{7\nu(G) - 4}{26} \tag{1.19}$$

holds for this class of graphs, and thereby obtained $\frac{7}{26}$ as an asymptotic lower bound for independence ratios. They also established the existence of arbitrarily large graphs that satisfy the bound $\frac{3}{11}$ with equality. The tight independence ratio bound therefore lies in a narrow interval; there has been no subsequent improvement upon their results.

Griggs and Murphy [11] developed an efficient algorithm for finding independent sets of size at least $\frac{5}{14}\nu(G)$ for any non-regular $G \in \mathcal{G}_3\,(3)$. This algorithm is easily modified to work for 3-regular graphs by removing an edge before applying it. It is clear that removing edges from a graph, while holding order constant tends to increase the independence ratio, but they are the only authors who have attempted to quantify this relationship. They established relationships between the edge densities of graphs, which are defined by $\frac{\mathrm{e}(G)}{\nu(G)}$, and independence ratios.

In his 1998 dissertation, Sandberg [20], a student of Griggs, introduced two bounds for independence numbers that follow from variations of size bounds.

He showed that each $G \in \mathcal{G}_5\,(5)$ satisfies

$$\alpha(G) \geq \frac{11\nu(G) - 6}{52} \tag{1.20}$$

and obtained $\frac{1}{5}$ as an independence ratio bound for members of $\mathcal{G}_5\,(5)$, in agreement with ABT. He also showed that

$$\alpha(G) \geq \frac{8\nu(G) - 1}{45} \tag{1.21}$$

for each $G \in \mathcal{G}_6\,(6)$. By applying this bound and using special arguments for graphs of order at most 17, he was able to obtain

$$\mu(G) \geq \frac{3}{17} \tag{1.22}$$

for each member of that class of graphs. He also exhibited a member of $\mathcal{G}_6\,(6)$ that has order 11 and independence two.

In 2001 Heckman and Thomas [12] published a simple proof of Staton's $\frac{5}{14}$ independence ratio bound, but there have been no new bounds published since Sandberg's 1998 dissertation.

1.3 An Overview of This Dissertation

This dissertation addresses some open problems pertaining to size and independence ratio bounds and their relationship to Ramsey-type numbers. Some general results concerning size bounds are presented in Chapter 2. In that chapter, it is shown that the size bounds (1.16) and (1.17) introduced by Fraughnaugh and Locke [9] are the first two terms in an infinite sequence of size bounds, each of which holds for all graphs. This sequence is specified and the range of independence ratios for which each bound is tight is established.

The techniques for establishing size bounds that are used in this dissertation are an abstraction and refinement of the techniques used by Fraughnaugh and Locke [9]. Chapter 3 contains an informal overview of the proof inductive method used here, followed by a rigorous development of the proof techniques. This material provides the foundation upon which all of the theory developed in this dissertation is built. For this reason, it should be considered to be an integral part of that research.

Two size bounds are established in Chapter 4. These two proofs, which are relatively simple, and yet yield previously unpublished results, are intended to provide the reader with examples of how the proof techniques developed in Chapter 3 are used in this dissertation.

In Chapter 5, it is shown that, if $m \geq 3$, $G \in \mathcal{G}_m(m)$, G is connected and G is not the graph $G(10, 2)$, that is depicted in Figure 1.2, then

$$\mu(G) \geq \frac{3m-6}{3m^2-7m+4}. \tag{1.23}$$

For each $m \geq 7$, this yields an improvement over the previous best known bound of $\mu \geq \frac{1}{m}$, and when $m = 5$, with the exception of $G(10, 2)$, we obtain $\frac{9}{44}$ as an independence ratio bound. This is an improvement upon the previous best known bound of $\mu \geq \frac{1}{5}$. Techniques from Chapter 3 and a symmetry argument specific to the class of graphs in question are used to establish this result.

Chapters 6 and 7 are devoted to the establishment of size bounds for K_6 and K_7-free graphs, respectively.

In Chapter 8, issues pertaining to Fajtlowicz's Independence ratio bound

$$\mu(G) \geq \frac{2}{m+\Delta(G)} \tag{1.24}$$

are addressed. His tightness results for the case $3m-2\Delta = 5$ are briefly reviewed and new results for the case $3m - 2\Delta = 4$ are introduced.

Chapter 9 contains a summary of the material presented in this dissertation and suggestions for further study.

2. Size Bounds

In the first chapter, it was noted that upper bounds for Ramsey-type numbers follow from lower bounds for independence ratios, which, in turn, follow from lower bounds for size. Formalizations of these relationships are presented in Section 2.1. The traditional definition of size bounds, as presented in that section, is used throughout the following chapters but, when comparing two size bounds, as is done in this chapter, an alternative definition proves useful; this definition is presented in Section 2.2. An infinite collection of size bounds, each of which holds for all graphs, is constructed in Section 2.3, and it is shown that each is tight over a range of independence ratios. Issues pertaining to size bounds when a forbidden clique constraint is imposed are discussed in Section 2.4.

2.1 Size Bounds and Ramsey-Type Numbers

We begin by offering a formal definition of size bounds.

Definition 2.1 *Let $\mathcal{C}$ be a non-empty collection of graphs and assume that $a > 0$ and $b < 0$.*

1. *If $\mathrm{e}(G) \geq a\nu(G) + b\alpha(G)$ for each $G \in \mathcal{C}$, then a and b are said to yield a size bound for $\mathcal{C}$.*

2. *If there exists $G \in \mathcal{C}$ such that equality holds in (1), then the bound is said to hold tightly at G. In this case, the bound is also said to hold tightly at $(\nu(G), \alpha(G))$ and at $\mu(G)$.*

The conditions $a > 0$ and $b < 0$, as imposed in the preceding definition, are reasonable because increasing order while keeping independence fixed yields an increase in size, while increasing independence while holding order fixed corresponds to a decrease in size. For the remainder of this chapter, we assume that a and b satisfy these properties.

The following pair of lemmas are used to convert size bounds into upper bounds for Ramsey-Type numbers.

Lemma 2.2 *Let $a > 0$, $b < 0$ and $D \geq 1$, Assume that G satisfies the inequality stated in Definition 2.1.1 and $\Delta(G) \leq D$. Then*

$$\mu(G) = \frac{\alpha(G)}{\nu(G)} \geq -\frac{a - \frac{D}{2}}{b}. \tag{2.1}$$

Proof: It is clear that

$$\frac{D\nu(G)}{2} \geq \mathrm{e}\left(G\right). \tag{2.2}$$

Now combine Definition (2.1.1) and (2.2) and solve to obtain the desired inequality. ■

Lemma 2.3 *Let $a > 0$, $b < 0$, $m \geq 3$ and $D, \geq 1$, Assume that there exists a constant c such that*

$$\mu(G) \geq c, \quad \forall\, G \in \mathcal{G}_D\left(m\right). \tag{2.3}$$

Then

$$R_D(m, n) \leq \left\lfloor c^{-1}(n-1) \right\rfloor + 1. \tag{2.4}$$

Proof: It is clear that a (D, m, n)-critical graph can have order no larger than $\lfloor c^{-1}(n-1) \rfloor$. ■

When using size bounds to compute upper bounds for Ramsey-type numbers $R_D(m,n)$, we fix D and m and establish a relationship between n and $R_D(m,n)$. This is done by showing that a particular size bound holds for all members of $\mathcal{G}_D\,(m)$, applying Lemma 2.2, so as to obtain an independence ratio bound, and then applying Lemma 2.3.

2.2 Size Bound Operators

Consider the right hand side of the inequality stated in Definition 2.1.1. For fixed a and b we may view the sum

$$a\nu(G) + b\alpha(G) \tag{2.5}$$

as a function that takes orders and independence numbers of graphs as parameters. When considering collections of size bounds, it is convenient to express each of them in terms of a function that incorporates (2.5) into its definition. Furthermore, when comparing two size bounds, it is instructive to extend the domains of these functions, so as to allow for non-integer arguments. For these two reasons we introduce the following:

Definition 2.4 *Let $a > 0$ and $b < 0$.*

1. *Define* $\mathrm{BD} = \{(x,y) \in \mathbb{R}^2 \mid x, y \geq 0 \text{ and } y \leq x\}$. BD *is said to be the size bound domain.*

2. *Define $B_{a,b} : \mathrm{BD} \to \mathbb{R}$ by $B_{a,b}(x,y) = ax + by$. $B_{a,b}$ is said to be a size bound operator.*

3. *Given a graph G, the expression $\mathrm{e}\,(G) \geq B_{a,b}(\nu(G), \alpha(G))$ is said to be a size inequality.*

In this definition, we let x correspond to order and y to independence so that, given any graph G, we have $(\nu(G), \alpha(G)) \in \mathrm{BD}$. For fixed a and b, it is clear that

$$\mathrm{e}\,(G) \geq a\nu(G) + b\alpha(G) \iff \mathrm{e}\,(G) \geq B_{a,b}(\nu(G), \alpha(G)). \tag{2.6}$$

Therefore we have a natural correspondence between size bounds and size bound operators. The following lemma, whose proof is obvious, yields a fundamental relationship between two size bounds.

Lemma 2.5 *Let $a, c > 0$ and $b, d < 0$ such that $b \neq d$ and let $(x, y) \in \mathrm{BD}$. Then*

$$B_{a,b}(x, y) = B_{c,d}(x, y) \iff \frac{y}{x} = \frac{a-c}{d-b}. \tag{2.7}$$

Whenever two bounds are compared in this dissertation, it is always the case that $a > c$ exactly when $b < d$ and

$$0 < \frac{a-c}{d-b} \leq 1, \tag{2.8}$$

so that $\frac{a-c}{d-b}$ yields an independence ratio. Therefore, if $B_{a,b}$ and $B_{c,d}$ are equal exactly on a ray that is contained in BD, then one bound is dominant for independence ratios that lie above that ray and the other is dominant for independence ratios that lie below it.

When considering bounds of any sort, tightness is always an important issue. This problem is particularly difficult when considering size bounds, because, a complete answer to the question of tightness often requires the specification of an infinite collection of bounds (This problem will be addressed in the following two sections); furthermore there remain some unresolved issues that make the problem of simply defining a precise notion of tightness difficult. The following definition yields three different formalizations of this concept.

Definition 2.6 *Let $\mathcal{C}$ be a class of graphs and let $B_{a,b}$ be a size bound operator*

1. *Then $B_{a,b}$ is said to yield a $\mathcal{C}$-size bound if the inequality specified in Definition 2.4.3 holds for each $G \in \mathcal{C}$.*

2. *Assume that $B_{a,b}$ yields a $\mathcal{C}$-size bound and let $G \in \mathcal{C}$. Then $B_{a,b}$ is said to hold tightly at G if $\mathrm{e}\,(G) = B_{a,b}(\nu(G), \alpha(G))$. In this case we also say that $B_{a,b}$ holds tightly at $(\nu(G), \alpha(G))$ and at $\mu(G)$.*

If a bound holds tightly at (n, r), then it also holds tightly at the independence ratio $\frac{r}{n}$, but it appears that it might not hold tightly at (n_0, r_0) where $\frac{r_0}{n_0}$ represents $\frac{r}{n}$ reduced to lowest terms. Such situations will arise in the following chapters.

Fraughnaugh and Locke introduced the notions of (n, r)-graphs and (n, r)-mingraphs. The following definition associates those concepts with a particular class of graphs.

Definition 2.7 *Let $\mathcal{C}$ be a collection of graphs, let $n, r \in \mathbb{N}$ such that $0 < r \leq n$ and let $G \in \mathcal{G}$ such that $\nu(G) = n$ and $\alpha(G) = r$.*

1. *Then G is said to be an (n, r)-graph.*

2. *If $G \in \mathcal{C}$ then G is said to be a $(\mathcal{C}, n, r)$-graph.*

3. *If G is a $(\mathcal{C}, n, r)$-graph and $\mathrm{e}\,(G) \leq \mathrm{e}\,(H)$ whenever H is a $(\mathcal{C}, n, r)$-graph, then said to be a $(\mathcal{C}, n, r)$-mingraph.*

The characterization of mingraphs that correspond to particular sets of parameters is an important aspect of the study of Ramsey-type numbers.

2.3 Turan Bounds

It was noted in Chapter 1 that Fraughnaugh and Locke [9] established the following pair of size bounds for K_4-free graphs:

$$\mathrm{e}(G) \geq \nu(G) - \alpha(G) \tag{2.9}$$

and

$$\mathrm{e}(G) \geq 2\nu(G) - 3\alpha(G). \tag{2.10}$$

Since their proofs did not utilize the fact that only K_4-free graphs were being considered, they actually showed that these bounds hold for all graphs. They additionally obtained an interesting pair of tightness results, which we combine into a single theorem.

Theorem 2.8 *Let $r, n \in \mathbb{N}$ be such that $0 < r \leq n$.*

1. *There exists a $(\mathcal{G}, n, r)$-graph G for which (2.9) holds with equality exactly when*

$$\frac{1}{2} \leq \frac{r}{n} \leq 1 \tag{2.11}$$

and, in this case

$$(2r - n)\,\mathrm{K}_1 + (n - r)\,\mathrm{K}_2 \tag{2.12}$$

yields the unique $(\mathcal{G}, n, r)$-mingraph.

2. *There exists a $(\mathcal{G}, n, r)$-graph G for which (2.10) holds with equality exactly when*

$$\frac{1}{3} \leq \frac{r}{n} \leq \frac{1}{2} \tag{2.13}$$

and, in this case,

$$(3r - n)\,\mathrm{K}_2 + (n - 2r)\,\mathrm{K}_3 \tag{2.14}$$

yields the unique $(\mathcal{G}, n, r)$*-mingraph.*

This pair of size bounds suggests the following generalization. Establish the existence of an infinite sequence of size bound operators $\{B_{a_l,b_l}\}_{l=1}^{\infty}$ that satisfies the following four conditions:

1. B_{a_1,b_1} and B_{a_2,b_2} correspond to (2.9) and (2.10) respectively.
2. $\mathrm{e}\,(G) \geq B_{a_l,b_l}(\nu(G),\, \alpha(G)) \quad \forall\, G \in \mathcal{G}, \quad \forall\; l \geq 1.$
3. There is a $(\mathcal{G}, n, r)$-graph for which B_{a_l,b_l} holds tightly exactly when $\frac{1}{l+1} \leq \frac{r}{n} \leq \frac{1}{l}$.
4. In the case that $\frac{r}{n}$ satisfies the preceding inequality, there exist constants c and d such that tightness is yielded uniquely by a graph of the form $c\,\mathrm{K}_l + d\,\mathrm{K}_{l+1}$.

The remainder of this section is devoted to establishing that such a sequence of operators does indeed exist. In order to simplify notation, we shall write B_l in place of B_{a_l,b_l}.

It is clear that a graph is the unique $(\mathcal{G}, n, r)$-mingraph exactly when its complement is the unique graph of order n and clique number r of maximum size; Turan's theorem states that this graph is the Turan graph $\mathrm{T}_{n,r}$ (This graph was defined in (1.10.4). We immediately obtain the following corollary to Turan's Theorem:

Corollary 2.9 *Let $n, r \in \mathbb{N}$ such that $0 < r \leq n$. then $\overline{T}_{n,r}$ is the unique $(\mathcal{G}, n, r)$-mingraph.*

In order to satisfy condition 4, it is necessary that B_l hold tightly at both K_l and K_{l+1}. Therefore the coefficients a_l and b_l are computed by solving the following linear system.

$$\begin{pmatrix} l & 1 \\ l+1 & 1 \end{pmatrix} \begin{pmatrix} a_l \\ b_l \end{pmatrix} = \begin{pmatrix} \binom{l}{2} \\ \binom{l+1}{2} \end{pmatrix}. \tag{2.15}$$

We obtain

$$a_l = l \quad \text{and} \quad b_l = -\binom{l+1}{2}. \tag{2.16}$$

Notice that when $l = 1$ and $l = 2$, the resulting size bound operators correspond to the size bounds (2.9) and (2.10), respectively.

We now show that each B_l yields a $\mathcal{G}$-size bound.

Lemma 2.10 *Let $l > 0$ and assume that $0 < n \leq r$. Then*

$$B_l(n, r) = B_{l+1}(n, r) \iff \frac{r}{n} = \frac{1}{l+1}, \tag{2.17}$$

and

$$B_l(n, r) > B_{l+1}(n, r) \iff \frac{r}{n} > \frac{1}{l+1}. \tag{2.18}$$

Proof:

$$B_l(n,r) - B_{l+1}(n,r) = ln - \binom{l+1}{2} r - \left((l+1)n - \binom{l+2}{2} r \right) \tag{2.19}$$

$$= -n + \left(\binom{l+2}{2} - \binom{l+1}{2} \right) r \tag{2.20}$$

$$= -n + \left(\frac{(l+2)(l+1)}{2} - \frac{(l+1)l}{2} \right) r \tag{2.21}$$

$$= -n + (l+1)r. \tag{2.22}$$

Clearly

$$-n + (l+1)r \geq 0 \iff \frac{r}{n} \geq \frac{1}{l+1}. \tag{2.23}$$

Now (2.17) follows by continuity. ∎

Theorem 2.11 *Let $G \in \mathcal{G}$ and let $l \geq 1$. Then B_l yields a $\mathcal{G}$-size bound that is tight at the independence ratio $\frac{r}{n}$ exactly when*

$$\frac{1}{l+1} \leq \frac{r}{n} \leq \frac{1}{l}. \tag{2.24}$$

Furthermore, if (2.24) is satisfied, then $\overline{T}_{n,r}$ is the unique $(\mathcal{G}, n, r)$-mingraph.

Proof: First assume that (2.24) holds and define c and d by

$$\begin{pmatrix} 1 & 1 \\ l & l+1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} r \\ n \end{pmatrix}. \tag{2.25}$$

We have

$$c = (l+1)r - n \quad \text{and} \quad d = n - lr. \tag{2.26}$$

Both c and d are integers, both are non-negative exactly when $\frac{1}{l+1} \leq \frac{r}{n} \leq \frac{1}{l}$ and they cannot be simultaneously equal to zero. Therefore

$$G = c\,\mathrm{K}_l + d\,\mathrm{K}_{l+1} \tag{2.27}$$

is a well defined graph. It is the complement of a Turan graph because it is a balanced sum of cliques. We also have

$$\nu(G) = lc + (l+1)d = l((l+1)r - n) + (l+1)(n - lr) = n \tag{2.28}$$

and

$$\alpha(G) = c + d = (l+1)r - n + n - lr = r. \tag{2.29}$$

Therefore

$$G = \overline{T}_{n,r}. \tag{2.30}$$

The bound B_l was defined so as to hold tightly for both K_l and K_{l+1}. Since the order (size, independence number) of a sum of graphs equals the sum of the orders (sizes, independence numbers) of the terms, it follows that B_l holds tightly at $\overline{T}_{n,r}$, and it follows from Corollary 2.9, that any other $(\mathcal{G}, n, r)$-graph must satisfy B_l with strict inequality. It has now been established that the inequality stated in Definition 2.4.3 holds whenever (2.24) is satisfied, so assume that

$$\frac{r}{n} \notin \left[\frac{1}{l+1}, \frac{1}{l}\right]. \tag{2.31}$$

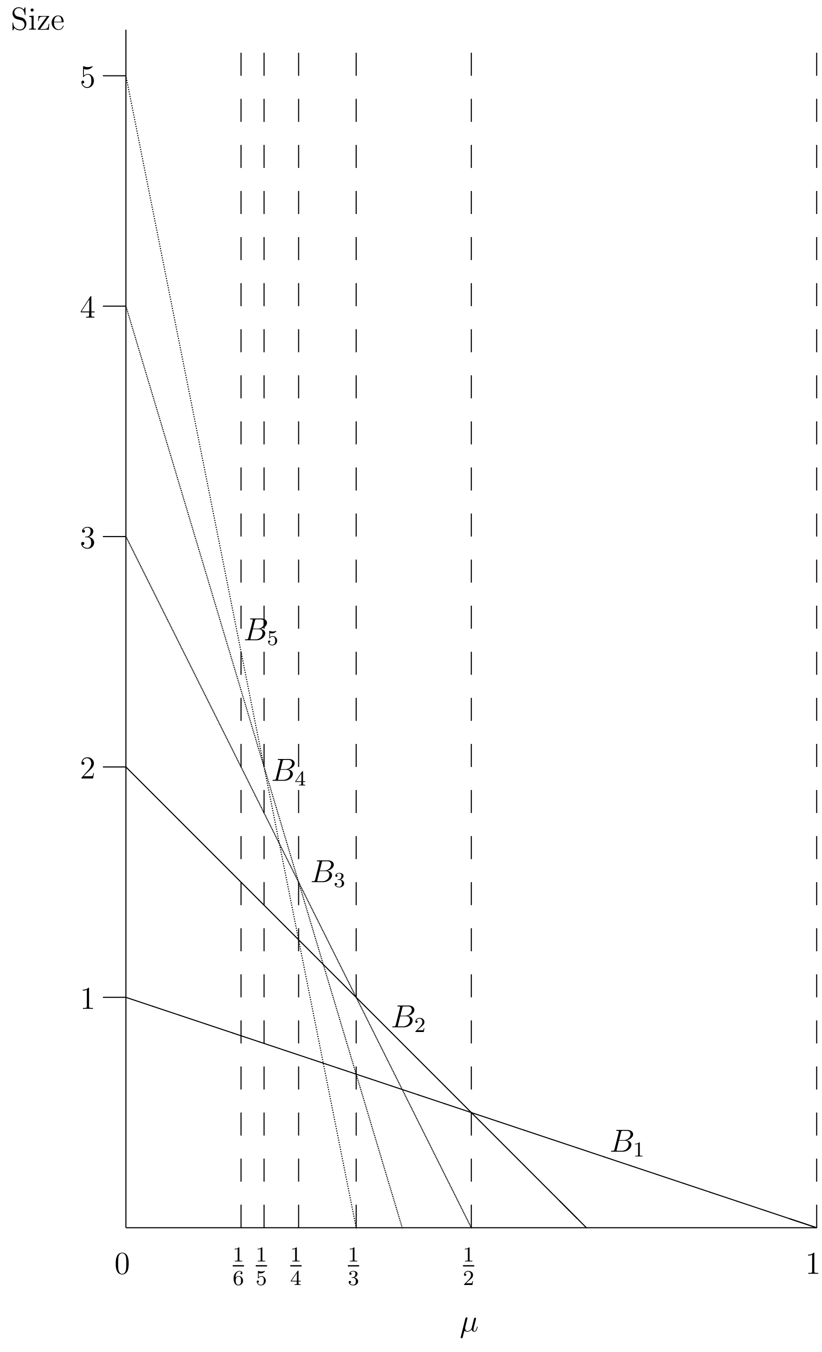
Say

$$\frac{1}{k+1} \leq \frac{r}{n} \leq \frac{1}{k}. \tag{2.32}$$

Now $B_k(n, r)$ yields a tight bound at independence ratio $\frac{r}{n}$, and it follows by Lemma 2.10 and a simple induction that B_l provides a bound that is not tight for independence ratios outside of the interval $[\frac{1}{l+1}, \frac{1}{l}]$. ■

If we restrict B_l to $\{(1, r) \mid 0 \leq r \leq 1\}$ and view it as a function of r, then we obtain a function that is evaluated at independence ratios. Figure 2.1 illustrates the graphs of such restrictions of B_l, $1 \leq l \leq 5$. Notice that $B_l = B_{l+1}$ at independence ratio $\frac{1}{l}$ and B_l dominates the other bounds over the interval $[\frac{1}{l+1}, \frac{1}{l}]$. This figure provides insight into Lemma 2.10 and the induction used in the proof of the preceding theorem.

Definition 2.12 *The size bound operators* $\{B_l\}_{l=1}^{\infty}$ *are said to be Turan size bound operators, and we refer to the associated size bounds as Turan size bounds.*



This figure illustrates graphs of Turan bounds $B_l, \quad 1 \leq l \leq 5$ restricted to the set $\{(1, r) \mid 0 \leq r \leq 1\}$. The graphs for each B_l and B_{l+1} cross when the independence ratio is equal to $\frac{1}{l}$, with B_l dominating B_{l+1} over smaller independence ratios. This figure provides an illustration of Theorem 2.11.

Figure 2.1: RESTRICTED TURAN SIZE BOUND OPERATORS

Let $(x, y) \in \mathrm{BD}$, and define

$$B(x, y) = \max_{l \geq 1} B_l(x, y). \tag{2.33}$$

Then, for each $G \in \mathcal{G}$,

$$\mathrm{e}\,(G) \geq B(\nu(G), \alpha(G)). \tag{2.34}$$

In addition, for each $(n, r) \in \mathrm{BD}$, $\overline{T}_{n,r}$ is the unique graph for which (2.34) holds with equality. We have constructed a tight, piecewise-linear lower bound for sizes of graphs. While this construction is relatively simple, the analogous problem, when a forbidden clique constraint is imposed, is extremely difficult.

2.4 Forbidden Cliques

We now consider the problem of obtaining size bounds for the class of K_m-free graphs. Fraughnaugh and Locke [9] showed that any K_4-free graph satisfies the size bound

$$\mathrm{e}\,(G) \geq 5\nu(G) - 12\alpha(G), \tag{2.35}$$

and it is easily verified that (2.35) does not hold for all graphs (simply consider K_4). They also showed that, if $\frac{1}{4} \leq \frac{r}{n} \leq \frac{1}{3}$ and r is even, then there exists a member of $\mathcal{G}_4\,(4)$ of order n and independence r that satisfies (2.35) with equality, but, in the case that r is odd, the issue of tightness remained open. The notion of tightness at the independence ratio $\frac{r}{n}$ does not necessarily coincide with the notion of tightness at (n, r), because, when r is odd we still obtain tightness at $\frac{r}{n}$ from tightness at $(2n, 2r)$. As long as this problem remains unresolved, it appears that a precise formulation of the notion of tightness, as it applies to size bounds, will remain a difficult problem.

We next show that, given $m \geq 3$, no size bound operator yields a tight size bound for arbitrarily small independence ratios as follows: Let $a > 0$ and $b < 0$ and consider the size bound operator $B_{a,b}$. We have

$$B_{a,b}(1, 0) = a, \tag{2.36}$$

while

$$B_l(1, 0) = l, \tag{2.37}$$

for each Turan operator B_l. Therefore for sufficiently small independence ratios, $B_{a,b}$ is strictly dominated by an infinite collection of Turan operators.

We wish to generalize the preceding construction, so as to obtain analogous results for K_m-free graphs. Notice that Turan operators satisfy the following three properties:

1. (Completeness) Given an independence ratio c, there exists a Turan operator that holds tightly at c.

2. (Minimality) If we exclude some B_l from the consideration, then there exists an independence ratio c_0 for which none of the remaining operators hold tightly.

3. (Uniqueness) No other collection of operators satisfies 1 and 2.

We have used the weaker notion of tightness in stating the preceding completeness condition, in order to gain generality. While this restriction is not required for Turan bounds, there are some results that suggest it may be necessary in other contexts.

Intuition suggests that, given $m \geq 3$, a unique, minimal complete sequence of size bound operators should exist for $\mathcal{G}(m)$, but there is no proof that this is so. However, in the case that the sequence $\left\{B_{a_l^m, b_l^m}\right\}_{l=1}^{\infty} = \{B_l^m\}_{l=1}^{\infty}$ does satisfy these properties, we may define

$$B^m(x, y) = \max_{l \geq 1} B_l^m(x, y), \tag{2.38}$$

so as to obtain a tight, piecewise-linear lower bound for sizes of graphs belonging to $\mathcal{G}(m)$. In addition, if each B_l^m holds tightly exactly for independence ratios between x_l^m and y_l^m, then analysis of the following four sequences could yield valuable information about Ramsey-type and Ramsey numbers.

$$\{a_l^m\}_{l=1}^{\infty}, \quad \{b_l^m\}\, l = 1^{\infty}, \quad \{x_l^m\}_{l=1}^{\infty}, \quad \{y_l^m\}_{l=1}^{\infty}. \tag{2.39}$$

Some of the authors cited in Chapter 1 have made contributions to this problem, and additional information is provided in this dissertation, but, at the present time, there is not enough data available to facilitate such an analytic approach.

3. Proof Techniques

It was shown in Chapter 2, that upper bounds for Ramsey-type numbers follow from size bounds via applications of Lemmas 2.2 and 2.3. With the exception of Turan bounds, which were also presented in that chapter, all of the size bounds established in this dissertation follow by an induction on order. This chapter provides the foundation for those proofs.

The basic graph-theoretic material presented in Chapter 1 is augmented, in Section 3.1, by the introduction of a collection of definitions that pertain specifically to the proof techniques developed in this chapter. Diagramming conventions for illustrating graphs are introduced in Section 3.2. Section 3.3 contains an informal overview of the inductive method, and Section 3.4 contains a rigorous exposition of the associated proof techniques.

3.1 Specialized Definitions

There is a complex hierarchy of cases and subcases associated with the inductive proof, and several of the subcases are specified by stating that a particular graph G is obtained by excluding certain configurations of edges from a clique. In the next definition, we introduce some notational conventions that pertain to excluded edges. Since it is natural to associate such structures with the complement of G, over-lines are included in the following notation in order to emphasize this relationship.

Definition 3.1 *Let* $G \in \mathcal{G}$

1. *Let* $u.v \in \mathrm{V}(G)$ *such that* $u \neq v$ *and* $uv \notin \mathrm{E}(G)$*. then* uv *is said to be a red edge of* G*. The set of all such edges is denoted by* $\overline{\mathrm{E}}(G)$*.*

2. *Define* $\overline{\mathrm{e}}(G) = \left|\overline{\mathrm{E}}(G)\right|$*.* $\overline{\mathrm{e}}(G)$ *is said to be the complementary size of* G*.*

3. *Let* $u \in \mathrm{V}(G)$ *and* $W \subseteq \mathrm{V}(G)$*. Define* $\overline{\mathrm{N}}_W(u) = \{w \in W \setminus u \mid w \not\leftrightarrow u\}$*.* $\overline{\mathrm{N}}_W(u)$ *is said to be the complementary open neighborhood of* u *relative to to* W*.*

4. *Let* $u \in \mathrm{V}(G)$ *and* $W \subseteq \mathrm{V}(G)$*. Define* $\overline{\mathrm{d}}_W(u) = \left|\overline{\mathrm{N}}_W(u)\right|$*,* $\overline{\mathrm{d}}_W(u)$ *is said to be the complementary degree of* u *relative to to* W*.*

5. *Let* $U, W \subseteq \mathrm{V}(G)$*. Define* $\overline{\mathrm{d}}_W(U) = \sum_{u \in U} \overline{\mathrm{d}}_W(u)$*.* $\overline{\mathrm{d}}_W(U)$ *is said to be the complementary degree sum of* U *relative to to* W*.*

Expression (3.1) provides an example of how a graph may be defined by excluding edges from a clique:

$$G = \mathrm{K}_8 \setminus \mathrm{E}(\mathrm{K}_{2,3}). \tag{3.1}$$

In this example, G is obtained by excluding the edges of the complete bipartite graph $\mathrm{K}_{2,3}$ from the clique K_5

We may also define a graph H by excluding one or more vertices from a graph G as in the following example:

$$H = G \setminus x. \tag{3.2}$$

When excluding vertices from a graph, it is clearly necessary to also exclude incident edges, but when excluding edges, their endpoints are retained.

The notion of edges being incident with vertices has already been introduced, and it follows naturally that edges may also be incident with sets of vertices. The next definition yields some variations on this concept.

Definition 3.2 *Let $G \in \mathcal{G}$ and let $W \subseteq \mathrm{V}(G)$.*

1. *Define* $\mathrm{Inc}(W) = \{uv \in \mathrm{E}(G) \mid \{u, v\} \cap W \neq \emptyset\}$. $\mathrm{Inc}(W)$ *is said to be the set of edges of G that are incident with W.*

2. *Define* $\mathrm{Int}(W) = \{uv \in \mathrm{E}(G) \mid u, v \in W\}$. $\mathrm{Int}(W)$ *is is said to be the set of edges of G that are internal to W.*

3. *Define* $\mathrm{Pend}(W) = \{uv \in \mathrm{E}(G) \mid |\{u, v\} \cap W| = 1\}$. $\mathrm{Pend}(W)$ *is is said to be the set of edges of G that are pendant to W.*

4. *Assum that $U \subseteq W$, and define* $\mathrm{Pend}(W; U) = \{wv \in \mathrm{Pend}(W) \mid \{u, v\} \cap U \neq \emptyset\}$. $\mathrm{Pend}(W; U)$ *is said to be the set of edges of G that are pendant to W at U.*

Definition 3.3 *Let $S \subseteq \mathrm{V}(G)$. Define*

$$\mathrm{DegEx}(S) = \mathrm{d}(S) - \delta(G)\,|S|\,. \tag{3.3}$$

$\mathrm{DegEx}(S)$ *is said to be the degree excess of S.*

The importance of pendant edges and degree excess will be established in Section 5.

The following pair of graphs play fundamental roles in the induction described in Section 3.3.

Definition 3.4 *Let $G \in \mathcal{G}$, and let $U \in \mathrm{Indep}\,(G)$. Define*

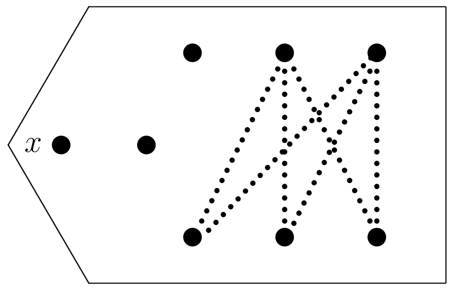
1. $F(U) = \langle(\mathrm{N}\,[U])\rangle$.

2. $H(U) = \langle(\mathrm{V}\,(G) \setminus \mathrm{N}\,[U])\rangle$.

3.2 Diagramming Conventions

In expression (3.1) we defined a graph by specifying the edges that did not belong to it, and occasions shall arise in which it is instructive to illustrate a graph by exhibiting such edges. In particular, when depicting graphs induced by the closed neighborhood of some $x \in \mathrm{V}\,(G)$, only red edges are specified, and it is assumed that all unspecified edges belong to $F(x)$. Such illustrations utilize "home plate" shaped boxes, such as the one depicted in Figure 3.1. We strictly adhere to the following conventions when using these boxes: Vertex x resides at the point of the box, members of $\mathrm{N}\,(x)$ reside to its right and non-edges between vertices are illustrated by dotted lines. For example, in the figure, $\mathrm{d}(x) = 7$ and exactly six edges are excluded from $F(x)$. Since these red edges are configured as the edges of the complete, bipartite graph $\mathrm{K}_{2,3}$, we see that $F(x)$ possesses the structure specified in expression (3.1).

If a collection of vertices is not contained in a pentagonal box, and both solid and dotted lines occur in the illustration, then a solid line joining two vertices indicates that there is an edge joining them, a dotted line indicates the absence of such an edge and, if there remain pairs of vertices with neither type of edge joining them, then we make no assumption as to whether or not those vertices are adjacent. If only solid lines occur in a diagram, then, as is customary, it is



This graph is a clique from which the edges of a complete bipartite graph have been removed. It has size 22.

Figure 3.1: $F(x) = \mathrm{K}_8 \setminus \mathrm{E}(\mathrm{K}_{2,3})$

assumed that a pair of vertices is adjacent exactly when there is a line joining them.

3.3 The Induction Method

Size bound operators were studied in Chapter 2, but from this point forward, we shall only consider size bounds of the form

$$\mathrm{e}\,(G) \geq a\nu(G) + b\alpha(G), \tag{3.4}$$

where $a > 0$, $b < 0$ and G belongs to a given collection of graphs $\mathcal{C}$, We shall additionally restrict our attention to the cases $\mathcal{C} = \mathcal{G}\,(m)$ and $\mathcal{C} = \mathcal{G}_D\,(m)$ where $m \geq 3$ and $D \geq 2$. It is clear that the empty graph belongs to any such class, and, since the order, size and independence number of this graph are each equal to zero, it clearly satisfies (3.4) with equality.

The proof technique described in this section is an abstraction of a technique used by Fraughnaugh and Locke [9]. When utilizing this technique, we consider a collection of graphs $\mathcal{C}$ and establish the size bound by induction on order. In all cases, the empty graph provides the base and the induction step is performed by removing the closed neighborhood of an independent set U, applying the induction hypothesis to $H(U)$ and using information about the structure of $F(U)$ to complete the induction (Recall that $F(U)$ and $H(U)$ were defined in Definition 3.4).

Since we always remove the closed neighborhood of U from G, any independent subset of $\mathrm{V}(H(U))$ can be extended to an independent subset of $\mathrm{V}(G)$ by adjoining U. Therefore we obtain

$$\alpha(G) \geq \alpha(H(x)) + |U|. \tag{3.5}$$

This inequality may be strict, but this does not create difficulties when applying the induction hypothesis because $b < 0$.

The induction step is partitioned into cases and subcases, with the value of $\delta(G)$ providing the coarsest partition. The initial choice for U is always a vertex of minimum degree x. We clearly have

$$|\mathrm{Inc}(\mathrm{N}[x])| \geq \binom{\delta+1}{2}. \tag{3.6}$$

As long as $\delta(G) \leq m-2$, this yields sufficiently many edges to complete the induction, but this is not generally so when $\delta(G) \geq m-1$. In such cases we have $\nu(F(x)) \geq m$ and, since we assume graphs to be K_m-free, it follows that

$$\alpha(F(x)) \geq 2. \tag{3.7}$$

We next consider independent sets U, where U is in subset of $\mathrm{N}(x)$ of size at least two. In practice, $|b|$ is considerably larger than $|a|$, so that we gain an advantage by increasing independence so long as the accompanying increase in size is not to great. We now identify a collection of independent sets, show that $\mathrm{N}[x]$ must contain at least one of them and then try to show that there are enough edges incident with the closed neighborhood of each so as to establish the induction. If this does not yield the bound, then it becomes necessary to consider additional choices for independent sets. Should this situation arise, the ensuing choices are context specific, and a complex hierarchy of subcases generally results.

3.4 Graph Parameters

In order to implement the induction described in the preceding section, it is necessary to quantify certain properties of graphs. In this section, we introduce five functions and a collection of lemmas that pertain to them. These functions and lemmas are used to manage the subcase hierarchy and establish individual subcases.

The following definition, hypothesis and conclusion provide a means of incorporating the induction hypothesis into the hypothesis of each of the lemmas proved in this section.

Definition 3.5 *Let $a > 0$ and $b < 0$, and define $\Gamma_{a,b}$ to be the class of graphs that satisfy inequality (3.4).*

Hypothesis 3.6 *Let G be a graph and assume that $H \in \Gamma_{a,b}$ whenever $H < G$*

Conclusion 3.7 $G \in \Gamma_{a,b}$.

Each lemma proved in this section includes Hypothesis 3.6, along with some additional assumptions in its hypothesis and asserts that Conclusion 3.7 follows.

It is therefore permissible to assume that the hypothesis of any such lemma does not hold. Statements of such assumptions, which immediately follow related lemmas, are numbered in the same manner as definitions and theorems so that they can be easily referenced. By appealing to these assumptions, we are able to focus our attention on configurations that require specialized arguments. Throughout this dissertation, these assumptions are not considered to be in force, unless they are explicitly invoked.

We assume that a and b satisfy the constraints specified in Definition 3.5. Each of the following functions incorporates a and b into its definition and, throughout this chapter, each function is sub-scripted by these variables to emphasize that dependency. But, when these functions are applied in the following chapters, these values will be clear from context, and the subscripts will be omitted.

We next introduce some notational conventions associated with function arguments. The variables δ, n and r all represent non-negative integers and each function takes at least one of them as an argument in its definition. These functions are always used within the context where we are considering a graph G, and we have excluded the closed neighborhood of an independent set U from it. In this setting, these three variables correspond to $\delta(G)$, $|\mathrm{N}[U]|$ and $|U|$, respectively. The reader might find it instructive to compare arguments, as used in function definitions, to arguments, as used in the lemmas that immediately follow them.

The first function specifies the number of edges incident with $\mathrm{N}[U]$ that suffice to establish (3.4), given that this bound it is satisfied by $H(U)$.

Definition 3.8 *Define*

$$\zeta_{a,b}(n,r) = \lceil an + br \rceil. \tag{3.8}$$

Lemma 3.9 *Assume Hypothesis 3.6, and let* $U \in \mathrm{Indep}\,(G)$*. If*

$$|\mathrm{Inc}\,(\mathrm{N}\,[U])| \geq \zeta_{a,b}(|\mathrm{N}\,[U]\,|,\ |U|), \tag{3.9}$$

then Conclusion 3.7 follows.

Proof: Assume the hypothesis of the lemma. Then

$$\begin{aligned}
\mathrm{e}\,(G) &= \mathrm{e}\,(H(U)) + |\mathrm{Inc}\,(\mathrm{N}\,[U])| & (3.10)\\
&\geq \mathrm{e}\,(H(U)) + \zeta_{a,b}(|\mathrm{N}\,[U]\,|,\ |U|) & (3.11)\\
&\geq a\nu(H(U)) + b\alpha(H(U)) + a|\mathrm{N}\,[U]\,| + b\,|U| & (3.12)\\
&= a\nu(G) + b(\alpha(H(U)) + |U|) & (3.13)\\
&\geq a\nu(G) + b\alpha(G). & (3.14)
\end{aligned}$$

Notice 3.14 holds because

$$\alpha(G) \geq \alpha(H(U)) + |U| \quad \text{and} \quad b < 0. \tag{3.15}$$

■

Lemma 3.9 clearly justifies the following:

Assumption 3.10 *Suppose that Hypothesis 3.6 is in force, and* $U \in \mathrm{Indep}\,(G)$*. Then*

$$|\mathrm{Inc}\,(\mathrm{N}\,[U])| < \zeta_{a,b}(|\mathrm{N}\,[U]|\,, |U|). \tag{3.16}$$

It is clear that

$$|\mathrm{Inc}\,(\mathrm{N}\,[U])| \geq \left\lceil \frac{\delta(G)\,|\mathrm{N}\,[U]|}{2} \right\rceil. \tag{3.17}$$

This observation motivates the following:

Definition 3.11 *Define*

$$\psi_{a,b}(\delta, n, r) = \left\lceil \frac{n\delta}{2} \right\rceil - \zeta_{a,b}(n, r). \tag{3.18}$$

Lemma 3.12 *Assume Hypothesis 3.6 and let* $U \in \mathrm{Indep}\,(G)$*. If*

$$\psi_{a,b}(\delta(G), |\mathrm{N}\,[U]|\,, |U|) \geq 0, \tag{3.19}$$

then Conclusion 3.7 follows.

Proof: Apply (3.17) and Lemma 3.9. ■

Assumption 3.13 *Assume Hypothesis 3.6 and let* $U \in \mathrm{Indep}\,(G)$*. Then*

$$\psi_{a,b}(\delta(G), |\mathrm{N}\,[U]|\,, |U|) < 0. \tag{3.20}$$

When $\psi(\delta(G), |\mathrm{N}\,[U]|\,, |u|) < 0$, we obtain a measure of the amount by which Lemma 3.12 fails to yield Conclusion 3.7. This value is used in Definition 3.23 and, in the event that U consists of a single vertex of minimum degree x, it also yields an upper bound for the values of $\overline{\mathrm{e}}\,(F(x))$ that require specialized argument.

Lemma 3.14 *Assume Hypothesis 3.6 and let* $x \in \mathrm{MinDeg}(G)$*. If*

$$\overline{\mathrm{e}}\,(F(x)) \geq -\psi_{a,b}(\delta(G), \delta(G) + 1, 1), \tag{3.21}$$

then Conclusion 3.7 follows.

Proof: Assume the hypothesis and let $v \in \mathrm{N}(x)$. Since $\mathrm{d}(x) = \delta(G)$, v must have at least as many neighbors outside of $\mathrm{N}[x]$ as it has non-neighbors within that set. Therefore

$$|\mathrm{Pend}(\mathrm{N}[x])| \geq 2\,\overline{\mathrm{e}}\,(F(x)). \tag{3.22}$$

This implies

$$\begin{aligned}
|\mathrm{Inc}(\mathrm{N}[x])| &= |\mathrm{Int}(\mathrm{N}[x])| + |\mathrm{Pend}(\mathrm{N}[x])| && (3.23)\\
&\geq \binom{\delta(G)+1}{2} - \overline{\mathrm{e}}\,(F(x)) + 2\,\overline{\mathrm{e}}\,(F(x)) && (3.24)\\
&\geq \binom{\delta(G)+1}{2} - \psi_{a,b}(\delta(G), \delta(G)+1, 1) && (3.25)\\
&= \zeta_{a,b}(\delta(G), \delta(G)+1, 1). && (3.26)
\end{aligned}$$

Now apply Lemma 3.9. ■

Assumption 3.15 *Assume Hypothesis 3.6 and let* $x \in \mathrm{MinDeg}(G)$. *Then*

$$\overline{\mathrm{e}}\,(F(x)) < -\psi_{a,b}(\delta(G), \delta(G)+1, 1). \tag{3.27}$$

The next lemma and assumption allow us to exclude closed neighborhoods of x from consideration if there are a large number of edges pendant to them.

Lemma 3.16 *Assume Hypothesis 3.6 and let* $x \in \mathrm{MinDeg}(G)$. *If*

$$|\mathrm{Pend}(\mathrm{N}[x])| \geq -2\psi_{a,b}(\delta(G), \delta(G)+1, 1) - 1, \tag{3.28}$$

then Conclusion 3.7 follows.

Proof: Assume the hypothesis. First notice that Assumption 3.15 implies

$$-\,\overline{\mathrm{e}}\,(F(x)) - 1 \geq \psi_{a,b}(\delta(G), \delta(G)+1, 1). \tag{3.29}$$

Now calculate

$$
\begin{aligned}
|\mathrm{Inc}\,(\mathrm{N}\,[x])| &= |\mathrm{Pend}\,(\mathrm{N}\,[x])| + |\mathrm{Int}\,(\mathrm{N}\,[x]))| && (3.30)\\
&\geq -2\psi_{a,b}(\delta(G), \delta(G)+1, 1) - 1 + \mathrm{e}\,(F(x)) && (3.31)\\
&\geq -2\psi_{a,b}(\delta(G), \delta(G)+1, 1) + \binom{\delta(G)+1}{2}\\
&\qquad - \overline{\mathrm{e}}\,(F(x)) - 1 && (3.32)\\
&\geq -2\psi_{a,b}(\delta(G), \delta(G)+1, 1) + \binom{\delta(G)+1}{2}\\
&\qquad +\psi_{a,b}(\delta(G), \delta(G)+1, 1) && (3.33)\\
&= \binom{\delta(G)+1}{2} - \psi_{a,b}(\delta(G), \delta(G)+1, 1) && (3.34)\\
&= \zeta_{a,b}(\delta(G), \delta(G)+1, 1). && (3.35)
\end{aligned}
$$

In this calculation, we have invoked Assumption 3.15 in order to obtain (3.33). Now apply Lemma 3.9. ∎

Assumption 3.17 *Assume Hypothesis 3.6 and let* $x \in \mathrm{MinDeg}(G)$*. Then*

$$|\mathrm{Pend}\,(\mathrm{N}\,[x])| < -2\psi_{a,b}(\delta(G), \delta(G)+1, 1) - 1. \tag{3.36}$$

If $x \in \mathrm{MinDeg}(G)$, then

$$|\mathrm{Inc}\,(\mathrm{N}\,[x])| \geq \binom{\delta(G)+1}{2}, \tag{3.37}$$

and we may view $\psi(\delta(G), \delta(G)+1, 1)$ as a function of $\delta(G)$. When the hypothesis of the following lemma is satisfied, this function is monotonically decreasing and inequality (3.4) can then be established for several values of $\delta(G)$ in one step.

Lemma 3.18 *Assume Hypothesis 3.6 and let* $k \in \mathbb{N}$ *such that*

$$\delta(G) \leq k \leq a - 1, \tag{3.38}$$

and

$$\psi_{a,b}(k, k+1, 1) \geq 0. \tag{3.39}$$

Then Conclusion 3.7 follows.

Proof: If $\delta(G) = k$, then it suffices to apply Lemma 3.12. Next assume the hypothesis and let $\delta(G) = k - 1$. Upon removing ceiling operations we obtain

$$\psi_{a,b}(k, k+1, 1) = \binom{k+1}{2} - \lceil a(k+1) + b \rceil \tag{3.40}$$
$$\leq \binom{k+1}{2} - ak - a - b \tag{3.41}$$

and

$$\psi_{a,b}(k-1, k, 1) = \binom{k}{2} - \lceil ak + b \rceil \tag{3.42}$$
$$> \binom{k}{2} - ak - b - 1. \tag{3.43}$$

Therefore

$$\psi_{a,b}(k-1, k, 1) - \psi_{a,b}(k, k+1, 1) > \binom{k}{2} - ak - b - 1 \tag{3.44}$$
$$- \left(\binom{k+1}{2} - ak + a - b\right) \tag{3.45}$$
$$= a - 1 - k \tag{3.46}$$
$$\geq 0. \tag{3.47}$$

This establishes the lemma for the case $\delta(G) = k - 1$. Now iteratively apply the same argument using successively smaller values of $\delta(G)$. ■

There shall arise situations in which we evaluate $\psi_{a,b}(\delta(G), |\mathrm{N}[U]|, |U|)$ where $\delta(G)$ and $|U|$ are held fixed and $|\mathrm{N}[U]|$ is allowed to vary. In such instances, we are essentially evaluating a function of the single variable $|\mathrm{N}[U]|$.

The next lemma asserts that this function is monotonically decreasing whenever its hypothesis is satisfied.

Lemma 3.19 *Let $k, r, n, \delta \in \mathbb{N}$ such that $r \leq n$ and $k < n$. Assume that*

$$a \geq \frac{\delta}{2} + \frac{3}{2}. \tag{3.48}$$

Then

$$\psi_{a,b}(\delta, k, r) \geq \psi_{a,b}(\delta, n, r). \tag{3.49}$$

Proof: The conclusion of the lemma is clearly equivalent to

$$\psi_{a,b}(\delta(G), k, r) - \psi_{a,b}(\delta(G), n, r) \geq 0. \tag{3.50}$$

In order to establish this inequality we need only prove the lemma for the case $k = n - 1$ and apply an induction.

It follows directly from the definitions of $\psi_{a,b}$ and $\zeta_{a,b}$ that

$$\begin{aligned} \psi_{a,b}(\delta(G), n, r) &= \left\lceil \frac{\delta(G)}{2} n \right\rceil - \lceil an + br \rceil & (3.51) \\ &\leq \frac{\delta(G)}{2} n + \frac{1}{2} - an - br & (3.52) \end{aligned}$$

and

$$\begin{aligned} \psi_{a,b}(\delta(G), n-1, r) &= \left\lceil \frac{\delta(G)}{2}(n-1) \right\rceil - \lceil a(n-1) + br \rceil & (3.53) \\ &> \frac{\delta(G)}{2} n - \frac{\delta(G)}{2} - an + a - br - 1. & (3.54) \end{aligned}$$

The following calculation serves to complete the proof.

$$
\begin{aligned}
\psi_{a,b}(\delta(G), n-1, r) - \psi(\delta(G), n, r) > \frac{\delta(G)}{2}n - \frac{\delta(G)}{2} \\
-an + a - br - 1 \\
- \left(\frac{\delta(G)}{2}n + \frac{1}{2} - an - br \right) & \quad (3.55) \\
= a - \frac{\delta(G)}{2} - \frac{3}{2} & \quad (3.56) \\
\geq 0. & \quad (3.57)
\end{aligned}
$$

■

In the csse that the hypothesis of Lemma 3.19 is satisfied, Definition 3.20 yields notation for the largest value of $|\mathrm{N}[U]|$ for which $\psi(\delta(G), |\mathrm{N}[U]|, |U|)$ is non-negative, and Assumption 3.22 allow us to restrict our attention to closed neighborhoods of larger size.

Definition 3.20 *Assume that the hypothesis of Lemma 3.19 is satisfied. Define*

$$\eta_{a,b}(\delta, r) = \max \{ n \mid \psi_{a,b}(\delta, n, r) \geq 0 \}. \quad (3.58)$$

Lemma 3.21 *Assume Hypothesis 3.6, assume that*

$$a \geq \frac{\delta}{2} + \frac{3}{2} \quad (3.59)$$

and let $U \in \mathrm{Indep}(G)$*. If*

$$|\mathrm{N}[U]| \leq \eta_{a,b}(\delta(G), |U|), \quad (3.60)$$

then Conclusion 3.7 follows.

Proof: Assume the hypothesis and apply Lemmas 3.19 and 3.12. ■

Lemma 3.21 clearly justifies the following:

Assumption 3.22 *Assume hypothesis 3.6, and assume that* $a \geq \frac{\delta}{2} + \frac{3}{2}$ *and* $U \in \mathrm{Indep}\,(G)$. *Then*

$$|\mathrm{N}\,[U]| > \eta(\delta(G), |U|). \tag{3.61}$$

Assumption 3.22 allows us to disregard cases where $|\mathrm{N}\,[U]|$ is small. The next function allows us to disregard independent subsets of $\mathrm{N}\,[x]$ that have large closed neighborhoods.

It follows directly from the definition of $\psi_{a,b}$ that, should $\mathrm{N}\,[x]$ induce a clique, the existence of $-\psi_{a,b}(\delta(G), \delta(G)+1, 1)$ edges pendant to $\mathrm{N}\,[x]$ implies the existence of $\zeta_{a,b}(\delta(G)+1, 1)$ edges incident with $\mathrm{N}\,[x]$. It therefore follows from Assumption 3.10 that

$$|\mathrm{Pend}\,(\mathrm{N}\,[x])| \leq -\psi_{a,b}(\delta(G), \delta(G)+1, 1) - 1 \tag{3.62}$$

whenever $\langle \mathrm{N}\,[x] \rangle = \mathrm{K}_{\delta+1}$. As we successively remove edges from $F(x)$, we increment the number of pendant edges that are required in order to obtain $\zeta_{a,b}(\delta(G)+1, 1)$ incident edges. Lemma 3.24 allows us to assume the existence of a bound for the number of edges pendant to $\mathrm{N}\,[x]$ at a set $S \subseteq \mathrm{N}\,(x)$ that depends of the number of red edges that have both endpoints in that set. The utility of this lemma follows from the fact that the existence of red edges that have one or both endpoints in $\mathrm{N}\,[x] \setminus S$ does not invalidate the bound. This is so because the degree of each $v \in \mathrm{N}\,(x)$ is at least as large as that of x, so that v must have at least as many neighbors outside of $\mathrm{N}\,[x]$ as it has non-neighbors within that set. Therefore, for each red edge that has an endpoint in $\mathrm{N}\,(x) \setminus S$, there must be at least one edge pendant to $\mathrm{N}\,[x]$ at that endpoint. We formalize this argument as follows:

Definition 3.23 *Let $k > 0$. Define*

$$\rho_{a,b}(\delta, k) = \max\left\{0, -\psi_{a,b}(\delta, \delta+1, 1) - 1 + k\right\}. \tag{3.63}$$

Lemma 3.24 *Assume that Hypothesis 3.6 holds. Let $x \in \mathrm{MinDeg}(G)$ and $S \subseteq \mathrm{N}(x)$. Let $k > 0$ such that*

$$\bar{\mathrm{e}}(\langle S \rangle) \leq k. \tag{3.64}$$

If

$$|\mathrm{Pend}(\mathrm{N}[x]; S)| \geq \rho_{a,b}(\delta(G), k) + 1, \tag{3.65}$$

then Conclusion 3.7 follows.

Proof: Assume the hypothesis. It is clear that we need only consider the case

$$-\psi(\delta(G), \delta(G)+1, 1) > 0. \tag{3.66}$$

The hypothesis of this lemma involves red edges that have both endpoints in S, but we must also consider red edges that have at most one endpoint in S. We denote this set by T. Now we have

$$|\mathrm{Int}(\mathrm{N}[x])| \geq \binom{\delta(G)+1}{2} - (k + |T|). \tag{3.67}$$

Assume that $v \in \mathrm{N}(x)$. We clearly have $\mathrm{d}(v) \geq \mathrm{d}(x)$. Therefore

$$|\mathrm{Pend}(\mathrm{N}[x]; v)| \geq \bar{\mathrm{d}}_{\mathrm{N}[x]}(v), \tag{3.68}$$

and it follows that

$$|\mathrm{Pend}(\mathrm{N}[x]; \mathrm{N}[x] \setminus S)| \geq |T|. \tag{3.69}$$

Now (3.67) and (3.69) yield

$$\begin{aligned}
|\mathrm{Inc}\,(\mathrm{N}\,[x])| &= |\mathrm{Int}\,(\mathrm{N}\,[x])| + |\mathrm{Pend}\,(\mathrm{N}\,[x])| && (3.70)\\
&= |\mathrm{Int}\,(\mathrm{N}\,[x])| + |\mathrm{Pend}\,(\mathrm{N}\,[x]\,;S)| \\
&\quad\quad + |\mathrm{Pend}\,(\mathrm{N}\,[x]\,;\mathrm{N}\,[x]\setminus S)| && (3.71)\\
&\geq \binom{\delta(G)+1}{2} - k - |T| + |\mathrm{Pend}\,(\mathrm{N}\,[x]\,;S)| + |T| && (3.72)\\
&\geq \binom{\delta(G)+1}{2} - k + \rho_{a,b}(\delta(G),k) + 1 && (3.73)\\
&= \binom{\delta(G)+1}{2} - \psi_{a,b}(\delta(G),\delta(G)+1,1) && (3.74)\\
&= \zeta_{a,b}(\delta(G)+1,1). && (3.75)
\end{aligned}$$

Apply Lemma 3.9. ■

We are now able to assume the following pair of bounds:

Assumption 3.25 *Assume Hypothesis 3.6. Let* $x \in \mathrm{MinDeg}(G)$ *and* $S \subseteq \mathrm{N}\,(x)$*. Assume that* $\overline{\mathrm{e}}\,(G(S)) \leq k$*. Then*

1. $|\mathrm{Pend}\,(\mathrm{N}\,[x]\,;S)| \leq \rho_{a,b}(\delta(G),k)$.

2. $|\mathrm{N}\,[S]| \leq |\mathrm{N}\,[x]| + \rho_{a,b}(\delta(G),k)$.

Lemma 3.24 justifies (1), and then (2) follows because

$$|\mathrm{M}\,[S;x]| \leq |\mathrm{Pend}\,(\mathrm{N}\,[x]\,;S)|\,. \quad (3.76)$$

While the existence of red edges that have endpoints outside of S does not affect the calculation of ρ, it may be necessary to consider the existence of additional red edges that have both end points in S. For example, if G is K_m-free and $\mathrm{d}(x) = \delta(G) = m$, then it is necessary that $\mathrm{N}\,[x]$ contain either three independent vertices or a disjoint pair of independent sets of size two. Now assume that $\alpha(F(x)) = 2$ and $\{p_0, q_0\}$, $\{p_1, q_1\}$ are independent and disjoint. If we let $S = \{p_0, q_0, p_1, q_1\}$, then it is possible for there to be two additional red edges, say p_0q_1 and p_1, q_0, that have both endpoints in S without increasing independence. Therefore we may have as many as $\rho(\delta(G), 4)$ edges pendant to $\mathrm{N}\,[x]$ at S. Given this configuration, symmetry allows us to assume that

$$|\mathrm{Pend}\,(\mathrm{N}\,[x]\,; p_0, q_0)| \leq \left\lfloor \frac{\rho(\delta(G), 4)}{2} \right\rfloor. \tag{3.77}$$

This situation will arise frequently on the following chapters.

There are two ways in which Lemma 3.12 can be sharpened. The calculation of $\psi_{a,b}$ is based on the assumptions that all members of $\mathrm{N}\,[U]$ have degree equal to $\delta(G)$, and all edges incident with $\mathrm{N}\,[U]$ are internal. The notions of pendant edges and degree excess were introduced in Definitions 3.2 and 3.3, respectively. The following definition, lemma and assumption provide a means for exploiting these concepts.

Definition 3.26 *Define*

$$\lambda_{a,b}(\delta, n, r) = 2\zeta_{a,b}(n, r) - \delta n - 1. \tag{3.78}$$

Lemma 3.27 *Assume Hypothesis 3.6, and let $U \in \mathrm{Indep}\,(G)$. If*

$$\mathrm{DegEx}\,(\mathrm{N}\,[U]) + |\mathrm{Pend}\,(\mathrm{N}\,[U])| \geq \lambda_{a,b}(\delta(G), |\mathrm{N}\,[U]|\,, |U|), \tag{3.79}$$

then Conclusion 3.7 follows.

Proof: Assume the hypothesis. It follows from Definition 3.3 that

$$\mathrm{d}(\mathrm{N}\,[U]) = \delta n + \mathrm{DegEx}\,(\mathrm{N}\,[U]) \tag{3.80}$$

and Definition 3.26 implies

$$\delta n = 2\zeta_{a,b}(|\mathrm{N}[U]|, |U|) - \lambda_{a,b}(\delta(G), |\mathrm{N}[U]|, |U|) - 1. \tag{3.81}$$

Each pendant edge contributes one to $\mathrm{d}(\mathrm{N}[U])$. Therefore

$$\begin{aligned} 2|\mathrm{Int}(\mathrm{N}[u])| &= \delta n + \mathrm{DegEx}(\mathrm{N}[U]) - |\mathrm{Pend}(\mathrm{N}[U])| && (3.82) \\ &= 2\zeta_{a,b}(|\mathrm{N}[U]|, |U|) - \lambda_{a,b}(\delta(G), |\mathrm{N}[U]|, |U|) - 1 \\ &\quad + \mathrm{DegEx}(\mathrm{N}[U]) - |\mathrm{Pend}(\mathrm{N}[U])|. && (3.83) \end{aligned}$$

If we were to have

$$\mathrm{DegEx}(\mathrm{N}[U]) + |\mathrm{Pend}(\mathrm{N}[U])| = \lambda_{a,b}(\delta(G), |\mathrm{N}[U]|, |U|), \tag{3.84}$$

then it would follow that

$$2|\mathrm{Int}(\mathrm{N}[u])| = 2(\zeta_{a,b}(|\mathrm{N}[U]|, |U|) - |\mathrm{Pend}(\mathrm{N}[U])|) - 1, \tag{3.85}$$

and this is clearly impossible. The hypothesis therefore allows us to assume that

$$\mathrm{DegEx}(\mathrm{N}[U]) + |\mathrm{Pend}(\mathrm{N}[U]) \geq \lambda_{a,b}(\delta(G), |\mathrm{N}[U]||, |U|) + 1. \tag{3.86}$$

The following calculation utilizes

$$\begin{aligned} |\mathrm{Inc}(\mathrm{N}[U])| &= |\mathrm{Pend}(\mathrm{N}[U])| + |\mathrm{Int}(\mathrm{N}[U])| && (3.87) \\ &= |\mathrm{Pend}(\mathrm{N}[U])| + \frac{\delta n + \mathrm{DegEx}(\mathrm{N}[U]) - |\mathrm{Pend}(\mathrm{N}[U])|}{2}. && (3.88) \end{aligned}$$

Notice that (3.88) implies (3.91) and (3.81) implies (3.93).

$$
\begin{aligned}
& \zeta_{a,b}(|\mathrm{N}[U]|, |U|) \leq |\mathrm{Inc}(\mathrm{N}[U])| && (3.89)\\
\iff & 2\zeta_{a,b}(|\mathrm{N}[U]|, |U|) \leq 2|\mathrm{Inc}(\mathrm{N}[U])| && (3.90)\\
\iff & 2\zeta_{a,b}(|\mathrm{N}[U]|, |U|) \leq 2|\mathrm{Pend}(\mathrm{N}[U])| + \delta n \\
& \quad +\mathrm{DegEx}(\mathrm{N}[U]) - |\mathrm{Pend}(\mathrm{N}[U])| && (3.91)\\
\iff & 2\zeta_{a,b}(|\mathrm{N}[U]|, |U|) \leq \delta n + \mathrm{DegEx}(\mathrm{N}[U]) + |\mathrm{Pend}(\mathrm{N}[U])| && (3.92)\\
\iff & 2\zeta_{a,b}(|\mathrm{N}[U]|, |U|) \leq 2\zeta_{a,b}(|\mathrm{N}[U]|, |U|) - \lambda_{a,b}(\delta(G), |\mathrm{N}[U]|, |U|) \\
& \quad -1 + \mathrm{DegEx}(\mathrm{N}[U]) + |\mathrm{Pend}(\mathrm{N}[U])| && (3.93)\\
\iff & \lambda_{a,b}(\delta(G), |\mathrm{N}[U]|, |U|) + 1 \leq \mathrm{DegEx}(\mathrm{N}[U]) + |\mathrm{Pend}(\mathrm{N}[U])|. && (3.94)
\end{aligned}
$$

Now apply Lemma 3.9. ■

Assumption 3.28 *Assume Hypothesis 3.6 and let* $U \in \mathrm{Indep}(G)$*. Then*

$$\mathrm{DegEx}(\mathrm{N}[U]) + |\mathrm{Pend}(\mathrm{N}[U])| < \lambda_{a,b}(\delta(G), |\mathrm{N}[U]|, |U|). \tag{3.95}$$

The following lemma, which generalizes a technique introduced by Fraughnaugh and Locke [9], provides for the establishment of size bounds when $\delta(G)$ is sufficiently large.

Lemma 3.29 *Assume Hypothesis 3.6 and let* $x \in \mathrm{MinDeg}(G)$*. If* $\delta(G) \geq a$*, then Conclusion 3.7 follows.*

Proof: Assume the hypothesis of the lemma and consider the graph $G \setminus x$.

Now

$$\begin{aligned}
\mathrm{e}\,(G) &= \mathrm{d}(x) + \mathrm{e}\,(G \setminus x) && (3.96)\\
&\geq a + a(\nu(G-1) + b\alpha(G \setminus x) && (3.97)\\
&\geq a\nu(G) + b\alpha(G). && (3.98)
\end{aligned}$$

The last inequality follows because

$$\alpha(G) \geq \alpha(G \setminus x) \quad \text{and} \quad b < 0. \tag{3.99}$$

■

A considerable amount of technical information has been introduced in this chapter, and the reader has yet to be presented with an application; two relatively simple size bounds are proved in the next chapter.

4. Two Elementary Size Bounds

Two size bounds are established in this chapter. Each holds tightly for a clique and a graph that Fajtlowicz [5] calls a butterfly (The topics of butterflies and generalizations of these two bounds will be studied in Chapter 8). The more difficult of the two proofs is presented in Section 4.1. This proof includes explanations of how the techniques developed in Chapter 3 are being applied and how they generalize to other proofs. The proof presented in Section 4.2, which is quite simple, is written in the style of the proofs presented in the following chapters. These two proofs are included in order to present the reader with original results while providing relatively uncomplicated applications of the techniques developed in the preceding chapter..

4.1 A Size Bound for $\mathcal{G}(5)$

We show that any K_5-free graph G satisfies the size bound

$$\mathrm{e}(G) \geq \frac{13}{2}\nu(G) - 20\alpha(G), \tag{4.1}$$

and we show that this bound holds tightly for independence ratios between $\frac{1}{5}$ and $\frac{1}{4}$. The imposition of the maximum degree constraint, $\Delta(G) \leq 5$, then yields $\frac{1}{5}$ as an independence ratio bound for $\mathcal{G}_5(5)$. The size bound and tightness result are original results, while the independence ratio bound is well known.

Theorem 4.1

1. *Let $G \in \mathcal{G}(5)$. Then* $\mathrm{e}(G) \geq \frac{13}{2}\nu(G) - 20\alpha(G)$.
2. *Let $G \in \mathcal{G}_5(5)$. Then* $\mu(G) \geq \frac{1}{5}$.

3. *Let* $\frac{1}{5} \leq \frac{r}{n} \leq \frac{1}{4}$. *Then there exists* $G \in \mathcal{G}_5(5)$ *such that* $\nu(G) = n$, $\alpha(G) = r$, *and* $\mathrm{e}(G) = \left\lceil \frac{13}{2} n \right\rceil - 20r$.

4. *Let* $k \geq 2$. *Then*

$$R_5(5,k) = \begin{cases} 5(k-1)+1 & \text{if } k \text{ is odd} \\ 5(k-1) & \text{if } k \text{ is even} \end{cases}. \tag{4.2}$$

Proof: We prove (1) by induction on $\nu(G)$. This inequality clearly holds for the empty graph. Now assume that G is not empty and (1) holds for members of $\mathcal{G}(5)$ of smaller order. We consider cases determined by $\delta(G)$, and, in each case, we assume that $x \in \mathrm{MinDeg}(G)$.

Case I $(\delta \leq 3)$

We begin with the required parameter evaluations.

Parameter Evaluation 4.2

1. $\zeta(4,1) = \frac{13}{2} \cdot 4 - 20 = 6$.
2. $\psi(3,4,1) = \binom{4}{2} - \zeta(4,1) = 0$.

We first assume that $\delta(G) = 3$, so that

$$|\mathrm{N}[x]| = 4. \tag{4.3}$$

The induction hypothesis asserts that (1) holds for $H(x)$, and Evaluation 4.2.1 asserts that the theorem follows for G if there are at least $\zeta(4,1) = 6$ edges incident with $\mathrm{N}[x]$. $F(x)$ is of minimum size uniquely when it is equal to K_4, and, in that case, we obtain exactly the required number of edges (This fact is reflected in Evaluation 4.2.2). Therefore (1) follows when $\delta(G) = 3$, by an application of Lemma 3.12. Now, since the hypothesis of Lemma 3.18 is satisfied, $\psi(\delta(G), \delta(G)+1, 1)$ is decreasing and it follows that (1) also holds for smaller values of $\delta(G)$.

Case II $\quad (\delta = 4)$

The following parameter evaluations are required for this case.

Parameter Evaluation 4.3

1. $\zeta(5,1) = \left\lceil \frac{13}{2} \cdot 5 \right\rceil - 20 \cdot 1 = 13.$

2. $\psi(4,5,1) = \binom{5}{2} - \zeta(5,1) = -3.$

3. $\rho(4,1) = 3.$

4. $\zeta(8,2) = \left\lceil \frac{13}{2} \cdot 8 - 20 \cdot 2 \right\rceil = 12.$

5. $\psi(4,8,2) = \frac{4 \cdot 8}{2} - \zeta(8,2) = 4.$

6. $\eta(4,2) = 8$

Evaluation 4.3.1 asserts that (1) holds for G if there are at least 13 edges incident with $\mathrm{N}[x]$, but the minimum degree condition guarantees the existence of only 10 such edges (Notice that Evaluation 4.3.2 specifies this disparity). We must consider an alternative independent set. Since $|\mathrm{N}[x]| = 5$, and G is K_5-free, $\mathrm{N}(x)$ must contain a pair of nonadjacent vertices p and q.

It follows from Evaluation 4.3.3 and Assumption 3.25 that

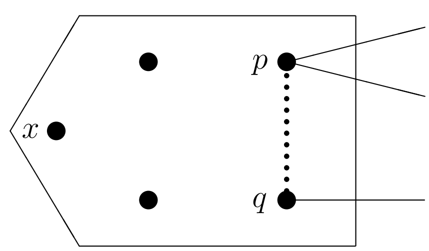
$$|\mathrm{N}[p,q]| \leq |\mathrm{N}[x]| + \rho(4,1) = 8. \tag{4.4}$$

If this inequality does not hold, then we obtain at least $\zeta(5,1) = 13$ edges incident with $\mathrm{N}[x]$ and (1) follows by Lemma 3.9.

Since $\delta(G) = 4$ and $|\{p, q\}| = 2$, we are considering the function $\psi(4, |\mathrm{N}[p,q]|, 2)$. We obtain a positive valuation for this function when equality holds in (4.4). Therefore Lemma 3.12 yields (1) in this case. Now, because the hypothesis of Lemma 3.19 is satisfied, $\psi(4, |\mathrm{N}[p,q]|, 2)$ is a decreasing function of its second argument. In fact, we have

$$\psi(4, n+1, 2) \leq \psi(4, n, 2) - 4, \tag{4.5}$$

and we obtain positive valuations whenever $|\mathrm{N}[p_0, q_0]| < 8$, so that (1) again follows. Figure 4.1 illustrates this case.



The edge pq has been excluded from $\mathrm{N}[x]$. Evaluation 4.3.3 and Assumption 3.25 yield $|\mathrm{M}[p,q;x]| \leq 3$. This implies $|\mathrm{N}[p,q]| \leq 8 = \eta(4,2)$.

Figure 4.1: $\delta(G) = 4$

In this example, the bound for $|\mathrm{N}[p,q]|$, as stated in (4.4), yields a positive valuation for $\psi(4, |\mathrm{N}[p,q]|, 2)$, and this case follows via an application of Lemma

3.19. But, in the following chapters, it will often occur that the bound for $|\mathrm{N}[U]|$, as stated by the analogue of (4.4), yields a negative valuation for ψ. In such cases, we determine the largest value of $|\mathrm{N}[U]|$ that yields a positive valuation for $\psi(\delta(G), |\mathrm{N}[U]|, |U|)$ (Recall that we denote this value $\eta(\delta(G), |U|)$), and apply Lemma 3.21. This serves to establish the size bound for values of $|\mathrm{N}[U]|$ that are no larger than $\eta(\delta(G), |U|)$. Next, context specific arguments are used for each value of $|\mathrm{N}[U]|$ between $\eta(\delta(G), |U|) + 1$ and the upper bound specified by the analogue of (4.4). When making such arguments, it is clearly essential that $\delta(G)$ and $|U|$ remain fixed.

Case III $(\delta = 5)$

We begin by calculating the required parameters.

Parameter Evaluation 4.4

1. $\zeta(6,1) = \dfrac{13}{2} \cdot 6 - 20 = 19.$
2. $\psi(5,6,1) = \dbinom{6}{2} - \zeta(6,1) = -4.$
3. $\rho(5,3) = -\psi(5,6,1) - 1 + 3 = 6.$
4. $\rho(5,4) = -\psi(5,6,1) - 1 + 4 = 7.$
5. $\zeta(9,2) = \left\lceil \dfrac{13}{2} \cdot 9 - 20 \cdot 2 \right\rceil = 19.$
6. $\psi(5,9,2) = \left\lceil \dfrac{5 \cdot 9}{2} \right\rceil - \zeta(9,2) = 4.$
7. $\eta(5,2) = 9.$
8. $\zeta(14,3) = \dfrac{13}{2} \cdot 14 - 20 \cdot 3 = 31.$

9. $\psi(5, 14, 3) = \left\lceil \dfrac{5 \cdot 14}{2} \right\rceil - \zeta(14, 3) = 4.$

10. $\eta(5, 3) = 14.$

Since G is K_5-free, $\mathrm{N}\,(x)$ must contain either three independent vertices or a disjoint pair of independent sets, each of size two. This is so because the only remaining possibility is to exclude the edges of a star, and, because $|\mathrm{N}\,[x]| = 6$, the members of $\mathrm{N}\,[x]$ other than the center would then induce K_5. We next partition the current case based upon these two possibilities. Notice that we have offered an explanation as to why the next level of partitioning takes all possibilities into consideration. Since subcase hierarchies tend to become quite complex, the argument that all possible subcases have been considered consists of several such brief explanations, each offered immediately prior to the next partition of the current case or subcase.

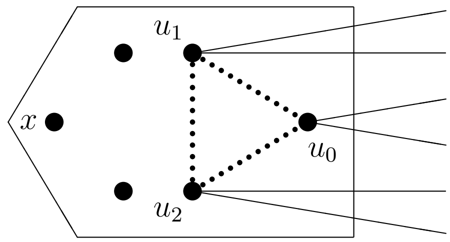
Case III.A $\quad (\alpha(F(x)) \geq 3)$

Assume that $U = \{u_i\}^{i \in 3} \in \mathrm{Indep}\,(F(x))$. Assumption 3.25 and Evaluation 4.4.3 yield

$$|\mathrm{N}\,[U]| \leq |\mathrm{N}\,[x]| + \rho(5, 3) = 12. \tag{4.6}$$

The bound now follows by Evaluation 4.4.10 and Lemma 3.21. Figure 4.2 illustrates the configuration under consideration.

It is possible that $\alpha(F(x)) > 3$, but there is no need to consider this eventuality. We simply let U consist of any three independent vertices in $\mathrm{N}\,(x)$, apply Assumption 3.25, so as to obtain a bound for the size of $\mathrm{N}\,[U]$, and then apply Lemma 3.21 as we have done here. When the function ρ was defined in Chapter 3, it was established that the existence of red edges that have one or



The set U is independent and has no more than $\rho(3) = 6$ neighbors outside of $\mathrm{N}[x]$. This implies $|\mathrm{N}[U]| \leq 12 < \eta(5, 3)$.

Figure 4.2: $\delta(G) = 5$, $\alpha(F(x)) = 3$

both endpoints outside of the set S, as defined in that discussion, do not affect the valuation of $\rho(\delta(G), \overline{\mathrm{e}}(\langle S \rangle))$. Therefore we simply equate the set U, in this argument, with that S, and it follows that the existence of larger independent sets is irrelevant. In general, it suffices to determine the smallest value of $|U|$ for which Lemma 3.21 yields the size bound, and establish that case. The bound will then hold for independent sets of larger size.

Case III.B $\quad (\alpha(F(x)) = 2)$

Now assume that $\alpha(F(x)) = 2$, and $\mathrm{N}(x)$ contains a disjoint pair of independent sets $\{p_0, q_0\}$ and $\{p_1, q_1\}$. We establish a bound for the number of edges pendant to $\{p_0, q_0, p_1, q_1\}$, assume that no more than half of them are incident with $\{p_0, q_0\}$ and then obtain a bound for $|\mathrm{N}[p_0, q_0]|$.

It is possible to have two additional red edges with both endpoints in

$\{p_0, q_0, p_1, q_1\}$ without obtaining $\alpha(F(x)) = 3$. Therefore we may assume that

$$|\mathrm{Pend}\left(\mathrm{N}\left[x\right]; p_0, q_0, p_1, q_1\right)| \leq \rho(5,4) \tag{4.7}$$

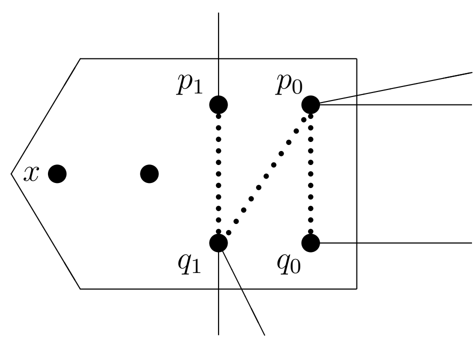
and, by symmetry,

$$|\mathrm{Pend}\left(\mathrm{N}\left[x\right]; p_0, q_0\right)| \leq \left\lfloor \frac{\rho(5,4)}{2} \right\rfloor. \tag{4.8}$$

Using Evaluation 4.4.4, we obtain

$$|\mathrm{N}\left[p_0, q_0\right]| \leq |\mathrm{N}\left[x\right]| + \left\lfloor \frac{\rho(5,4)}{2} \right\rfloor = 9 < \eta(5,2). \tag{4.9}$$

Now apply Lemma 3.21 to complete the proof for this case. Figure 4.3 illustrates one possible configuration.



In this example, $|\mathrm{M}\left[p_0.q_0; x\right]| \leq 3$ so that $|\mathrm{N}\left[p_0, q_0\right]| \leq 9 = \eta(5,2)$, and the bound follows.

Figure 4.3: $\delta(G) = 5$, $\alpha(F(x)) = 2$

The preceding argument is generally used when $F(x)$ is configured as stated, but there is an alternate proof for this particular case. Evaluation 4.4.2 and

Assumption 3.15 yield

$$\overline{\mathrm{e}}\,(F(x)) \leq 3 < \rho(5,4), \tag{4.10}$$

as depicted in the figure, but, since this approach does not work in general, we simply invoke (4.7). Notice that, in the current situation, both techniques yield the same bound for $|\mathrm{N}\,[p_0, q_0]|$.

Case IV $(\delta = 6)$

We may again assume that $\mathrm{N}\,[x]$ contains either three independent vertices or a disjoint pair of independent sets, each of size two. Although neither of these conditions suffice to make G K_5-free, it is still necessary that one or the other of them hold. The following partition into subcases is based upon these two possibilities. Since this case is similar to Case III, additional explanations are omitted.

We first calculate the pertinent parameters.

Parameter Evaluation 4.5

1. $\zeta(7,1) = \left\lceil \frac{13}{2} \cdot 7 - 20 \right\rceil = 26.$

2. $\psi(6,7,1) = \binom{7}{2} - \zeta(7,1) = -5.$

3. $\rho(6,3) = -\psi(6,7,1) - 1 + 3 = 7.$

4. $\rho(6,4) = \psi(6,7,1) - 1 + 4 = 8.$

5. $\zeta(11,2) = \left\lceil \frac{13}{2} \cdot 11 - 20 \cdot 2 \right\rceil = 32.$

6. $\psi(6,11,2) = \frac{6}{2} \cdot 11 - \zeta(11,2) = 1.$

7. $\eta(6,2) = 11.$

8. $\zeta(14,3) = \frac{13}{2} \cdot 14 - 20 \cdot 3 = 31.$

9. $\psi(6,14,3) = \frac{6 \cdot 14}{2} - \zeta(14,3) = 11.$

10. $\eta(6,3) = 14.$

Case IV.A $(\alpha(F(x)) \geq 3)$

Assume that $U = \{u_i\}^{i \in 3} \in \text{Indep}\,(F(x))$. Assumption 3.25 and Evaluation 4.5.3 imply

$$|\text{N}\,[U]| \leq |\text{N}\,[x]| + \rho(6,3) = 14 = \eta(6,3). \tag{4.11}$$

Now apply Lemma 3.21.

Case IV.B $(\alpha(F(x)) = 2)$

We assume that $\alpha(F(x)) = 2$, and let $\{p_0.q_0\}$ and $\{p_1, q_1\}$ be a pair of disjoint, independent subsets of $\text{N}\,(x)$. Evaluation 4.5.4, Assumption 3.25 and $\alpha(F(x)) = 2$ imply

$$|\text{Pend}\,(\text{N}\,[x]\,; p_0, q_0, p_1, q_1)| \leq \rho(6,4) = 8, \tag{4.12}$$

and symmetry allows us to assume that

$$|\text{N}\,[p_0, q_0]| \leq |\text{N}\,[x]| + 4 = 11 = \eta(6,2). \tag{4.13}$$

Lemma 3.21 yields the desired result.

Case V $(\delta \geq 7)$

Apply Lemma 3.29.

In this case we have decreased order by one and size by at least $\left\lceil \frac{13}{2} \right\rceil$, while we have not increased independence. Now the induction follows readily from the linearity of the bound.

This completes the proof of (1), and we immediately obtain (2) by an application of Lemma 2.2.

We next establish (3). Given positive integers n and r such that $\frac{1}{5} \leq \frac{r}{n} \leq \frac{1}{4}$, we wish to show that there exists a graph $G \in \mathcal{G}_5\,(5)$ of order n and independence r that satisfies (1) with equality. We consider two cases based on the parity of n.

Case I (n is even)

It is easily verified that K_4 and $G(10,2)$ both satisfy the (1) with equality. Therefore so must any sum of the form

$$c\,\mathrm{K}_4 + d\,G(10,2). \tag{4.14}$$

Let $n = 2k$, and solve the following linear system.

$$\begin{pmatrix} 1 & 2 \\ 4 & 10 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} r \\ 2k \end{pmatrix}. \tag{4.15}$$

This yields

$$c = 5r - 2k \quad \text{and} \quad d = k - 2r. \tag{4.16}$$

Both numbers are integers, and we have

$$c \geq 0 \iff \frac{1}{5} \leq \frac{r}{n} \tag{4.17}$$

and

$$d \geq 0 \iff \frac{r}{n} \leq \frac{1}{4}. \tag{4.18}$$

It is now clear that

$$G = c\,\mathrm{K}_4 + d\,G(10,2) \tag{4.19}$$

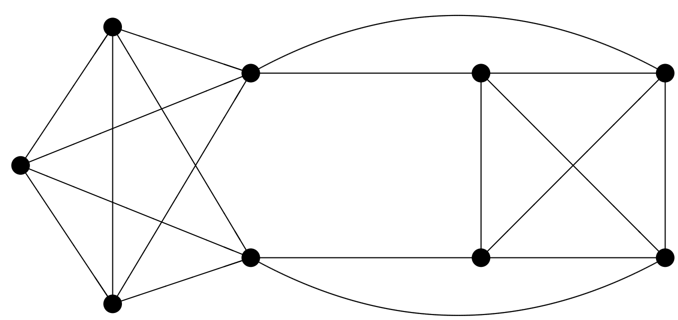
is a well defined graph. G is a member of $\mathcal{G}_5(5)$, and it is constructed so as to have the required order and independence. This establishes tightness when n is even.

Case II (n is odd)

The graph in Figure 4.4 has order nine, independence two and size 19. We refer to this graph as $G(9,2)$. Since

$$\frac{13}{2} \cdot 9 - 20 \cdot 2 = \frac{37}{2}, \tag{4.20}$$

we see that $G(9,2)$ fails to satisfy (1) with equality, but exceeds this bound by only one half.



G(9,2)

Figure 4.4: THE UNIQUE $(\mathcal{G}_5(5), 9, 2)$-MINGRAPH

Let $n = 2k + 9$ and define

$$G = G(9,2) + c\,\mathrm{K}_4 + d\,G(10,2), \tag{4.21}$$

where

$$c = 5r - 2k - 10 \quad \text{and} \quad d = k - 2r + 4. \tag{4.22}$$

These coefficients are obtained by solving the following linear system:

$$\begin{pmatrix} 1 & 2 \\ 4 & 10 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} r-2 \\ 2k \end{pmatrix}. \tag{4.23}$$

In this case, we have used one copy of $G(9,2)$ and applied the preceding construction to obtain a graph of order $n-9$ and independence $r-2$.

This establishes tightness for independence ratios between $\frac{1}{5}$ and $\frac{1}{4}$. Tightness does not hold for independence ratios larger than $\frac{1}{4}$ because the Turan bound, $\mathrm{e}\,(G) \geq 4\nu(G) - 10\alpha(G))$, is strictly dominant in that case. The problem of tightness remains open for independence ratios smaller than $\frac{1}{5}$.

Finally, we establish (4). As mentioned in Chapter 1, it follows from the combined work of ABT and Fajtlowicz that $\frac{1}{5}$ yields a tight independence ratio bound for $\mathcal{G}_5\,(5)$, and $G(10,2)$ is the unique connected graph for which tightness holds. Therefore all of the information required for the computation of the Ramsey-type numbers $R_5(5,n)$ was available at that time, but this result has never been published. These numbers are computed as follows:

First notice that Lemma 2.3 implies

$$R_5(5,k) \leq 5(k-1)+1. \tag{4.24}$$

We now consider the following two cases:

Case I $\quad$ (k is odd)

Let $k = 2j+1$. We have

$$\nu(jG(10,2)) = 10j = 5(k-1) \tag{4.25}$$

and

$$\alpha(jG(10,2)) = 2j < k. \tag{4.26}$$

The existence of this critical graph establishes tightness.

Case II (k is even)

Let $k = 2j$. In order for (4.24) to hold with equality, a critical graph must have order $5(2j-1)$; furthermore, the independence ratio of each component must equal $\frac{1}{5}$. Since $G(10,2)$ is the unique connected member of $\mathcal{G}_5\,(5)$ that possesses the required independence ratio, the order of any critical graph must be a multiple of 10, and we obtain a contradiction. Therefore

$$R_5(5,k) \leq 5(k-1). \tag{4.27}$$

Now the graph

$$(j-1)G(10,2) + \mathrm{K}_4 \tag{4.28}$$

is critical. ■

It will be shown in Chapter 5, that any connected $G \in \mathcal{G}_5\,(5)$ other than $G(10,2)$ satisfies the size bound

$$\mathrm{e}\,(G) \leq 7\nu(G) - 22\alpha(G) \tag{4.29}$$

and the independence ratio bound

$$\mu(G) \geq \frac{9}{44}. \tag{4.30}$$

4.2 A Size Bound for $\mathcal{G}\,(3)$

Fraughnaugh and Locke [9] established the size bounds (1.16) and (1.17). Both hold for triangle-free graphs, and the first holds with equality for triangle-free graphs when independence ratios are assumed to be at least $\frac{1}{2}$, but neither holds tightly for smaller independence ratios. In this section we establish a size bound that holds for all triangle-free graphs, and, moreover, holds with equality for independence between $\frac{2}{5}$ and $\frac{1}{2}$.

Theorem 4.6

1. *Let* $G \in \mathcal{G}(3)$. *Then* $\mathrm{e}(G) \geq 3\nu(G) - 5\alpha(G)$.
2. *Let* $G \in \mathcal{G}_2(3)$. *Then* $\mu(G) \geq \dfrac{2}{5}$.
3. *Let* $\dfrac{2}{5} \leq \dfrac{r}{n} \leq \dfrac{1}{2}$. *Then there exists* $G \in \mathcal{G}_2(3)$ *such that* $\nu(G) = n$, $\alpha(G) = r$ *and* $\mathrm{e}(G) = 3n - 5r$.
4. *Let* $k \geq 2$. *Then* $R_2(3,k) = \left\lfloor \dfrac{5}{2}(k-1) \right\rfloor + 1$.

Proof:

We prove (1) by induction on order. The empty graph clearly provides the base for the induction. Now let $G \in \mathcal{G}_2(3)$ such that $\nu(G) > 0$, and assume that (1) holds for members of $\mathcal{G}_2(3)$ of smaller order. The proof is partitioned into cases based on minimum degree, and, in each case, we assume that $x \in \mathrm{MinDeg}(G)$.

Case I $(\delta(G) \leq 1)$

The following parameter evaluations are applicable:

Parameter Evaluation 4.7

1. $\zeta(2,1) = 3 \cdot 2 - 5 = 1$.
2. $\psi(1,2,1) = \dbinom{2}{2} - \zeta(2,1) = 0$.

Evaluation 4.7.2 and Lemma 3.12 yield the desired result when $\mathrm{d}(x) = \delta(G) = 1$, and it follows when $\mathrm{d}(x) = \delta(G) = 0$, because the hypothesis of Lemma 3.18 is satisfied.

Case II $(\delta(G) = 2.)$

The following parameter evaluations are used in the proof of this case.

Parameter Evaluation 4.8

1. $\zeta(3,1) = 3 \cdot 3 - 5 = 4$

2. $\psi(2,3,1) = \binom{3}{2} - \zeta(3,1) = -1.$

3. $\rho(2,1) = -\psi(2,3,1) - 1 + 1 = 1.$

4. $\zeta(4,2) = 3 \cdot 4 - 5 \cdot 2 = 2.$

5. $\psi(2,4,2) = \frac{2 \cdot 4}{2} - \zeta(4,2) = 2.$

It follows from Evaluation 4.8.2 that it does not suffice to simply consider $\mathrm{N}[x]$. We let $\mathrm{N}(x) = \{p, q\}$, and note that $p \not\leftrightarrow q$ because G is triangle-free. It follows from Evaluation 4.8.3 and Assumption 3.25 that

$$|\mathrm{N}[p,q]| \leq |\mathrm{N}[x]| + \rho(2,1) = 4. \tag{4.31}$$

Now (1) follows by Evaluation 4.8.5 and Lemma 3.12 win the case that equality holds in (4.31), and it holds for closed neighborhoods of smaller size by Lemma 3.19.

Case III $(\delta(G) \geq 3)$

Apply Lemma 3.29.

This completes the proof of (1). Now (2) follows upon the imposition of the maximum degree constraint, $\Delta(G) \leq 2$, and an application of Lemma 2.2.

We next establish (3). Assume that $r, n \in \mathbb{N}$ such that $1 \leq r < n$ and

$$\frac{2}{5} \leq \frac{r}{n} \leq \frac{1}{2}. \tag{4.32}$$

Since K_2 and C_5 both satisfy (1) with equality, so must sums of these two graphs. Define c and d by

$$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} r \\ n \end{pmatrix}. \tag{4.33}$$

We obtain

$$c = 5r - 2n \quad \text{and} \quad d = n - 2r. \tag{4.34}$$

Both c and d are integers, and they satisfy the inequalities

$$c \geq 0 \iff \frac{r}{n} \geq \frac{2}{5}, \tag{4.35}$$

and

$$d \geq 0 \iff \frac{r}{n} \leq \frac{1}{2}. \tag{4.36}$$

Therefore

$$G = c\,\mathrm{K}_2 + d\,\mathrm{C}_5 \tag{4.37}$$

is a well defined graph, it clearly belongs to $G \in \mathcal{G}_2\,(3)$, and we have

$$\nu(G) = n \quad \text{and} \quad \alpha(G) = r \tag{4.38}$$

by construction. This completes the proof of (3).

We finally consider (4). Since the bound yielded by (2) and Lemma 2.3 coincides with the right side of (4), it suffices to exhibit critical graphs.

Case I (k is odd)

Let $k = 2j + 1$. It is easily verified that $j\,\mathrm{C}_5$ is critical.

Case II (k is even)

Let $k = 2j$. Now $k - 1$ is odd and the floor operation in (4) has an effect. In this case $(j - 1)\,\mathrm{C}_5 + \mathrm{K}_2$ yields a critical graph. ■

5. Size Bounds for $\mathcal{G}_m(m)$

Improvements upon existing independence ratio bounds for $\mathcal{G}_m(m)$ are established in this Chapter. Pertinent historical information from Chapter 1 is briefly reviewed in Section 5.1, and the bounds are established in Section 5.2. These results are compared to the bound $\mu \geq \frac{1}{m}$ in Section 5.3. Section 5.4 contains an explanation of how the coefficients for these bounds were obtained.

5.1 Introduction

In 1976, Albertson, Bollobas and Tucker (ABT) [1] showed that, given $m \geq 3$, if G is a connected, K_m-free, m-regular graph, then

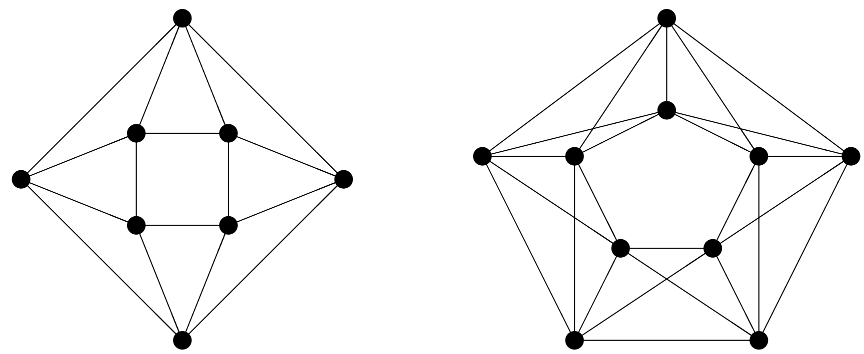
$$\mu(G) \geq \frac{1}{m}. \tag{5.1}$$

They also showed that this inequality is strict unless G is one of the two graphs illustrated in Figure 5.1.

Two years later, Fajtlowicz [4] improved upon this result by showing that

$$\mu(G) \geq \frac{2}{m + \Delta(G)} \tag{5.2}$$

for each $G \in \mathcal{G}(m)$. Upon assuming that $\Delta(G) \leq m$, we immediately obtain an extension of (5.1) to non-regular graphs. In that same article he asserted that $\frac{1}{m}$ does not yield a tight bound if $m \geq 7$ or if $m \in \{4, 5\}$ and the two graphs depicted in Figure 5.1 are excluded from consideration, but he did not offer a proof, nor did he state that specific bounds had been calculated.



G(8,2) G(10,2)

These are the only two connected, members of K_m, $m \geq 3$ for which the independence ratio bounds $\mu \geq \frac{1}{m}$ hold with equality.

Figure 5.1: THE CRITICAL GRAPHS OF ABT

In the next section, it will be shown that, if $m \geq 3$, $G \in \mathcal{G}_m(m)$, G is connected and $G \neq G(10.2)$, then

$$\mu(G) \geq \frac{3m-6}{3m^2-7m+4}. \tag{5.3}$$

If $m = 3$, or $m = 6$, then the resulting bounds are weaker then those obtained by Staton [21] and Sandberg [20], and, in the case that $m = 4$, the bound $\mu \geq \frac{1}{4}$ again follows. We obtain $\mu \geq \frac{9}{44}$ as a bound for connected members of $\mathcal{G}_5(5)$ other than $G(10,2)$ and, for each $m \geq 7$, (5.3) yields a bound that is strictly larger than $\frac{1}{m}$. Therefore the cases $m = 5$ and $m \geq 7$ are of primary interest.

5.2 Establishment of Bounds for $\mathcal{G}_m(m)$

Each of the independence ratio bounds proved in this section follow from a size bound. We begin by defining the coefficients.

Definition 5.1 *Let $m \geq 3$ and define*

$$a_m = 2m - 3 \quad \textit{and} \quad b_m = -\frac{3}{2}m^2 + \frac{7}{m} - 2. \tag{5.4}$$

We next introduce the corresponding independence ratio bounds.

Definition 5.2 *Let $m \geq 3$ and define*

$$\beta_m = \frac{3m-6}{3m^2 - 7m + 4}. \tag{5.5}$$

Theorem 5.3 *Let $m \geq 3$ and assume that that G is a connected member of $\mathcal{G}_m\,(m)$ and, in the case that $m = 5$, additionally assume that $G \neq G(10, 2)$. Then*

1. $\mathrm{e}\,(G) \geq a_m \nu(G) + b_m \alpha(G)$.
2. $\mu(G) \geq \beta_m$.
3. $R_m(m,n) \leq \left\lfloor \dfrac{3m^2 - 7m + 4}{3m - 6}\,(n-1) \right\rfloor + 1$.

Proof: First notice that (2) follows from (1) by Lemma 2.2, and (3) follows from (2) by Lemma 2.3.

We prove (1) by induction on order. This inequality clearly holds for the empty graph, so let $\nu(G) > 0$, and assume that it holds for all connected members of $\mathcal{G}_m\,(m)$ of smaller order. Throughout this proof we assume that $x \in \mathrm{MinDeg}(G)$.

Case I $\quad (\delta \leq m - 2)$

The case $\delta = m - 2$ follows by Evaluation 5.4.2, and Lemma 3.12, and then Lemma 3.18 yields the result for smaller values of δ.

Parameter Evaluation 5.4

1.

$$\zeta(m-1,1) = (m-1)a_m + b_m \tag{5.6}$$
$$= (m-1)(2m-3) - \frac{3}{2}m^2 + \frac{7}{2}m - 2 \tag{5.7}$$
$$= 2m^2 - 5m + 3 - \frac{3}{2}m^2 + \frac{7}{2}m - 2 \tag{5.8}$$
$$= \frac{1}{2}m^2 - \frac{3}{2}m + 1. \tag{5.9}$$

2.

$$\psi(m-2,m-1,1) = \binom{m-1}{2} - \zeta(m-1,1) \tag{5.10}$$
$$= \frac{1}{2}m^2 - \frac{3}{2}m + 1 - \left(\frac{1}{2}m^2 - \frac{3}{2}m + 1\right) \tag{5.11}$$
$$= 0. \tag{5.12}$$

Case II $(\delta = m-1)$

We begin with the standard calculations.

Parameter Evaluation 5.5

1.

$$\zeta(m,1) = a_m m + b_m \tag{5.13}$$
$$= 2m^2 - 3m - \frac{3}{2}m^2 + \frac{7}{2}m - 2 \tag{5.14}$$
$$= \frac{1}{2}m^2 + \frac{1}{2}m - 2. \tag{5.15}$$

2.

$$\psi(m-1,m,1) = \binom{m}{2} - \zeta(m,1) \tag{5.16}$$
$$= \frac{1}{2}m^2 - \frac{1}{2}m - \frac{1}{2}m^2 - \frac{1}{2}m + 2 \tag{5.17}$$
$$= -m + 2 \tag{5.18}$$
$$> 0. \tag{5.19}$$

3. $\rho(m-1,1) = -\psi(m-1,m,1) = m-2.$

Since $|\mathrm{N}[x]| = m$, $\mathrm{N}(x)$ must contain a pair of non-adjacent vertices, p and q. Assumption 3.25 and Evaluation 5.5.3 imply

$$|\mathrm{M}[p,q;x]| \leq \rho(m-1,1) = m-2, \tag{5.20}$$

and therefore

$$|\mathrm{N}[p,q]| \leq |\mathrm{N}[x]| + m - 2 = 2m-2. \tag{5.21}$$

It is clear that

$$\nu(H(p,q)) = \nu(G) - |\mathrm{N}[p,q]| \tag{5.22}$$

and

$$\alpha(H(p,q)) \leq \alpha(G) - 2. \tag{5.23}$$

The proof of this case now follows upon showing that

$$\psi(m-1,2m-2,2) > 0, \tag{5.24}$$

and then applying Lemma 3.19. The following evaluations yield (5.24).

Parameter Evaluation 5.6

1.

$$\zeta(2m-2,2) = (2m-3)(2m-2) + 2\left(-\frac{3}{2}m^2 + \frac{7}{2}m - 2\right) \tag{5.25}$$
$$= 4m^2 - 10m + 6 - 3m^2 + 7m - 4 \tag{5.26}$$
$$= m^2 - 3m + 2. \tag{5.27}$$

2.

$$\psi(m-1,2m-2,2) = \frac{(m-1)(2m-2)}{2} - \zeta(2m-2,2) \tag{5.28}$$
$$= m^2 - 2m + 1 - (m^2 - 3m + 2) \tag{5.29}$$
$$= m - 1. \tag{5.30}$$

Case III $(\delta = m)$

This final case is considerably more difficult than its predecessors. We first show that the bound follows unless the closed neighborhood of each vertex induces a particular graph, and then use a symmetry argument to establish that case.

Parameter Evaluation 5.7

1.

$$\zeta(m+1,1) = a_m(m+1) + b_m \tag{5.31}$$
$$= (2m-3)(m+1) - \frac{3}{2}m^2 + \frac{7}{2}m - 2 \tag{5.32}$$
$$= 2m^2 - m - 3 - \frac{3}{2}m^2 + \frac{7}{2}m - 2 \tag{5.33}$$
$$= \frac{1}{2}m^2 + \frac{5}{2}m - 5. \tag{5.34}$$

2.

$$\psi(m, m+1, 1) = \binom{m+1}{2} - \zeta(m+1, 1) \tag{5.35}$$
$$= \frac{1}{2}m^2 + \frac{1}{2}m - \frac{1}{2}m^2 - \frac{5}{2}m + 5 \tag{5.36}$$
$$= -2m + 5. \tag{5.37}$$

3. $\rho(m, 3) = -\psi(m-1, m, 1) - 1 + 3 = 2m - 3.$

4. $\rho(m, 4) = -\psi(m-1, m, 1) - 1 + 4 = 2m - 2.$

Since G is K_m-free, $\mathrm{N}\,[x]$ must contain an independent set of size three or a pair of disjoint independent sets, each of size two. This observation yields the following partition into subcases.

Case III.A $\quad (\alpha(F(x)) \geq 3)$

Let $U \in \mathrm{Indep}\,(F(x))$ and assume that $|U| = 3$. Assumption 3.25 and Evaluation 5.7.3 imply

$$|\mathrm{N}\,[U]| \leq |\mathrm{N}\,[x]| + \rho(m, 3) = 3m - 2. \tag{5.38}$$

We also have

$$\alpha(H(U)) \leq \alpha(G) - 3. \tag{5.39}$$

Now calculate

Parameter Evaluation 5.8

1.

$$\zeta(3m-2, 3) = (2m-3)(3m-2) + \left(-\frac{3}{2}m^2 + \frac{7}{2}m - 2\right)3 \tag{5.40}$$
$$= 6m^2 - 13m + 6 - \frac{9}{2}m^2 + \frac{21}{2}m - 6 \tag{5.41}$$
$$= \frac{3}{2}m^2 - \frac{5}{2}m. \tag{5.42}$$

2.

$$\psi(m, 3m-2, 3) = \left\lceil \frac{m(3m-2)}{2} \right\rceil - \zeta(3m-2, 3) \tag{5.43}$$

$$= \left\lceil \frac{3}{2}m^2 - m - \frac{3}{2}m^2 + \frac{5}{2}m \right\rceil \tag{5.44}$$

$$= \left\lceil \frac{3}{2}m \right\rceil . \tag{5.45}$$

An application of Lemma 3.21 now yields the bound for the current subcase.

Case III.B $(\alpha(F(x)) = 2$ and $\{p_0, q_0\}, \{p_1, q_1\} \in \operatorname{Indep}(G))$

We assume that $\alpha(F(x)) = 2$ and $\{p_0, q_0\}$ and $\{p_1, q_1\}$ are a disjoint pair of independent subsets of $\mathrm{N}(x)$. Now p_0q_0 and p_1q_1 are red edges, each has both endpoints in $\{p_0, q_0, p_1, q_1\}$ and it is possible to have two additional red edges, say p_0q_1 and p_1q_0, without obtaining $\alpha(F(x)) = 3$. Assumption 3.25 and Evaluation 5.7.4 yield

$$|\operatorname{Pend}(\mathrm{N}[x]; p_0, q_0, p_1, q_1)| \leq 2m - 2, \tag{5.46}$$

and, by symmetry, we may assume that

$$|\mathrm{M}[p_0, q_0; x]| \leq |\mathrm{M}[p_1, q_1; x]| . \tag{5.47}$$

Therefore Evaluation 5.7.4 implies

$$|\mathrm{N}[p_0, q_0]| \leq |\mathrm{N}[x]| + \frac{\rho(m, 4)}{2} = 2m. \tag{5.48}$$

We apply the induction hypothesis to $H(p_0, q_0)$ and consider the structure of $F(p_0, q_0)$.

Parameter Evaluation 5.9

1.

$$\zeta(2m,2) = (2m-3)2m + \left(-\frac{3}{2}m^2 + \frac{7}{2}m - 2\right)2 \tag{5.49}$$
$$= 4m^2 - 6m - 3m^2 + 7m - 4 \tag{5.50}$$
$$= m^2 + m - 4. \tag{5.51}$$

2.

$$\psi(m,2m,2) = \frac{2m^2}{2} - \zeta(2m,2) \tag{5.52}$$
$$= m^2 - m^2 - m + 4 \tag{5.53}$$
$$= -m + 4. \tag{5.54}$$

Lemma 3.12 yields the desired result in the case that $m \leq 4$. Therefore we may assume

$$m \geq 5. \tag{5.55}$$

The following evaluations and Lemma 3.19 suffice to establish the theorem in the case that $|\mathrm{N}[p_0, q_0]| \leq 2m - 1$.

Parameter Evaluation 5.10

1.

$$\zeta(2m-1,2) = (2m-3)(2m-1) + \left(-\frac{3}{2}m^2 + \frac{7}{2}m - 2\right)2 \tag{5.56}$$
$$= 4m^2 - 8m + 3 - 3m^2 + 7m - 4 \tag{5.57}$$
$$= m^2 - m - 1. \tag{5.58}$$

2.

$$\psi(m,2m-1,2)=\left\lceil\frac{m(2m-1)}{2}\right\rceil-\zeta(2m-1,2) \tag{5.59}$$
$$=\left\lceil m^2-\frac{m}{2}-m^2+m+1\right\rceil \tag{5.60}$$
$$=\left\lceil\frac{m}{2}\right\rceil+1. \tag{5.61}$$

We have shown that the theorem follows unless,

$$|\mathrm{N}\left[p,q\right]|\geq 2m \tag{5.62}$$

for each non-adjacent pair $p,q\in\mathrm{N}\left(x\right)$. For each such pair we have

$$|\mathrm{Pend}\left(\mathrm{N}\left[x\right];p,q\right)|\geq|\mathrm{M}\left[p,q;x\right]|\geq m-1. \tag{5.63}$$

It now follows from (5.46) that

$$|\mathrm{M}\left[p_0,q_0;x\right]|=|\mathrm{M}\left[p_1,q_1;x\right]|=m-1. \tag{5.64}$$

We introduce the following:

Notation 5.11

1. $|\mathrm{M}\left[p_0;x\right]|=k$.

2. $|\mathrm{M}\left[q_0;x\right]|=m-1-k=l$.

3. $k\geq l$.

If inequality (5.46) were to be strict, then (5.63) would not hold for p_0 and q_0, and the bound would follow. We therefore assume that

$$\bar{\mathrm{e}}\left(\langle p_0,q_0,p_1,q_1\rangle\right)=4. \tag{5.65}$$

In order for this equality to hold, it is necessary that

$$p_i \not\leftrightarrow q_j, \quad \forall\, i, j \in 2. \tag{5.66}$$

Now (5.64) and (5.66) imply

$$|\mathrm{M}\,[p_1; x]| = k \quad \text{and} \quad |\mathrm{M}\,[q_1; x]| = l. \tag{5.67}$$

Next, m-regularity implies

$$|\mathrm{Pend}\,(\mathrm{N}\,[x]\,; v)| = \overline{\mathrm{d}}_{\mathrm{N}[x]}(v), \quad \forall\, v \in \mathrm{N}\,(x)\,. \tag{5.68}$$

Therefore we may denote the members of $\mathrm{N}\,(x)$ that are not neighbors of p_0 by

$$Q = \{q_j\}^{j \in k}, \tag{5.69}$$

and the members of $\mathrm{N}\,(x)$ that are not neighbors of q_0 by

$$P = \{p_i\}^{i \in l}. \tag{5.70}$$

It follows from $\alpha(F(x)) = 2$ that P and Q induce cliques. Inequality (5.63) implies

$$|\mathrm{Pend}\,(q_j; x)| \geq l,\ \forall\, j \in k, \tag{5.71}$$

and we obtain

$$\overline{\mathrm{e}}\,(\mathrm{N}\,[x]) \geq kl. \tag{5.72}$$

Recall that $-\psi(m, m+1, 1) = 2m - 5$, so that Assumption 3.15 implies

$$\overline{\mathrm{e}}\,(F(x)) \leq 2m - 6. \tag{5.73}$$

We now consider possible values for k. Since $p_0, p_1 \in P$, we must have

$$k \leq m - 3. \tag{5.74}$$

If equality holds, then we obtain $kl = 2m - 6$, and if not, then $l \leq k$ implies $kl > 2m - 6$. Therefore, in order to satisfy (5.73), it is necessary that

$$k = m - 3 \quad \text{and} \quad l = 2 \tag{5.75}$$

The second equality implies

$$|\mathrm{M}\,[q_j; x]| = 2, \, \forall\, j \in k. \tag{5.76}$$

In order to satisfy all of these constraints it is necessary that

$$F(x) = \mathrm{K}_{m+1} \setminus \mathrm{E}(\mathrm{K}_{2,k-3}). \tag{5.77}$$

This configuration is illustrated in Figure 5.2.

We introduce the following:

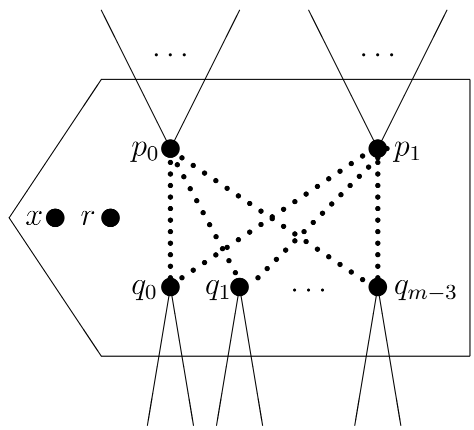
Definition 5.12 *Let $m \geq 5$ and let $G \in \mathcal{G}_m(m)$. Then G is said to be exceptional if it is m-regular, given any vertex x, $F(x)$ has the structure specified in (5.77) and*

$$\mathrm{M}\,[p_i; x] \cap \mathrm{M}\,[q_j; x] = \emptyset, \quad \forall\, i \in k, \; \forall\, j \in l. \tag{5.78}$$

If the closed neighborhood of some vertex were to induce a graph whose structure is different from that specified in (5.77), then we could consider that vertex in place of x and the theorem would follow by the previous arguments. We have justified the following:

Assumption 5.13 *G is exceptional.*

The following notational conventions will be in effect for the remainder of the proof. Figure 5.3 illustrates this notation for the case $m = 7$.

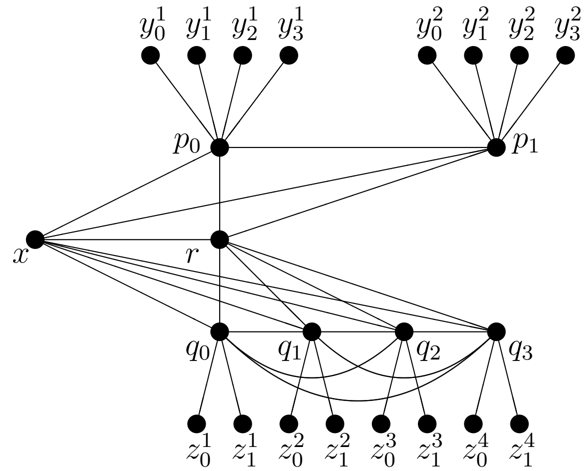


This figure illustrates the structure of the graph induced by closed neighborhood of a vertex of an exceptional graph. The dotted lines indicate red edges, and there are exactly $2m - 6$ of them. These edges correspond to the edges of a complete bipartite graph with partite sets $P = \{p_i\}^{i \in 2}$ and $Q = \{q_j\}^{j \in m-3}$. p_0 and p_1 each have $m - 3$ neighbors outside of $\mathrm{N}[x]$ and each q_j has two such neighbors.

Figure 5.2: THE EXCEPTIONAL CONFIGURATION

Notation 5.14 *Let $x \in \mathrm{V}(G)$ and consider the vertices of G that are of distance two or less from it. Definition 5.12 implies the existence of a red $\mathrm{K}_{2,m-3}$ for $F(x)$. We denote the partite set of size two by $P = \{p_i\}^{i \in 2}$ and the partite set of size l by $Q = \{q_j\}^{j \in l}$. Regularity implies that for each $i \in 2$ p_i must have exactly $l = m - 3$ neighbors outside of $\mathrm{N}[x]$; we denote each such set by $Y_i = \{y^i_s\}^{s \in l}$. Similarly, for each $j \in l$, we let $Z_j = \{z^j_t\}^{t \in 2}$ denote the neighbors of q_j that do not reside in $\mathrm{N}[x]$. Finally, we let r denote the unique member of $\mathrm{N}(x)$ that does not belong to either P or Q.*

When we remove $\mathrm{N}[p_0, q_0]$ from G, we exclude $2m$ vertices and reduce the independence of the remaining graph by at least two. Therefore the bound



This figure illustrates the configuration of vertices of distance two or less from x for an exceptional graph when $m = 7$. Clearly $\mathrm{N}(x) = \{r, p_0, p_1, q_0, q_1, q_2, q_3\}$ and $\mathrm{N}[r] = \mathrm{N}[x]$. $P = \{p_0, p_1\}$, $Q = \{q_0, q_1, q_2, q_3\}$, both P and Q induce cliques and there are no edges joining members of these two sets. For each $i \in 2$, $Y_i = \{y_s^i\}^{s \in 4}$, and, for each $l \in 4$, $Z_l = \{z_t^l\}^{t \in 2}$.

Figure 5.3: $\mathrm{N}[\mathrm{N}[x]]$ - PRIOR TO MERGE

follows by Evaluation 5.9.1 and Lemma 3.9 if

$$|\mathrm{Inc}(\mathrm{N}[p_0, q_0])| \geq m^2 + m - 4. \tag{5.79}$$

$F(x)$ is obtained by excluding $2m - 6$ edges from K_{m+1} and this yields

$$|\mathrm{Pend}(\mathrm{N}[x])| = 4m - 12. \tag{5.80}$$

Therefore

$$|\mathrm{Inc}(\mathrm{N}[x])| = \binom{m+1}{2} + 2m - 6 \tag{5.81}$$

$$= \frac{1}{2}m^2 + \frac{1}{2}m + 2m - 6 \tag{5.82}$$

$$= \frac{1}{2}m^2 + \frac{5}{2}m - 6. \tag{5.83}$$

Upon applying (5.79) and (5.83), we see that the theorem follows if

$$|\mathrm{Inc}\,(\mathrm{M}\,[p_0, q_0; x]) \setminus \mathrm{Inc}\,(\mathrm{N}\,[x])| \geq m^2 + m - 4 - \left(\frac{1}{2}m^2 + \frac{5}{2}m - 6\right) \tag{5.84}$$

$$= \frac{1}{2}m^2 - \frac{3}{2}m + 2. \tag{5.85}$$

Next, we let n denote the number of edges that have one endpoint in $\mathrm{N}\,[x]$ and the other in $\mathrm{M}\,[p_0, q_0; x]$. Since G is m-regular,

$$\mathrm{d}(\mathrm{M}\,[p_0, q_0; x]) = m(m-1). \tag{5.86}$$

Therefore

$$|\mathrm{Inc}\,(\mathrm{M}\,[p_0, q_0; x]) \setminus \mathrm{Inc}\,(\mathrm{N}\,[x])| \geq \frac{m(m-1) - n}{2}. \tag{5.87}$$

It follows immediately from Lemma 5.15, whose proof is deferred until the end of this section, that

$$n = 2m - 2. \tag{5.88}$$

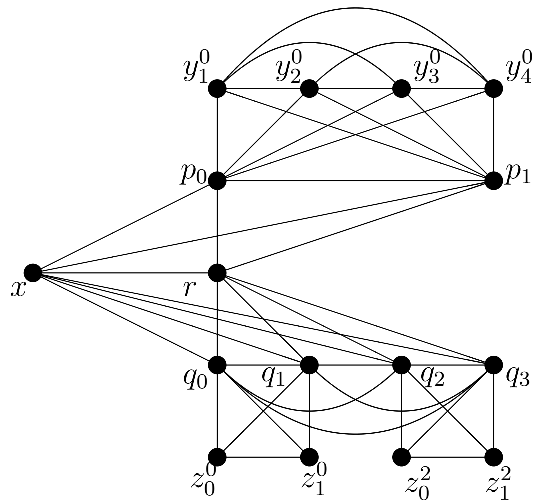
Lemma 5.15 *Assume that G is exceptional. Then*

1. *$Y_0 = Y_1$, and this set induces K_{m-3}.*

2. *There is a complete matching of Q that satisfies the property, that whenever $q_{j_0} \neq q_{j_1}$, we have $Z_{j_0} = Z_{j_1}$ if q_{j_0} and q_{j_1} are matched and $Z_{j_0} \cap Z_{j_1} = \emptyset$ otherwise. Furthermore each Z_j induces K_2.*

This lemma implies that each vertex of distance two from x is specified twice in Figure 5.3; Figure 5.4 illustrates the effect of merging them.

Since $|Q| = m - 3$, and the members of Q are matched, we see that there are no exceptional graphs when m is even. This observation and (5.55) justify



Comparison of this figure to Figure 5.3 illustrates the effect of applying Lemma 5.15. Now each vertex of distance two from x has been specified only once, and each has two neighbors in $\mathrm{N}(x)$. We have $Y_0 = Y_1$ with the members of this set labeled as members of Y_0. Also, in this example, $Z_0 = Z_1$ and $Z_2 = Z_3$.

Figure 5.4: $\mathrm{N}[\mathrm{N}[x]]$ - AFTER MERGE

the following:

$$m > 5 \quad \text{and} \quad m \text{ is odd.} \tag{5.89}$$

Substitution of (5.88) into (5.87) yields

$$|\mathrm{Inc}\,(\mathrm{M}\,[p_0, q_0; x]) \setminus \mathrm{Inc}\,(\mathrm{N}\,[x])| \geq \frac{(m-1)(m-2)}{2} \tag{5.90}$$

$$= \binom{m-1}{2} \tag{5.91}$$

$$= \frac{1}{2}m^2 - \frac{3}{2}m + 1, \tag{5.92}$$

and we fail to attain the required number of edges, as specified in (5.85), by one.

But, this situation can arise only if

$$\langle \mathrm{M}\,[p_0, q_0; x] \rangle = \mathrm{K}_{m-1}. \tag{5.93}$$

This is so because m-regularity requires that, each member of $\mathrm{M}[p_0, q_0; x]$ has exactly as many neighbors outside of $\mathrm{N}[p_0, q_0]$ as it has non-neighbors within $\mathrm{M}[p_0, q_0; x]$. Therefore we obtain two edges pendant to $\mathrm{N}[p_0, q_0]$ at $\mathrm{M}[p_0, q_0; x]$ for each edge excluded from $\langle \mathrm{M}[p_0, q_0; x] \rangle$. In this argument, we have used the facts that each $v \in \mathrm{M}[p_0 . q_0; x]$ has two neighbors in $\mathrm{N}[x]$, and there are $m - 2$ vertices in $\mathrm{M}[p_0, q_0; x]$ other then v.

We have now arrived at the point in the proof where the case $m = 5$ requires special treatment. We first assume that $m \neq 5$. By symmetry, Lemma 5.15 allows us to assume that $Z_0 = Z_1$ and therefore $Z_2 \neq Z_0$. We apply the preceding argument with q_0 replaced by q_2, and we see that the bound follows unless $\langle \mathrm{M}[p_0, q_2; x] \rangle = \mathrm{K}_{m-1}$, so assume that this is so. This implies

$$\{p_0, p_1, y_1^0, y_2^0, \ldots, y_{m-4}^0, z_0^0, z_1^0, z_0^2, z_1^2\} \subseteq \mathrm{N}\left(y_0^0\right) \tag{5.94}$$

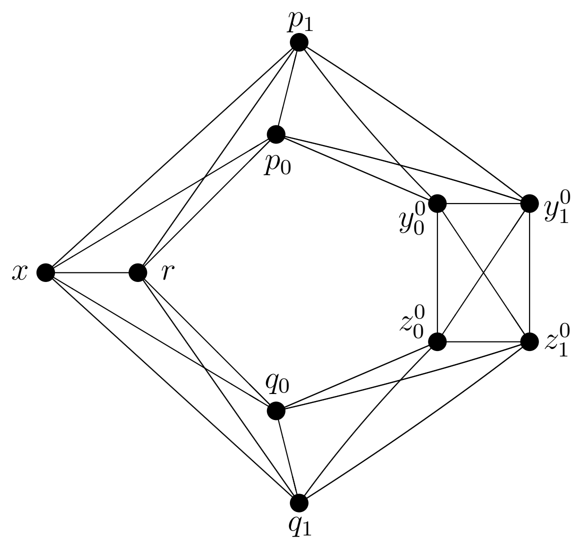
and we obtain a contradiction to $\Delta(G) = m$. The theorem now follows whenever $m \neq 5$.

Next, we let $m = 5$, and examine how the preceding proof fails for this case. Let $x \in \mathrm{V}(G)$ and consider the structure of $F(x)$. This graph is depicted in Figure 5.5.

Using Notation 5.14, we have

$$\mathrm{N}(x) = \{r, p_0, p_1, q_0, q_1\}. \tag{5.95}$$

By construction, $\mathrm{N}[r] = \mathrm{N}[x]$, and the red edges of $F(x)$ are exactly those that have one endpoint in $\{p_0, p_1\}$ and the other in $\{q_0, q_1\}$. Both p_0 and p_1 have y_0^0 and y_1^0 as neighbors that reside outside of $\mathrm{N}[x]$ and, similarly, q_0 and q_1 have z_0^0 and z_1^0 as neighbors that reside outside of $\mathrm{N}[x]$. It has been established that



This is the 5-regular graph obtained by ABT. This graph is the unique connected member of $\mathcal{G}_m\,(m)$ whose independence ratio is smaller than $\frac{9}{44}$.

Figure 5.5: THE 5-REGULAR EXCEPTION

(1) fails to hold only if $\langle y_0^0, y_1^0, z_0^0, z_1^0\rangle = \mathrm{K}_4$, and, unlike the case $m \geq 7$, there is no alternative choice for q_0. Since the construction of this counterexample is forced, any connected graph that fails to satisfy (1) must contain $G(10,2)$ as a subgraph. But, because this graph is 5-regular, it cannot be contained in a larger connected graph. It is therefore the unique exception to the theorem. Notice that

$$a_5 = 7, \quad b_5 = -22 \quad \text{and} \quad \beta_m = \frac{9}{44}, \tag{5.96}$$

yield

$$7\nu(G(10,2)) - 22\alpha(G(10,2)) = 26, \tag{5.97}$$

while

$$e\left(G(10,2)\right) = 25 \quad \text{and} \quad \mu(G(10,2)) = \frac{1}{5}. \tag{5.98}$$

■

The following proof of Lemma 5.15 serves to complete the proof of Theorem 5.3.

Proof: (Lemma 5.15) This proof utilizes Notation 5.14, and we again refer to Figure 5.3. The symmetry of exceptional graphs is used in showing that each vertex of distance two from x in that figure has actually been depicted twice. It has already been shown that, given any vertex of an exceptional graph, its closed neighborhood must induce $\mathrm{K}_{m+1} \setminus \mathrm{E}\left(\mathrm{K}_{2,m-3}\right)$. In particular, this property holds for each member of $\mathrm{N}\left(x\right)$; there are three cases to consider. First notice that

$$\mathrm{N}\left[r\right] = \mathrm{N}\left[x\right]. \tag{5.99}$$

Next, consider the closed neighborhood of one of the p_i, say p_0. We have

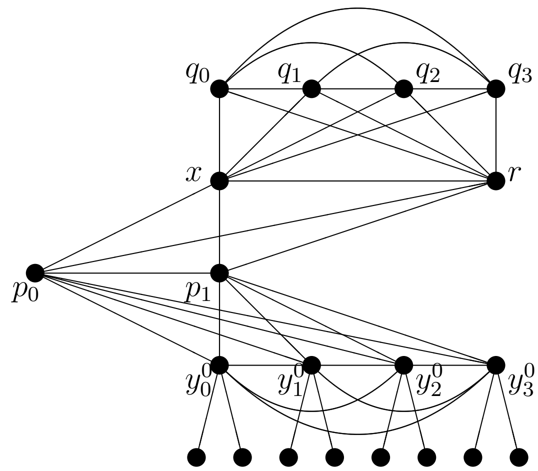
$$\mathrm{N}\left(p_0\right) = \{x, r, p_1, y_0^0, y_1^0, \ldots, y_{l-1}^0\}. \tag{5.100}$$

Since

$$x, r \not\leftrightarrow y_s^0, \quad \forall\, s \in l, \tag{5.101}$$

it follows that $\{x, r\}$ is the partite set of size two and $\{y_s^0\}^{s \in l}$ is the partite set of size $m-3$. Y_0 now resides in the position originally occupied by Q, so that it must induce K_l. This leaves p_1 as the unique vertex whose closed neighborhood coincides with that of p_0, and this implies

$$Y_0 = Y_1. \tag{5.102}$$



$p_1 \leftrightarrow y_l^0$, $\forall\, l \in 4$ so that $Y_0 = Y_1$. Since this set plays the roll of Q in Figure 5.3, it must induce K_{m-3}.

Figure 5.6: $\mathrm{N}\left[\mathrm{N}\left[p_0\right]\right]$

This configuration is illustrated in Figure 5.6.

Finally, consider $\mathrm{N}\left[q_0\right]$.

$$\mathrm{N}\left(q_0\right) = \left\{x, r, q_1, q_2, \ldots, q_{l-1}\right\}, \tag{5.103}$$

and, since

$$\langle\{x, r, q_0, \ldots, q_{l-1}\}\rangle = \mathrm{K}_{m-1}, \tag{5.104}$$

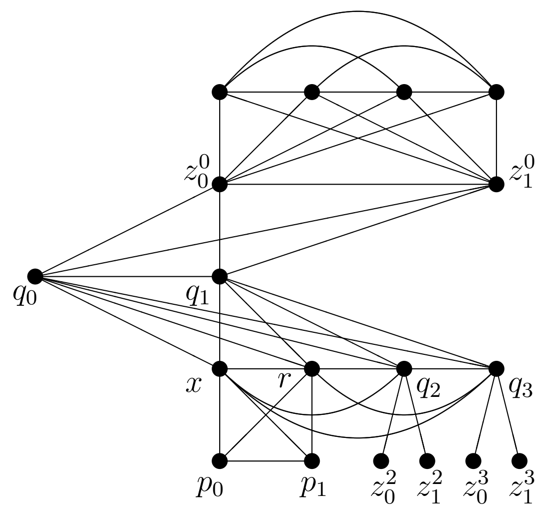
we see that $\{z_0^0, z_1^0\}$ is the only possible choice for the partite set of size two, and hence, $z_0^0 \leftrightarrow z_1^0$. Next, $x, r \not\leftrightarrow z_0^0, z_1^0$ implies that x and r are members of the partite set of size $m-3$. The remaining $m-5$ members of this set must belong to $\{q_1, \ldots, q_{l-1}\}$. By symmetry, we may assume that

$$\{x, r, q_2, q_3, \ldots, q_{l-1}\} \tag{5.105}$$

is the partite set of size $m-3$ and therefore

$$\mathrm{N}[q_1] = \mathrm{N}[q_0]. \tag{5.106}$$

This equality implies $Z_0 = Z_1$. Now, because the remaining q_i belong to the partite set of size $m-3$, none of them have neighbors in Z_0. We now match q_0 with q_1 because their closed neighborhoods coincide. Figure 5.7 illustrates this configuration.



z_0^0 and z_0^1 constitute the partite set of size two and therefore must be adjacent to each other. Since both are also adjacent to q_1 we see that $Z_0 = Z_1$. We also obtain this symmetry upon noting that x and r belong to the partite set of size $m-3$ and they share p_0 and p_1 as neighbors that do not reside in $\mathrm{N}[q_0]$.

Figure 5.7: $\mathrm{N}[\mathrm{N}[q_0]]$

We repeat the process of matching unpaired members of Q until all pairings have been obtained. At each step, we pair q_j with q_l, if $Z_j = Z_l$. We ultimately arrive at the configuration described in the lemma. Figure 5.4 illustrates the final result for the case $m = 7$. ■

5.3 Comparison to ABT

It follows immediately form

$$m\beta_m \to 1 \tag{5.107}$$

that $\{\beta_m\}_{m=3}^{\infty}$ is asymptotic to $\left\{\frac{1}{m}\right\}_{m=3}^{\infty}$, but this observation does not yield a measure of how much of an improvement we have obtained over $\mu \geq \frac{1}{m}$. If we let c_m denote the tight independence ratio bound for $\mathcal{G}_m(m)$, then it is necessary to also know the asymptotic behavior of these bounds, and no such information is available. Figure 5.1 contains information concerning β_m for various values of m.

Table 5.1: Comparison of β_m to $\frac{1}{m}$

m	3	4	5	6	7	8
β_m	$\frac{3}{10}$	$\frac{1}{4}$	$\frac{9}{44}$	$\frac{6}{35}$	$\frac{15}{102}$	$\frac{9}{70}$
$m\beta_m$	0.90000	1.00000	1.02277	1.02857	1.02941	1.02857
m	10	20	100	200	500	1000
β_m	$\frac{12}{117}$	$\frac{54}{1064}$	$\frac{147}{14652}$	$\frac{297}{59302}$	$\frac{747}{373252}$	$\frac{499}{498834}$
$m\beta_m$	1.02564	1.01504	1.00328	1.00165	1.00067	1.00033

This table contains data pertaining to the asymptotic behavior of $\{\beta_m\}_{m=3}^{\infty}$ relative to $\left\{\frac{1}{m}\right\}_{m=3}^{\infty}$. Notice that $m\beta_m$ is maximized when $m = 7$.

5.4 The Origin of the Bound

We now address the issue of how the values

$$a_m = 2m - 3 \quad \text{and} \quad b_m = -\frac{3}{2}m^2 + \frac{7}{2}m - 2 \tag{5.108}$$

were obtained. First notice that K_{m-1} is K_m-free and $m-2$ regular. It is easily verified that this graph satisfies

$$\mathrm{e}\left(\mathrm{K}_{m-1}\right) = a_m(m-1) + b_m. \tag{5.109}$$

We next construct a graph, which we denote by $G(2m-1, 2)$, that also satisfies this bound with equality. The unique graph of order $2m-1$ and independence two of minimum size is $\mathrm{K}_m + \mathrm{K}_{m-1}$, but this graph is clearly not K_m-free. We modify it to make it K_m-free by selecting a pair of vertices p and q from the component of order m and removing the edge joining them. The resulting graph is now K_m-free but it has independence three. To correct this problem, we add edges joining each vertex of the component of order $m-1$ to exactly one of p or q, and we do this in such a way that the number of edges incident with each of p and q is balanced. The resulting graph has order $2m-1$, independence two and size m^2-m-1. In order to obtain a bound that is satisfied by both graphs, we simply solve the following linear system.

$$\begin{pmatrix} m-1 & 1 \\ 2m-1 & 2 \end{pmatrix} \begin{pmatrix} a_m \\ b_m \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(m^2-m) \\ m^2-m-1 \end{pmatrix}, \tag{5.110}$$

and obtain the values specified in Definition 5.1.

The preceding construction yields

$$\Delta(G(2m-1,2)) = m-2+\left\lceil \frac{m-1}{2} \right\rceil, \tag{5.111}$$

and this value is strictly larger than m whenever $m \geq 6$. Therefore $G(2m-1,2)$ does not satisfy the hypothesis of the theorem in such cases. The rational for using this graph in the definition of the bound was that it is K_m-free and has only “a few” more edges than $\mathrm{K}_m + \mathrm{K}_{m-1}$. When first considering this problem, it was immediately clear that the size bounds obtained in this way yielded improved independence ratio bounds, but there was some question as to whether or not the size bounds would actually hold until a proof was obtained.

6. K_6-Free Graphs

In this chapter, we establish two size bounds that pertain to K_6-free graphs. The bound established in Section 6.1 holds for members of $\mathcal{G}_6\,(6)$, while the bound established in Section 6.2 holds for all members of $\mathcal{G}\,(6)$.

6.1 A Size Bound for $\mathcal{G}_6\,(6)$

The size bound presented in this section holds for $\mathcal{G}_6\,(6)$, and it holds with equality for independence ratios between $\frac{2}{11}$ and $\frac{1}{5}$.

Theorem 6.1 *Let $G \in \mathcal{G}_6\,(6)$. Then*

1. $\mathrm{e}\,(G) \geq 10\nu(G) - 40\alpha(G)$.
2. $\mu(G) \geq \dfrac{7}{40}$.
3. $R_6(6, n) \leq \left\lfloor \dfrac{40}{7}(n-1) \right\rfloor + 1$
4. *If $r, n \in \mathbb{N}$, such that $\dfrac{2}{11} \leq \dfrac{r}{n} \leq \dfrac{1}{5}$, then there exists a graph $G \in \mathcal{G}_6\,(6)$ such that $\nu(G) = n$, $\alpha(G) = r$ and $\mathrm{e}\,(G) = 10\nu(G) - 40\alpha(G)$.*

Proof: The proof of (1) proceeds by induction on order, with the empty graph providing the base case. We assume that $G \in \mathcal{G}_6\,(6)$ such that $\nu(G) > 0$, and (1) holds for members of $\mathcal{G}_6\,(6)$ of smaller order. We partition the induction step into cases based upon the value of $\delta(G)$, and, in each case, we assume that $x \in \mathrm{MinDeg}(G)$,

Case I $\quad (\delta(G) \leq 4)$

Evaluation 6.2.2 and Lemma 3.12 yield (1) when $\delta(G) = 4$. Then an application of Lemma 3.18 completes the proof for smaller minimum degrees.

Parameter Evaluation 6.2

1. $\zeta(5,1) = 10 \cdot 5 - 40 = 10.$

2. $\psi(4,5,1) = \binom{5}{2} - \zeta(5,1) = 0.$

Case II $\quad (\delta(G) = 5)$

The following parameter evaluations pertain to this case.

Parameter Evaluation 6.3

1. $\zeta(6,1) = 10 \cdot 6 - 40 = 20.$

2. $\psi(5,6,1) = \binom{6}{2} - \zeta(6,1) = -5.$

3. $\rho(5,1) = -\psi(5,6,1) = 5.$

4. $\rho(5,3) = 7.$

5. $\rho(5,4) = 8.$

6. $\zeta(10,2) = 10 \cdot 10 - 40 \cdot 2 = 20.$

7. $\psi(5,10,2) = \dfrac{5 \cdot 10}{2} - \zeta(10,2) = 5.$

8. $\zeta(11,2) = 10 \cdot 11 - 40 \cdot 2 = 30.$

9. $\lambda(5,11,2) = 2\zeta(11,2) - 5 \cdot 11 - 1 = 4$

10. $\eta(5,2) = 10.$

11. $\zeta(16,3) = 10 \cdot 16 - 40 \cdot 3 = 40.$

12. $\psi(5,16,3) = \dfrac{5 \cdot 16}{2} - \zeta(16,3) = 0.$

13. $\eta(5,3) = 16$.

First notice that $|\mathrm{N}[x]| = 6$. Since G is K_6-free, $\mathrm{N}(x)$ must contain an independent set of size at least two. If $\alpha(F(x)) \geq 3$, or $\mathrm{N}(x)$ contains a disjoint pair of independent vertices, then the bound follows immediately from the preceding evaluations. If neither of these conditions hold, then the red edges of $F(x)$ must form a star, and it is necessary to consider subcases.

Case II.A $(\alpha(F(x)) \geq 3)$

Let $U \in \mathrm{Indep}(F(x))$ such that $|U| = 3$. Assumption 3.25 and Evaluation 6.3.4 yield

$$|\mathrm{N}[U]| \leq |\mathrm{N}[x]| + \rho(5,3) = 13. \tag{6.1}$$

This case now follows by Evaluation 6.3.13 and Lemma 3.21.

Case II.B $(\alpha(F(x))) = 2$, $\{p_0, q_0\}, \{p_1, q_1\} \in \mathrm{Indep}(F(x)))$

Now assume that $\alpha(F(x) = 2$ and $\{p_0, q_0\}, \{p_1, q_1\} \in \mathrm{Indep}(F(x))$ such that $\cap_{i \in 2} \{p_i, q_i\} = \emptyset$. We have

$$\overline{\mathrm{e}}(\langle p_0, q_0, p_1, q_1 \rangle) \leq 4, \tag{6.2}$$

and symmetry allows us to assume that

$$|\mathrm{M}[p_0, q_0; x]| \leq |\mathrm{M}[p_1, q_1; x]|. \tag{6.3}$$

Now Assumption 3.25 and Evaluation 6.3.5 imply

$$|\mathrm{N}[p_0, q_0]| \leq |\mathrm{N}[x]| + \frac{\rho(5,4)}{2} = 10. \tag{6.4}$$

This case follows by Evaluation 6.3.10 and Lemma 3.21.

Case II.C $\quad (F(x) = \mathrm{K}_6 \setminus \mathrm{E}(\mathrm{K}_{1,k}))$

The only remaining possibility is that the excluded edges of $F(x)$ form a star. Notation 6.4, which is used in the proof of this case, is illustrated in Figure 6.3.

Notation 6.4 *Assume that* $x \in \mathrm{MinDeg}(G)$ *and* $F(x)$ *is the clique* K_6 *from which the edges of a star have been excluded.*

1. *Let* p *denote the center of the star.*

2. $Q = \{q_i\}^{i \in |Q|}$ *denotes the points of the star.*

3. $\mathrm{M}[p; x] = Y = \{y_j\}^{j \in |Y|}$.

4. $\mathrm{M}[q_i; x] = Z_i = \{z^i_k\}^{k \in |Z_i|}, \ \forall\, i \in |Q|$.

5. *If* $j, k \in |Q|$ *such that* $j < k$*, then* $|Z_j| \leq |Z_k|$.

6. $Y \cup Z_0 = W = \{w_l\}^{l \in |W|}$.

Let $u \in \mathrm{N}(x)$. It follows from

$$\delta(G) = \Delta(G) - 1 = \mathrm{d}(x) \leq \mathrm{d}(u) \leq \Delta(G) \tag{6.5}$$

that

$$\overline{\mathrm{d}}_{\mathrm{N}[x]}(u) \leq |\mathrm{Pend}\,(\mathrm{N}[x]\,; u)| \leq \overline{\mathrm{d}}_{\mathrm{N}[x]}(u) + 1. \tag{6.6}$$

In particular,

$$|Q| \leq |\mathrm{Pend}\,(\mathrm{N}[x]\,; p)| \leq |Q| + 1, \tag{6.7}$$

and

$$1 \leq |\mathrm{Pend}\,(\mathrm{N}[x]\,; q_i)| \leq 2, \quad \forall\, i \in |Q|\,. \tag{6.8}$$

We now consider the structure of $F(p, q_0)$. Let $i \in |Q|$. Assumption 3.22 and Evaluation 6.3.10 imply

$$|\mathrm{N}[p, q_i]| \geq 11, \tag{6.9}$$

and therefore

$$|\mathrm{M}[p, q_i; x]| \geq 5. \tag{6.10}$$

We may assume that

$$3 \leq |\mathrm{M}[p; x]| \leq 4. \tag{6.11}$$

The first inequality follows from (6.8) and (6.10). In order to establish the second, notice that each red edge of $F(x)$ has exactly one member of Q as an endpoint. Therefore (6.8) implies

$$|\mathrm{Int}(\mathrm{N}[x])| + |\mathrm{Pend}(\mathrm{N}[x]; Q)| \geq \binom{6}{2} = 15. \tag{6.12}$$

If the second inequality in (6.11) does not hold, then we obtain

$$|\mathrm{Inc}(\mathrm{N}[x])| \geq 20 = \zeta(6, 1), \tag{6.13}$$

and the bound follows by Lemma 3.9. Figure 6.1 illustrates this situation.

Inequality (6.11) justifies the next partition into subcases.

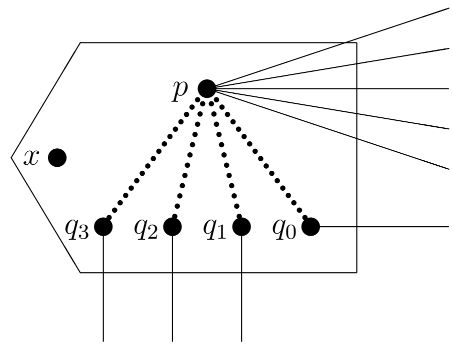
Case II.C.1 $\quad (|\mathrm{M}[p; x]| = 3)$

First notice that (6.7) implies

$$|Q| \geq 2. \tag{6.14}$$

It follows from the defining property for this subcase, (6.10) and Notation 6.4.5 that

$$|\mathrm{M}[q_i; x]| = 2, \quad \forall\, i \in |Q|. \tag{6.15}$$



In this example, $|\mathrm{Pend}\,(\mathrm{N}\,[x]\,;p)| = 5$ and $\bar{\mathrm{e}}\,(F(x)) = |Q| = |\mathrm{Pend}\,(\mathrm{N}\,[x]\,;Q)| = 4$. This yields $|\mathrm{Inc}\,(\mathrm{N}\,[x])| = 20 = \zeta(6,1)$, and the bound follows by Lemma 3.9.

Figure 6.1: $|\mathrm{Pend}\,(\mathrm{N}\,[x]\,;p)| = 5$

This yields

$$|\mathrm{Inc}\,(\mathrm{N}\,[x])| \geq \binom{6}{2} + |Q| + 3 \geq 20, \tag{6.16}$$

and the theorem again follows by Lemma 3.9. We refer the reader to Figure 6.2 for an illustration of this case.

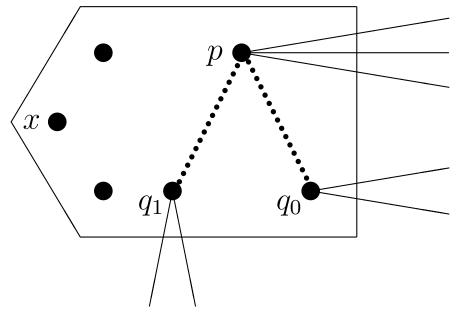
Case II.C.2 $\quad (|\mathrm{M}\,[p;x]| = 4)$

Figure 6.3 illustrates the following argument. It follows from (6.12) and the defining property of this subcase that

$$|\mathrm{Inc}\,(\mathrm{N}\,[x])| \geq 19, \tag{6.17}$$

and Evaluation 6.3.1 and Assumption 3.10 imply that equality must hold. This is possible only if

$$|\mathrm{M}\,[q_i;x]| = 1, \quad \forall\, i \in |Q|\,. \tag{6.18}$$



There are three edges pendant to $\mathrm{N}[x]$ at p. Since $\delta(G) = \Delta(G) - 1$, p must have at least two non-neighbors in $\mathrm{N}(x)$, Evaluation 6.3.10 and Notation 6.4.5 require that $|\mathrm{M}[\mathrm{N}[x]; q_i]| \geq 2$ for each q_i. This implies $|\mathrm{Inc}(\mathrm{N}[x])| \geq 20$.

Figure 6.2: $|\mathrm{M}[p; x]| = 3$

Next, $|\mathrm{M}[p; x]| = 4$ and $|\mathrm{M}[p, q_0; x]| = 5$ imply

$$z_0^0 \not\leftrightarrow p. \tag{6.19}$$

If we were to have $Y \subseteq \mathrm{N}[z_0^0]$, it would follow that

$$\left|\mathrm{N}\left[p, z_0^0\right]\right| \leq 10, \tag{6.20}$$

and the bound would immediately follow by Lemma 3.21 and Evaluation 6.3.10.

Therefore, by symmetry, we may assume

$$z_0^0 \not\leftrightarrow y_0. \tag{6.21}$$

Now we have

$$\left\{x, y_0, z_0^0\right\} \in \mathrm{Indep}(G). \tag{6.22}$$

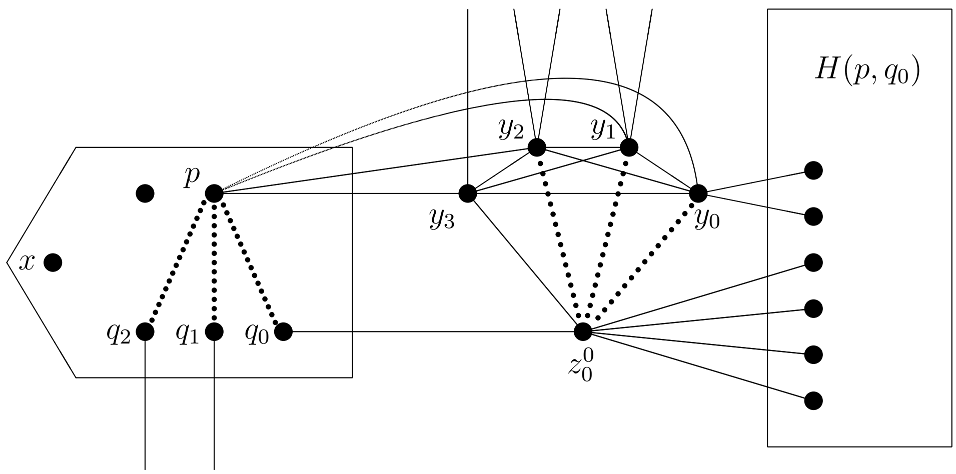
In this case, Evaluation 6.3.13 and Assumption 3.22 imply

$$\left|\mathrm{N}\left[x, y_0 . z_0^0\right]\right| \geq 17. \tag{6.23}$$

This inequality and $|\mathrm{N}\,[p, q_0]| = 11$ imply

$$|\mathrm{Pend}\,(\mathrm{N}\,[p, q_0])| \geq 6, \tag{6.24}$$

and now the theorem follows by Evaluation 6.3.9 and Lemma 3.27.



$\{x, y_0, z_0\}$ is independent, so that we may assume $|\mathrm{N}\,[x, y_0.z_0]| > \eta(5, 3) = 16$. This yields $|\mathrm{Pend}\,(\mathrm{N}\,[p, q_0])| \geq 6 > \lambda(5, 11, 2)$.

Figure 6.3: $|\mathrm{M}\,[p; x]| = 5$

Case III $(\delta(G) = 6)$

Now $|\mathrm{N}\,[x]\,| = 7$. In order for G to be K_6-free, it is necessary that $\mathrm{N}\,(x)$ contain either three independent vertices or a disjoint pair of independent sets of size two. If $\mathrm{N}\,(x)$ contains four independent vertices, then the theorem follows immediately from the following evaluations.

Parameter Evaluation 6.5

1. $\zeta(7, 1) = 10 \cdot 7 - 40 = 30.$

2. $\psi(6,7,1) = \binom{7}{2} - \zeta(6,1) = -9.$

3. $\rho(6,3) = 11.$

4. $\rho(6,4) = 12.$

5. $\rho(6,6) = 14.$

6. $\zeta(11,2) = 10 \cdot 11 - 40 \cdot 2 = 30.$

7. $\psi(6,11,2) = \frac{6 \cdot 11}{2} - \zeta(11,2) = 3.$

8. $\zeta(12,2) = 10 \cdot 12 - 40 \cdot 2 = 40.$

9. $\lambda(6,12,2) = 2\zeta(12,2) - 6 \cdot 12 - 1 = 7.$

10. $\zeta(13,2) = 10 \cdot 13 - 40 \cdot 2 = 50.$

11. $\lambda(6,13,2) = 2\zeta(13,2) - 6 \cdot 13 - 1 = 21.$

12. $\eta(6,2) = 11.$

13. $\zeta(17,3)10 \cdot 17 - 40 \cdot 3 = 50.$

14. $\psi(6,17,3) = \frac{6 \cdot 17}{2} - \zeta(17,3) = 1.$

15. $\zeta(18,3) = 10 \cdot 18 - 40 \cdot 2 = 60.$

16. $\lambda(6,18,3) = 2\zeta(18,3) - 6 \cdot 18 - 1 = 11.$

17. $\eta(6,3) = 17$

18. $\zeta(22,4) = 10 \cdot 22 - 40 \cdot 4 = 60.$

19. $\psi(6, 22, 4) = \dfrac{6 \cdot 22}{2} - \zeta(12, 2) = 6.$

20. $\eta(6, 4) = 22.$

First notice that 6-regularity implies

$$|\mathrm{Pend}\,(\mathrm{N}\,[x]\,; v)| = \overline{\mathrm{d}}_{\mathrm{N}[x]}(v), \quad \forall\, v \in \mathrm{N}\,(x)\,. \tag{6.25}$$

Case III.A $\quad (\alpha(F(x)) \geq 4)$

Assume that $U \in \mathrm{Indep}\,(F(x))$ such that $|U| = 4$. Since $\overline{\mathrm{e}}\,(\langle U \rangle) = 6$, it follows from Evaluation 6.5.5 and Assumption 3.25 that

$$|\mathrm{N}\,[U]| \leq |\mathrm{N}\,[x]| + 14 = 21. \tag{6.26}$$

Now apply Evaluation 6.5.20 and Lemma 3.21.

Case III.B $\quad (\alpha((F(x)) = 3)$

Assume that $U \in \mathrm{Indep}\,(F(x))$ such that $|U| = 3$. We adopt Notation 6.6 for this case. Figure 6.5 illustrates this notation, and yields insight into the following proof.

Notation 6.6 *Assume that* $U \in \mathrm{Indep}\,(F(x))$ *such that* $|U| = 3$*. Members of* $\mathrm{N}\,(x)$ *and* $\mathrm{M}\,[U; x]$ *are labeled as follows:*

1. $U = \{u_i\}^{i \in 3}.$

2. $\mathrm{N}\,(x) \setminus U = \{v_j\}^{j \in 3}.$

3. $\mathrm{M}\,[u_i; x] = W_i = \left\{w^i_j\right\}^{j \in |W_i|}, \quad \forall\, i \in 3.$

4. $i < k \implies |W_i| \leq |W_k| \quad \forall\, i, k \in l.$

Assumption 3.25 and Evaluation 6.5.3 imply

$$|\mathrm{N}\,[U]| \leq |\mathrm{N}\,[x]| + \rho(6,3) \leq 18, \tag{6.27}$$

while Assumption 3.22 and Evaluation 6.5.17 assert that equality must hold. More generally, if U' is an independent subset of $\mathrm{N}\,(x)$ of size three, then we assume that

$$|\mathrm{Pend}\,(\mathrm{N}\,[x]\,; U')| = |\mathrm{M}\,[U'; x]| = 11. \tag{6.28}$$

This implies

$$|\mathrm{N}\,[U']| = 18. \tag{6.29}$$

If either (6.28) or (6.29) fail to hold, then (1) follows immediately by letting $U = U'$.

Assumption 3.28 and Evaluation 6.5.16 imply that (1) follows if

$$|\mathrm{Pend}\,(\mathrm{N}\,[U])| \geq 11. \tag{6.30}$$

To this end, notice that Assumption 3.22 and Evaluation 6.5.20 imply

$$|\mathrm{Pend}\,(\mathrm{N}\,[U]\,; S)| \geq 5, \tag{6.31}$$

whenever $S \in \mathrm{Indep}\,(F(U))$ and $|S| = 4$.

We begin by considering the structure of $F(x)$. Observe that (6.25) and (6.28) imply

$$\sum_{i \in 3} \overline{\mathrm{d}}_{\mathrm{N}[x]}(u_i) = 11. \tag{6.32}$$

Since each red edge that has both endpoints in U contributes two to this sum, there must exist five additional red edges, each of which has one endpoint in U and the other in $\{v_j\}^{j \in 3}$. Assumption 3.15 and Evaluation 6.5.2 imply

$$\overline{\mathrm{e}}\,(\mathrm{N}\,[x]) \leq 8, \tag{6.33}$$

so that all red edges have now been accounted for, and, since $\alpha(F(x)) = 3$, each v_j must have at least one neighbor in U. By symmetry, we may assume that

$$\overline{\mathrm{d}}_{\mathrm{N}[x]}(v_0) = \overline{\mathrm{d}}_{\mathrm{N}[x]}(v_1) = 2 \quad \text{and} \quad \overline{\mathrm{d}}_{\mathrm{N}[x]}(v_2) = 1. \tag{6.34}$$

Either v_0 and v_1 have the same pair of non-neighbors in U, or they have one common non-neighbor in U. These two possibilities are illustrated in Figure 6.4 (a) and (b), respectively. Notice that there are three independent sets of size three in both (a) and (b). In (a) all three sets contain u_1 and u_2, and each must have 11 neighbors outside of $\mathrm{N}[x]$. It is possible that $v_2 \not\leftrightarrow u_1$ instead of $v_2 \not\leftrightarrow u_2$ in (a), but this is the only change that is consistent with (6.28), and it does not yield a different structure. In (b), the sets $\{u_1, u_2, v_j\}$, $j \in 2$ have only 10 neighbors outside of $\mathrm{N}[x]$, and this contradicts (6.28). Since this situation cannot be altered by changing the adjacency of v_2, we may assume that $F(x)$ is configured as in Figure 6.4.a.

We next consider $\langle \mathrm{M}[U;x] \rangle$, and we again refer the reader to Figure 6.5. In order to satisfy both (6.28) and (6.29), it is necessary that

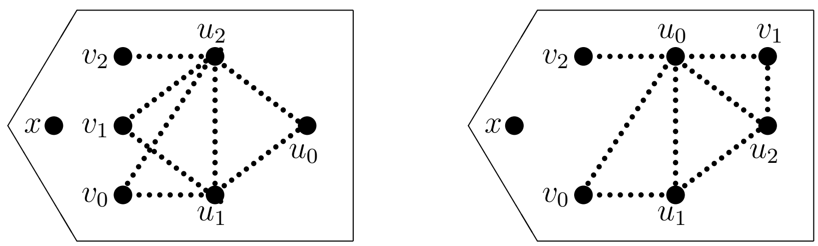
$$\mathrm{M}[U;x] = \cup_{i \in 3} W_i, \tag{6.35}$$

with the union disjoint. It now follows from the (6.25), and the fact that $F(x)$ has the structure illustrated in Figure 6.4.a that

$$|W_0| = 2, \quad |W_1| = 4 \quad \text{and} \quad |W_2| = 5. \tag{6.36}$$

If $\langle W_2 \rangle = \mathrm{K}_5$, then $\langle W_2 \cup \{u_2\} \rangle = \mathrm{K}_6$, but this is impossible. Therefore, by symmetry, we may assume that

$$w_0^2 \not\leftrightarrow w_1^2. \tag{6.37}$$



(a) (b)

This figure illustrates configurations for $\mathrm{N}[x]$ for which $\alpha(F(x)) = 3$. In (a), each set of three independent vertices must have 11 neighbors outside of $\mathrm{N}[U]$, while this property does not hold in (b).

Figure 6.4: $\alpha(F(x)) = 3$

In this case,

$$\{x, w_0^2, w_1^2\} \in \mathrm{Indep}(G), \tag{6.38}$$

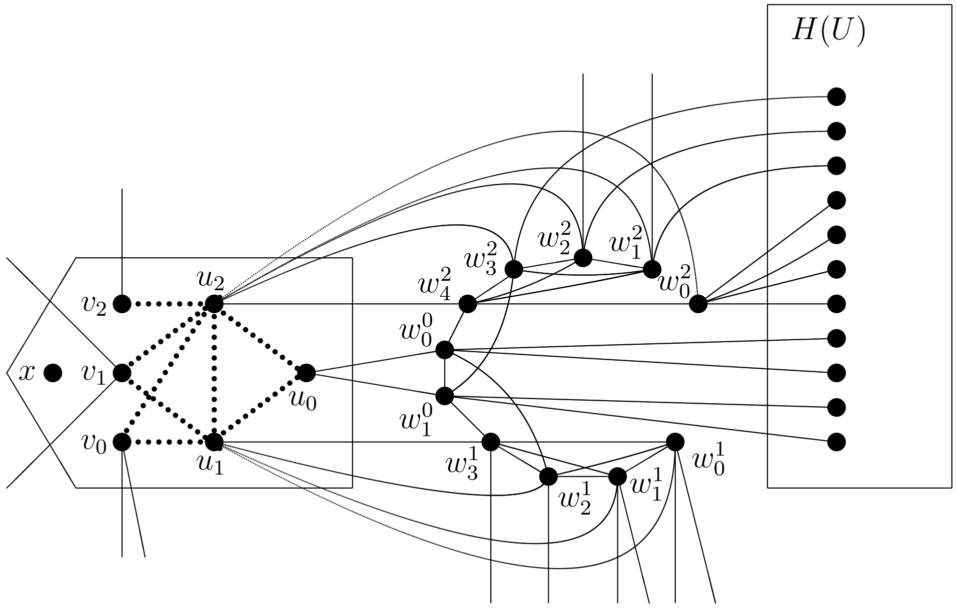
and, because $x, w_0^2, w_1^2 \in \mathrm{N}(u_2)$, we may assume that $F(u_2)$ also has the structure illustrated in Figure 6.4.a. Since x has no neighbors in W_2, it must correspond to u_2, and, by symmetry, we let w_0^2 and w_1^2 correspond to u_1 and u_0, respectively. We additionally let w_2^2, w_3^2 and w_4^2 correspond to v_0, v_1 and v_2, respectively. Now we have

$$\langle w_1^2, w_2^2, w_3^2, w_4^2 \rangle = \mathrm{K}_4, \quad w_0^2 \leftrightarrow w_4^2 \quad \text{and} \quad w_0^2 \not\leftrightarrow w_1^2, w_2^2, w_3^2, \tag{6.39}$$

and we see that $\langle W_2 \rangle$ is configured as in Figure 6.5.

Since the W_i are pairwise disjoint, we have

$$\{u_0, u_1, w_0^2, w_l^2\} \in \mathrm{Indep}(G), \quad 1 \le l \le 3. \tag{6.40}$$



There are $\lambda(6, 17, 3)$ edges pendant to $\mathrm{N}\,[U]$. The seven incident with W_2 are forced to exist because $w_0^2 \not\leftrightarrow w_1^2, w_2^2, w_3^2$. There are four pendant edges incident with Z_0, because it has been established that $|\mathrm{N}\,[Z_0] \cap (w_1 \cup W_2| \leq 4$

Figure 6.5: $\alpha(F(U)) = 3$

All of the neighbors of u_0 and u_1 reside within $\mathrm{N}\,[U]$, so that (6.31) yields

$$\left|\mathrm{Pend}\left(\mathrm{N}\,[U]\,; w_0^2, w_l^2\right)\right| \geq 5, \quad 1 \leq l \leq 3. \tag{6.41}$$

We have already accounted for two neighbors of w_0^2, and four neighbors for each of w_1^2, w_2^2 and w_3^2. Therefore

$$3 \leq \left|\mathrm{Pend}\left(\mathrm{N}\,[U]\,; w_0^2\right)\right| \leq 4 \tag{6.42}$$

and

$$1 \leq \left|\mathrm{Pend}\left(\mathrm{N}\,[U]\,; w_l^2\right)\right| \leq 2, \quad 1 \leq l \leq 3. \tag{6.43}$$

If

$$\left|\mathrm{Pend}\left(\mathrm{N}\,[U]\,; w_0^2\right)\right| = 3, \tag{6.44}$$

then, for each l, the second inequality in (6.43) must be an equality, and this implies

$$|\mathrm{Pend}\,(\mathrm{N}\,[U]\,; W_2)| \geq 9. \tag{6.45}$$

Otherwise

$$\left|\mathrm{Pend}\left(\mathrm{N}\,[U]\,; w_0^2\right)\right| = 4, \tag{6.46}$$

and we obtain

$$|\mathrm{Pend}\,(\mathrm{N}\,[U]\,; W_2)| \geq 7. \tag{6.47}$$

In either, case four additional edges pendant $\mathrm{N}\,[U]$ suffice to complete the proof of (1).

We next consider the structures of $\langle W_0 \rangle$ and $\langle W_1 \rangle$. If $w_0^0 \not\leftrightarrow w_1^0$, then, because the W_i are pairwise disjoint,

$$\left\{u_1, u_2, w_0^0, w_0^1\right\} \in \mathrm{Indep}\,(G)\,. \tag{6.48}$$

Now (6.31) and $\mathrm{N}\,[u_1, u_2] \subset \mathrm{N}\,[U]$ yield

$$\left|\mathrm{Pend}\left(\mathrm{N}\,[U]\,; w_0^0, w_0^1\right)\right| \geq 5, \tag{6.49}$$

and the bound follows. We may therefore assume that

$$w_0^0 \leftrightarrow w_0^1, \tag{6.50}$$

and, by a similar argument, we may assume that

$$\langle W_1 \rangle = \mathrm{K}_4. \tag{6.51}$$

It is clear that (1) follows if

$$|\mathrm{Pend}\,(\mathrm{N}\,[U]\,; W_0)| \geq 4. \tag{6.52}$$

Since w_0^0 and w_0^1 are adjacent to u_0, and to each other, it remains to account for eight edges incident with W_0. Each such edge is either pendant to $\mathrm{N}[U]$ or incident with $W_1 \cup W_2$. Therefore (6.52) is equivalent to

$$|\mathrm{N}[W_0] \cap (W_1 \cup W_2)| \leq 4. \tag{6.53}$$

We first consider edges that have one endpoint in W_0 and the other in W_1. Let $j \in 2$. We have

$$u_1 \not\leftrightarrow w_j^0, \tag{6.54}$$

because the W_i are pairwise disjoint. Now Evaluation 6.5.12 and Assumption 3.22 imply

$$\left|\mathrm{N}\left[u_1, w_j^0\right]\right| \geq 12, \tag{6.55}$$

and this implies

$$\left|\mathrm{N}\left[w_j^0\right] \cap W_1\right| \leq 2. \tag{6.56}$$

We first assume that equality holds, and, by symmetry, we let

$$w_0^1 \not\leftrightarrow w_j^0. \tag{6.57}$$

Now equality in (6.56) and $\langle W_1 \rangle = \mathrm{K}_4$ imply

$$\left|\mathrm{N}\left[w_0^1, w_j^0\right]\right| \leq 12. \tag{6.58}$$

Next

$$\left\{x, w_0^1, w_j^0\right\} \in \mathrm{Indep}\,(G)\,, \tag{6.59}$$

and

$$\{u_0, u_1\} \subseteq \mathrm{N}[x] \cap \mathrm{N}\left[w_0^1, w_j^0\right] \tag{6.60}$$

imply

$$\left|\mathrm{N}\left[x, w_i^0, w_0^1\right]\right| \leq 17 = \eta(6,3). \tag{6.61}$$

Since this contradicts Assumption 3.22, we obtain

$$\left|\mathrm{N}\left[w_j^0\right] \cap W_1\right| \leq 1, \tag{6.62}$$

and therefore

$$\left|\mathrm{N}\left[W_0\right] \cap W_1\right| \leq 2. \tag{6.63}$$

By a similar argument, we obtain

$$\left|\mathrm{N}\left[W_0\right] \cap \left\{w_1^2, w_2^2, w_3^2, w_4^2\right\}\right| \leq 2. \tag{6.64}$$

Now (6.53) follows unless

$$\mathrm{N}\left(w_0^2\right) \cap \left(W_0 \cup W_1\right) \neq \emptyset, \tag{6.65}$$

so assume that this condition holds. It now follows from $w_0^2 \leftrightarrow u_2, w_4^2$ and the first inequality of (6.42) that

$$\left|\mathrm{N}\left(w_0^2\right) \cap W_0\right| = 1, \tag{6.66}$$

and therefore

$$\left|\mathrm{Pend}\left(\mathrm{N}\left[U\right]; W_0\right)\right| \geq 3. \tag{6.67}$$

Now (6.44) holds, and, because this implies (6.45), we again obtain (6.30). This completes the proof for the case $\alpha(F(x)) = 3$.

Case III.C $\quad(\alpha((F(x)) = 2)$

Now $\alpha(F(x)) = 2$ and $\mathrm{N}\left[x\right]$ must contain a disjoint pair of independent sets of size two, $\{p_0, q_0\}$ and $\{p_1, q_1\}$. Before proceeding with the proof of this case, we introduce the requisite notation. This notation is illustrated in Figure 6.6.

Notation 6.7 *Suppose that* $\mathrm{N}[x]$ *contains a disjoint pair of independent sets* $\{p_0, q_0\}$ *and* $\{p_1, q_1\}$*. Define*

1. $\mathrm{M}[p_0; x] = Y_0 = \{y_s^0\}^{s \in |Y_0|}$.

2. $\mathrm{M}[q_0; x] = Z_0 = \{z_t^0\}^{t \in |Z_0|}$.

3. $Y_0 \cup Z_0 = W = \{w_l\}^{l \in |W|}$.

4. *Label these sets so that* $|\mathrm{M}[p_0, q_0; x]| \leq |\mathrm{M}[p_1, q_1; x]|$*, and* $|\mathrm{M}[q_0; x]| \leq |\mathrm{M}[p_0; x]|$.

Evaluation 6.5.12 and Assumption 3.22 imply

$$|\mathrm{M}[p, q; x]| \geq 5 \tag{6.68}$$

whenever $p, q \in \mathrm{N}(x)$ such that $p \not\leftrightarrow q$. Since $\alpha(F(x)) = 2$, it follows that

$$\overline{\mathrm{e}}(p_0, q_0, p_1, q_1) \leq 4. \tag{6.69}$$

Now it follows form Assumption 3.25, Evaluation 6.5.4 and Notation 6.7.4 that

$$|W| \leq \frac{\rho(6, 4)}{2} = 6. \tag{6.70}$$

Therefore

$$5 \leq |W| \leq 6. \tag{6.71}$$

This inequality justifies the next partition into subcases.

Case III.C.1 $(|W| = 5)$

In this case we have

$$|\mathrm{N}[p_0, q_0]| = 12. \tag{6.72}$$

We next consider subcases based upon whether or not W induces a clique.

Case III.C.1.a $(\langle W \rangle = \mathrm{K}_5)$

Since G is K_6-free, we may assume that

$$z_0^0 \not\leftrightarrow p_0. \tag{6.73}$$

Notation 6.7.4 implies

$$|Y_0| \geq 3, \tag{6.74}$$

and, because $Y_0 \subseteq \mathrm{N}(p_0) \cap \mathrm{N}[z_0^0]$, we obtain

$$\left|\mathrm{N}\left[p_0, z_0^0\right]\right| \leq 11 = \eta(6,2). \tag{6.75}$$

The result now follows by Lemma 3.21.

Case III.C.1.b $(\langle W \rangle \neq \mathrm{K}_5)$

Now assume that $w_0 \not\leftrightarrow w_1$. Then

$$\{x, w_0, w_1\} \in \mathrm{Indep}(G). \tag{6.76}$$

Evaluation 6.5.17 and Assumption 3.22 imply

$$|\mathrm{N}[x, w_0, w_1]| \geq 18, \tag{6.77}$$

and, since $\mathrm{N}[x] \subset \mathrm{N}[p_0, q_0]$, it follows that

$$|\mathrm{Pend}(\mathrm{N}[p_0, q_0]; w_0, w_1)| \geq 6. \tag{6.78}$$

Assume that $2 \leq l \leq 4$. If $W \subset \mathrm{N}[w_l]$, then

$$|\mathrm{Pend}(\mathrm{N}[p_0, q_0]; w_l)| \leq 1. \tag{6.79}$$

Therefore, by symmetry, we may assume that

$$|\mathrm{Pend}\,(\mathrm{N}\,[p_0, q_0]\,; w_0)| \geq |\mathrm{Pend}\,(\mathrm{N}\,[p_0, q_0]\,; w_l)|\,, \quad 1 \leq l \leq 4. \tag{6.80}$$

It follows from this inequality. (6.78) and the fact that w_0 has a neighbor in $\{p_0, q_0\}$, that

$$3 \leq |\mathrm{Pend}\,(\mathrm{N}\,[p_0, q_0]\,; w_0)| \leq 5. \tag{6.81}$$

The first inequality forces w_0 to have an additional non-neighbor in W. Say $w_0 \not\leftrightarrow w_2$. Inequality (6.78) also holds when w_1 is replaced by w_2, so that the second inequality in (6.81) implies

$$|\mathrm{Pend}\,(\mathrm{N}\,[p_0, q_0]\,; w_2)| \geq 1. \tag{6.82}$$

Finally, (6.78) and (6.82) imply

$$|\mathrm{Pend}\,(\mathrm{N}\,[p_0, q_0]\,; w_0, w_1, w_2)| \geq 7 = \lambda(6, 12, 2), \tag{6.83}$$

and the result follows by Lemma 3.27.

Case III.C.2 $(|W| = 6)$

If $p, q \in \mathrm{N}\,(x)$ such that $p \not\leftrightarrow q$, then we may assume that

$$|\mathrm{Pend}\,(\mathrm{N}\,[x]\,; p, q)| \geq |\mathrm{M}\,[p, q; x]| \geq 6, \tag{6.84}$$

for otherwise, Case III.C.1 would be applicable, while $\alpha(F(x)) = 2$ and (6.25) require that each member of $\mathrm{N}\,(x)$ must have exactly one non-neighbor in $\{p, q\}$. Therefore equality must hold across (6.84).

Now symmetry allows us to assume that $p_0 \not\leftrightarrow q_1$ and $q_0 \not\leftrightarrow p_1$. The following notation, which is illustrated in Figure 6.6, will be used for the remainder of the current subcase.

Notation 6.8 *Suppose that* $\mathrm{N}[x]$ *contains a disjoint pair of independent sets* $\{p_0, q_0\}$, $\{p_1, q_1\}$. *Define*

1. $Q = \{q_i\}^{i \in |Q|} = \{q \in \mathrm{N}(x) \mid q \not\leftrightarrow p_0\}$.

2. $P = \{p_j\}^{j \in |P|} = \{p \in \mathrm{N}(x) \mid p \not\leftrightarrow q_0\}$.

3. $|P| \leq |Q|$.

4. $\mathrm{M}[p_j; x] = Y_j = \{y_s^j\}^{s \in |Q|} \quad \forall\, j \in |P|$.

5. $\mathrm{M}[q_i; x] = Z_i = \{z_t^i\}^{t \in |P|} \quad \forall\, i \in |Q|$.

6. $Y_0 \cup Z_0 = W = \{w_l\}^{l \in |W|}$.

It follows from $\alpha(F(x)) = 2$ that both P and Q induce cliques, and equality across (6.84) implies that, for each $p_j \in P$, $q_i \in Q$,

$$p_j \not\leftrightarrow q_i, \quad |\mathrm{M}[p_j, q_i; x]| = ||\mathrm{N}[x]|\, p_j, q_i| = 6, \quad Y_j \cap Z_i = \emptyset. \tag{6.85}$$

We have now partitioned the members of $\mathrm{N}(x)$ into partite sets for a complete, red, bipartite graph. If both partite sets were to have size three, then we would have

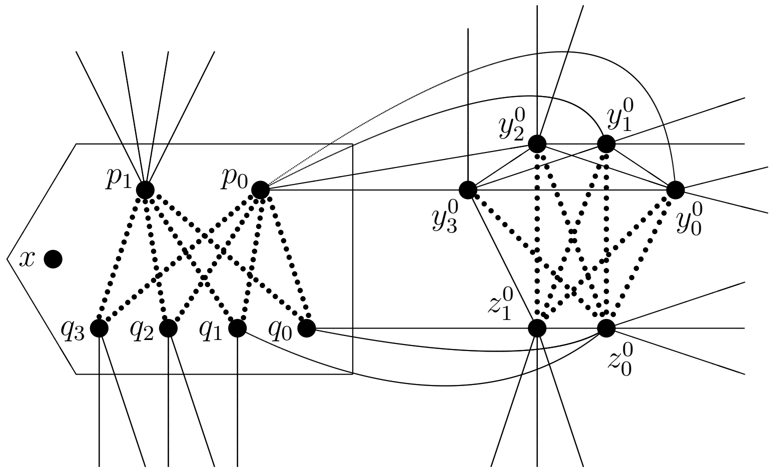
$$\overline{\mathrm{e}}\,(F(x)) = 9, \tag{6.86}$$

but, given Evaluation 6.5.2, this contradicts Assumption 3.15. Therefore Notation 6.8.3 yields

$$|P| = 2, \quad \text{and} \quad |Q| = 4. \tag{6.87}$$

We next consider configurations of vertices of distance two from x. Let $j \in 2$. If there exist $y_0^j, y_1^j \in Y_j$ such that $y_0^j \not\leftrightarrow y_1^j$, then

$$\left\{x, y_0^j, y_1^j\right\} \in \mathrm{Indep}\,(F(p_j))\,, \tag{6.88}$$



In this example $\mathrm{N}[x] = \mathrm{K}_7 - \mathrm{E}(\mathrm{K}_{2,4})$. We may assume that $G(Y_0) = \mathrm{K}_4$, because otherwise $\alpha(F(p_0)) \geq 3$ and this case has already been established. Similarly, we may assume that $G(Z_0) = \mathrm{K}_2$.

Figure 6.6: $F(x) = \mathrm{K}_7 \setminus \mathrm{E}(\mathrm{K}_{2,4})$

and Case (III.B) is applicable. We therefore assume that

$$\langle Y_j \rangle = \mathrm{K}_4, \quad \forall\, j \in 2. \tag{6.89}$$

Similarly

$$\langle Z_i \rangle = \mathrm{K}_2, \quad \forall\, i \in 4. \tag{6.90}$$

It follows from

$$|\mathrm{N}[p_0, q_0]| = 13, \tag{6.91}$$

Evaluation 6.5.11 and Lemma 3.28 that (1) follows if

$$|\mathrm{Pend}(\mathrm{N}[p_0, q_0])| \geq 21. \tag{6.92}$$

We first consider $\mathrm{Pend}(\mathrm{N}[p_0, q_0]\,; p_1)$. If $Y_0 = Y_1$, then

$$\langle Y_0 \cup \{p_0, p_1\} \rangle = \mathrm{K}_6, \tag{6.93}$$

and this is impossible. By symmetry, we may assume that

$$y_0^0 \not\leftrightarrow p_1. \tag{6.94}$$

If $|Y_0 \cap Y_1| = 3$, then

$$\left|\mathrm{N}\left[p_1, y_0^0\right]\right| \leq 11 = \eta(6,2), \tag{6.95}$$

and the theorem follows by Lemma 3.21. Next, assume that

$$y_0^0 \in Y_0 \cap Y_1 \quad \text{and} \quad |Y_0 \cap Y_1| \leq 2. \tag{6.96}$$

Then, since Y_0 and Y_1 both induce cliques and $y_0^0 \leftrightarrow p_0, p_1$, we obtain

$$\mathrm{d}(y_0^0) \geq 7. \tag{6.97}$$

We may therefore assume that

$$Y_0 \cap Y_1 = \emptyset, \tag{6.98}$$

and hence

$$|\mathrm{Pend}\,(\mathrm{N}\,[p_0, q_0]\,; p_1)| = 4. \tag{6.99}$$

Recall that

$$|\mathrm{Pend}\,(\mathrm{N}\,[x]\,; q_i)| = 2, \quad \forall\, i \in 4, \tag{6.100}$$

and let $t \in 2$. If $\mathrm{d}_Q(z_t^0) \geq 3$, then

$$\left|\mathrm{N}\,(x) \cap \mathrm{N}\left(z_t^0\right)\right| \geq 3. \tag{6.101}$$

This implies

$$\left|\mathrm{N}\left[x, z_t^0\right]\right| \leq 11 = \eta(6,2), \tag{6.102}$$

in contradiction to Assumption 3.22. Therefore

$$|\mathrm{N}\left[Z_0\right] \cap Q| \leq 4, \tag{6.103}$$

and hence,

$$4 \leq |\mathrm{Pend}\left(\mathrm{N}\left[p_0, q_0\right]; q_1, q_2, q_3\right)| \leq 6. \tag{6.104}$$

Upon taking (6.99) into consideration, we obtain

$$8 \leq |\mathrm{Pend}\left(\mathrm{N}\left[p_0, q_0\right]; p_1, q_1, q_2, q_3\right)| \leq 10. \tag{6.105}$$

We finally consider edges pendant to $\mathrm{N}\left[p_0, q_0\right]$ at $Y_0 \cup Z_0 = W$. Let $t \in 2$. If $Y_0 \subseteq \mathrm{N}\left(z_t^0\right)$, then

$$\left|\mathrm{N}\left[p_0, z_t^0\right]\right| \leq 10 < \eta(6,2), \tag{6.106}$$

and the bound again follows. Therefore let $s \in 4$, and assume that

$$y_s^0 \not\leftrightarrow z_t^0. \tag{6.107}$$

Now if

$$\left|\mathrm{N}\left[z_t^0\right] \cap Y_0\right| \geq 2, \tag{6.108}$$

then, since $\langle Y_0 \rangle = \mathrm{K}_4$, we have

$$\left|\mathrm{N}\left[z_t^0, y_s^0\right]\right| \leq 12. \tag{6.109}$$

We also have

$$\left\{x, z_t^0, y_s^0\right\} \in \mathrm{Indep}\left(G\right), \tag{6.110}$$

and

$$p_0, q_0 \in \mathrm{N}\left[x\right] \cap \mathrm{N}\left[z_t^0, y_s^0\right]. \tag{6.111}$$

But this implies

$$\left|\mathrm{N}\left[x, z_t^0, y_s^0\right]\right| \leq 17 = \eta(6,3), \tag{6.112}$$

and we again contradict Assumption 3.22. Therefore

$$\left|\mathrm{N}\left[z_t^0\right] \cap Y_0\right| \leq 1. \tag{6.113}$$

Now (6.110), Assumption 3.22 and Evaluation 6.5.17 yield

$$\left|\mathrm{N}\left[x, y_s^0, z_t^0\right]\right| \geq 18. \tag{6.114}$$

Since

$$\left|\mathrm{N}\left[p_0, q_0\right]\right| = 13 \tag{6.115}$$

and

$$\mathrm{N}\left[x\right] \subset \mathrm{N}\left[p_0, q_0\right], \tag{6.116}$$

it follows from (6.114) that

$$\left|\mathrm{Pend}\left(\mathrm{N}\left[p_0, q_0\right]; y_s^0, z_t^0\right)\right| \geq 5. \tag{6.117}$$

We next establish the existence of a collection of non-adjacent pairs y_s^0 and z_t^0, apply (6.117) to each, and take double counting into consideration to obtain

$$\left|\mathrm{Pend}\left(\mathrm{N}\left[p_0, q_0\right]; Y_0 \cup Z_0\right)\right| \geq 12. \tag{6.118}$$

Given inequality (6.105), this suffices to establish (6.92) unless

$$\left|\mathrm{Pend}\left(\mathrm{N}\left[U\right]; \mathrm{N}\left[x\right]\right)\right| = 8, \tag{6.119}$$

in which case, it will be shown that

$$\left|\mathrm{Pend}\left(\mathrm{N}\left[p_0, q_0\right]; Y_0 \cup Z_0\right)\right| \geq 14. \tag{6.120}$$

Let $s \in 4$. Since $\langle Y_0 \rangle = \mathrm{K}_4$ and $y_s^0 \leftrightarrow p_0$, we have

$$\left|\mathrm{Pend}\left(\mathrm{N}\left[p_0, q_0\right]; y_s^0\right)\right| \leq 2. \tag{6.121}$$

Similarly, if $t \in 2$, then $\left|\mathrm{N}\left[z_t^0\right] \cap \mathrm{N}\left[q_0\right]\right| \geq 2$ implies

$$\left|\mathrm{Pend}\left(\mathrm{N}\left[p_0, q_0\right]; z_t^0\right)\right| \leq 4. \tag{6.122}$$

First assume that

$$\left|\mathrm{Pend}\left(\mathrm{N}\left[p_0, q_0\right]; z_t^0\right)\right| = 4, \quad \forall\, t \in 2. \tag{6.123}$$

Then

$$z_t^0 \not\leftrightarrow y_s^0, \quad \forall\, t \in 2, \quad \forall\, s \in 4. \tag{6.124}$$

Now (6.117) implies

$$\left|\mathrm{Pend}\left(\mathrm{N}\left[p_0, q_0\right]; y_s^0\right)\right| \geq 1, \quad \forall\, s \in 4, \tag{6.125}$$

and (6.118) is satisfied. Next, by symmetry, assume that

$$\left|\mathrm{Pend}\left(\mathrm{N}\left[p_0, q_0\right]; z_0^0\right)\right| = 3. \tag{6.126}$$

It follows from (6.113) and (6.117) that

$$\left|\mathrm{Pend}\left(\mathrm{N}\left[p_0, q_0\right]; Y_0\right)\right| \geq 6. \tag{6.127}$$

We also have

$$\left|\mathrm{Pend}\left(\mathrm{N}\left[p_0, q_0\right]; z_1^0\right)\right| \geq 3, \tag{6.128}$$

and we again satisfy (6.118).

It only remains to assume (6.119). This can occur only if

$$\left|\mathrm{N}\left[z_t^0\right] \cap Q\right| = 2, \quad \forall\, t \in 2. \tag{6.129}$$

In this case

$$\left|\mathrm{Pend}\left(\mathrm{N}\left[p_0, q_0\right]; z_t^0\right)\right| = 3, \quad \forall\, t \in 2, \tag{6.130}$$

and furthermore, all neighbors of each z_t^0 have been accounted for, so that we obtain

$$z_t^0 \not\leftrightarrow y_s^0, \quad \forall\, t \in 2, \quad \forall\, s \in 4. \tag{6.131}$$

Now it follows from (6.117) and (6.130) that

$$\left|\mathrm{Pend}\left(\mathrm{N}\left[p_0, q_0\right]; Y_0\right)\right| = 8. \tag{6.132}$$

It has already been established that

$$\left|\mathrm{Pend}\left(\mathrm{N}\left[p_0, q_0\right]; Q\right)\right| \geq 8 \quad \text{and} \quad \left|\mathrm{Pend}\left(\mathrm{N}\left[p_0, q_0\right]; Z_0\right)\right| \geq 6. \tag{6.133}$$

Therefore

$$\left|\mathrm{Pend}\left(\mathrm{N}\left[p_0, q_o\right]\right)\right| \geq 22. \tag{6.134}$$

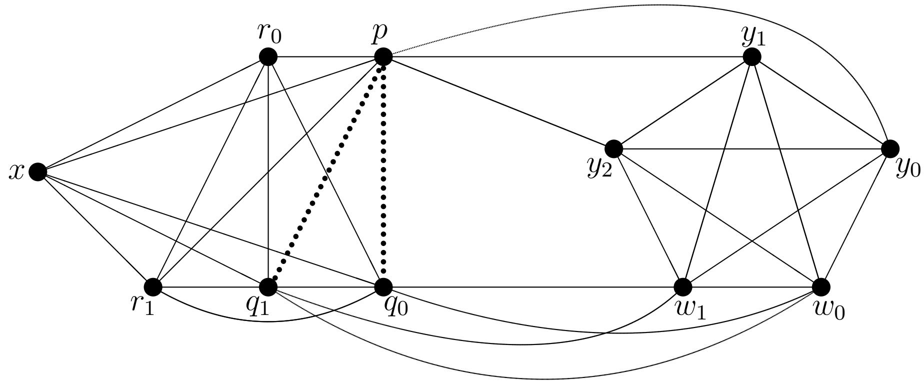
This completes the proof of Theorem (1). Now (2) follows by Lemma 2.2 and (3) follows by Lemma 2.3.

We now address the issue of tightness. The bound specified in (1) was defined so as to hold for K_5 and the graph $G(11, 2)$, which was first exhibited by Sandberg [20] in his 1998 dissertation; this graph, is depicted in Figure 6.7. It is clear that any sum consisting of copies of these two graphs must also satisfy this bound with equality. Assume that $n, r \in \mathbb{N}$, such that

$$\frac{2}{11} \leq \frac{r}{n} \leq \frac{1}{5}. \tag{6.135}$$

We wish to find non-negative integers c and d that yield

$$\nu(c\,\mathrm{K}_5 + dG(11, 2)) = n \tag{6.136}$$



This is the unique $(\mathcal{G}_6\left(6\right), 11, 2)$-mingraph.

Figure 6.7: $G(11, 2)$

and

$$\alpha(c\,\mathrm{K}_5 + dG(11,2)) = r. \tag{6.137}$$

These coefficients are derived by solving the following linear system:

$$\begin{pmatrix} 1 & 2 \\ 5 & 11 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} r \\ n \end{pmatrix}. \tag{6.138}$$

We obtain

$$c = 11r - 2n \quad \text{and} \quad d = n - 5r. \tag{6.139}$$

Clearly both c and d are integers, with

$$c \geq 0 \iff \frac{r}{n} \geq \frac{2}{11} \tag{6.140}$$

and

$$d \geq 0 \iff \frac{r}{n} \leq 15. \tag{6.141}$$

■

We have exhibited the unique size bound for $\mathcal{G}_6(6)$ that holds tightly for independence ratios between $\frac{2}{11}$ and $\frac{1}{5}$, and we have obtained $\frac{7}{40}$ as a lower bound for independence ratios for this class of graphs. It should be noted that Sandberg used a different approach to obtain the tighter independence ratio bound of $\mu(G) \geq \frac{3}{17}$.

6.2 A Size Bound for $\mathcal{G}(6)$

In this section, we establish a size bound that holds for all K_6-free graphs. Generalizations of this bound are discussed in Chapter 8.

Theorem 6.9 *Let $G \in \mathcal{G}(6)$. Then*

1. $\mathrm{e}(G) \geq \dfrac{17}{2}\nu(G) - \dfrac{65}{2}\alpha(G)$.
2. *If* $\Delta(G) \leq 7$, *then* $\mu(G) \geq \dfrac{2}{13}$.
3. $R_7(6, n) \leq \left\lfloor \dfrac{13}{2}(n-1) \right\rfloor + 1$.
4. *If* $r, n \in \mathbb{N}$, *such that* $\dfrac{2}{13} \leq \dfrac{r}{n} \leq \dfrac{1}{6}$ *and* r *and* n *have the same parity, then* $\exists\, G \in \mathcal{G}_7(6)$ *such that* $\nu(G) = n$, $\alpha(G) = r$ *and* $\mathrm{e}(G) = \dfrac{17}{2}\nu(G) - \dfrac{65}{2}\alpha(G)$.

Proof:

We prove (1) by induction on order, with the empty graph providing the base case. We assume that $G \in \mathcal{G}(6)$ such that $\nu(G) > 0$ and (1) holds for members of $\mathcal{G}(6)$ of smaller order. The proof is partitioned into cases based upon $\delta(G)$, and, in each case, we assume that $x \in \mathrm{MinDeg}(G)$.

Case I $(\delta(G) \leq 4)$

The following evaluations and Lemma 3.12 serve to establish the theorem when $\delta(G) = 4$, and an application of Lemma 3.18 yields the result for smaller minimum degrees.

Parameter Evaluation 6.10

1. $\zeta(5,1) = \dfrac{17}{2} \cdot 5 - \dfrac{65}{2} = 10.$
2. $\psi(4,5,1) = \dbinom{5}{2} - \zeta(5,1) = 0.$

Case II $(\delta(G) = 5)$

The proof for this case utilizes the following parameter evaluations.

Parameter Evaluation 6.11

1. $\zeta(6,1) = \left\lceil \dfrac{17}{2} \cdot 6 - \dfrac{65}{2} \right\rceil = 19.$
2. $\psi(5,6,1) = \dbinom{6}{2} - \zeta(6,1) = -4.$
3. $\rho(5,1) = -\psi(5,6,1) = 4.$
4. $\zeta(10,2) = \dfrac{17}{2} \cdot 10 - \dfrac{65}{2} \cdot 2 = 20.$
5. $\psi(5,10,2) = \dfrac{5 \cdot 10}{2} - \zeta(10,2) = 5.$
6. $\eta(5,2) = 10.$

In this case, there must exist $p, q \in \mathrm{N}\,(x)$. such that $p \not\leftrightarrow q$. Evaluation 6.11.3 and Assumption 3.25 yield

$$|\mathrm{N}\,[p,q]| \leq |\mathrm{N}\,[x]| + \rho(4) = 10 = \eta(5,2). \tag{6.142}$$

This case now follows by Lemma 3.19.

Case III $(\delta(G) = 6)$

We begin by presenting the requisite parameter evaluations.

Parameter Evaluation 6.12

1. $\zeta(7,1) = \frac{17}{2} \cdot 7 - \frac{65}{2} = 27.$
2. $\psi(6,7,1) = \binom{7}{2} - \zeta(7,1) = -6.$
3. $\rho(6,3) = 8.$
4. $\rho(6,4) = 9.$
5. $\zeta(11,2) = \left\lceil \frac{17}{2} \cdot 11 - \frac{65}{2} \cdot 2 \right\rceil = 29.$
6. $\psi(6,11,2) = \frac{6 \cdot 11}{2} - \zeta(11,2) = 4.$
7. $\eta(6,2) = 11.$
8. $\zeta(17,3) = \frac{17}{2} \cdot 17 - \frac{65}{2} \cdot 3 = 47.$
9. $\psi(6,17,3) = \frac{6 \cdot 17}{2} - \zeta(17,3) = 4.$
10. $\eta(6,3) = 17.$

Since $|\mathrm{N}[x]| = 7$ and G is K_6-free, $\mathrm{N}(x)$ must contain either three independent vertices or a disjoint pair of independent sets, each of size two. We therefore consider the following pair of subcases.

Case III.A $(\alpha(F(x)) \geq 3)$

Assume that $U \in \operatorname{Indep}(F(x))$ such that $|U| = 3$. It follows from Assumption 3.25 and Evaluation 6.12.3 that

$$|\mathrm{N}[U]| \leq |\mathrm{N}[x]| + \rho(6,3) = 15 < \eta(6,3). \tag{6.143}$$

This case now follows by Lemma 3.21.

Case III.B $(\alpha(F(x)) = 2)$

Assume that $\alpha(F(x)) = 2$ and $\{p_i, q_i\} \in \operatorname{Indep}(F(x)),\ \forall i \in 2.$ such that $\cap_{i\in 2} \{p_i, q_i\} = \emptyset$. It follows from $\alpha(F(x)) = 2$ that $\overline{\mathrm{e}}(\{p_0, q_0, p_1, q_1\}) \leq 4$. Now, by Assumption 3.25 and symmetry, we may assume that

$$|\mathrm{M}[p_0, q_0; x]| \leq \left\lfloor \frac{\rho(6,4)}{2} \right\rfloor = 4. \tag{6.144}$$

This implies

$$|\mathrm{N}[p_0, q_0]| \leq |\mathrm{N}[x]| + |\mathrm{M}[p_0, q_0; x]| = 11 = \eta(6,2). \tag{6.145}$$

Apply Lemma 3.21.

Case IV $(\delta(G) = 7)$

In this case, it suffices to consider independent subsets of $\mathrm{N}[x]$ of size no larger that three.

Parameter Evaluation 6.13

1. $\zeta(8,1) = \left\lceil \frac{17}{2} \cdot 8 - \frac{65}{2} \right\rceil = 36.$
2. $\psi(7,8,1) = \binom{8}{2} - \zeta(8,1) = -8.$
3. $\rho(7,3) = 10.$

4. $\zeta(13,2) = \dfrac{17}{2} \cdot 13 - \dfrac{65}{2} \cdot 2 = 46.$

5. $\psi(7,13,2) = \left\lceil \dfrac{7 \cdot 13}{2} \right\rceil - \zeta(13,2) = 0.$

6. $\eta(7,2) = 13.$

7. $\zeta(19,3) = \dfrac{17}{2} \cdot 19 - \dfrac{65}{2} \cdot 3 = 64.$

8. $\psi(7,19,3) = \left\lceil \dfrac{7 \cdot 19}{2} \right\rceil - \zeta(19,3) = 3.$

9. $\eta(7,3) = 19.$

Now $\mathrm{N}[x]$ contain either three independent vertices, or three pairwise disjoint independent sets of size two. While the first condition is not strong enough to ensure that $F(x)$ is K_6-free, it is still clear that one of these two conditions must hold.

Case IV.A $(\alpha(F(x)) \geq 3)$

Assume that $U \in \mathrm{Indep}(F(x))$ such that $|U| = 3$. Then, by Assumption 3.25,

$$|\mathrm{N}[U]| \leq |\mathrm{N}[x]| + \rho(7,3) = 18 < \eta(7,3). \tag{6.146}$$

Apply Lemma 3.21.

Case IV.B $(\alpha(F(x)) = 2)$

In order for $F(x)$ to be K_6-free, it is necessary that $\mathrm{N}[x]$ contain three pairwise disjoint independent sets, each of size two. Assume that $\{p_i, q_i\} \in \mathrm{Indep}(F(x))$, $i \in 3$ such that $\cap_{i \in 3} \{p_i, q_i\} = \emptyset$. Assumption 3.17 yields

$$|\mathrm{Pend}(\mathrm{N}[x])| \leq -2\psi(7,8,1) - 2 = 14, \tag{6.147}$$

and symmetry allows us to assume that

$$|\mathrm{Pend}\left(\mathrm{N}\left[x\right];p_0,q_0\right)| \leq \left\lfloor \frac{14}{3} \right\rfloor = 4. \tag{6.148}$$

This implies

$$|\mathrm{N}\left[p_0,q_0\right]| \leq 12 < \eta(7,2). \tag{6.149}$$

Apply Lemma 3.21.

Case V $\quad (\delta(G) = 8)$

We may again restrict our attention to independent subsets of $\mathrm{N}\left[x\right]$ of size at most three.

Parameter Evaluation 6.14

1. $\zeta(9,1) = \dfrac{17}{2} \cdot 9 - \dfrac{65}{2} = 44.$
2. $\psi(8,9,1) = \dbinom{9}{2} - \zeta(8,1) = -8.$
3. $\rho(8,3) = 10.$
4. $\zeta(14,2) = \dfrac{17}{2} \cdot 14 - \dfrac{65}{2} \cdot 2 = 54.$
5. $\psi(8,14,2) = \dfrac{8 \cdot 14}{2} - \zeta(14,2) = 2.$
6. $\eta(8,2) = 14.$
7. $\zeta(21,3) = \dfrac{17}{2} \cdot 21 - \dfrac{65}{2} \cdot 3 = 81.$
8. $\psi(8,21,3) = \dfrac{8 \cdot 21}{2} - \zeta(21,3) = 3.$
9. $\eta(8,3) = 21.$

Case V.A $(\alpha(F(x)) \geq 3)$

Assume that $U \in \operatorname{Indep}(F(x))$ such that $|U| = 3$. Apply Assumption 3.25 and Evaluation 6.14.3 to obtain

$$|\mathrm{N}[U]| \leq |\mathrm{N}[x]| + \rho(7,3) = 19 < \eta(8,3). \tag{6.150}$$

Apply Lemma 3.21.

Case V.C $(\alpha(F(x)) = 2)$

Since $\alpha(F(x)) = 2$, $\mathrm{N}[x]$ must contain four pairwise disjoint independent sets, $\{p_i, q_i\}$, $i \in 4$. Now symmetry and Assumption 3.17 allow us to assume that

$$|\operatorname{Pend}(\mathrm{N}[x]; p_0, q_0)| \leq \left\lfloor \frac{-2\psi(8,9,1) - 2}{4} \right\rfloor = 3. \tag{6.151}$$

This implies

$$|\mathrm{N}[p_0, q_0]| \leq 12 < \eta(8,2). \tag{6.152}$$

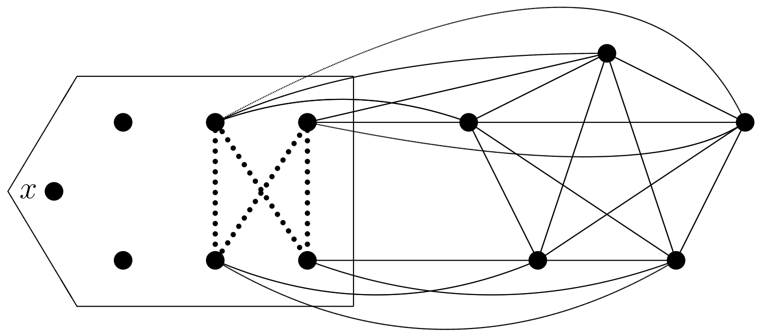
Lemma 3.21 again yields the desired result.

Case VI $(\delta(G) \geq 9)$

Apply Lemma 3.29.

This completes the proof of (1). Now (2) follows from (1) by Lemma 2.2, and (3) follows by Lemma 2.3. We next establish tightness. Consider the graph $G(12,2)$ that is depicted in Figure 6.8. This graph satisfies the following conditions:

$$\nu(G(12,2)) = 12, \quad \mathrm{e}(G(12,2)) = 37, \quad \alpha(G(12,2)) = 2, \quad \Delta(G(12,2)) = 7. \tag{6.153}$$



This is the unique $(\mathcal{G}(6), 12, 2)$-mingraph.

Figure 6.8: $G(12, 2)$

Since the size bound stated in (1) was defined so as to hold with equality for K_5 and $G(12,2)$, and it must also hold with equality for sums of these two graphs. Now we assume that

$$\frac{1}{6} \leq \frac{r}{n} \leq \frac{1}{5}, \tag{6.154}$$

and r and n have the same parity, and then we establish the existence of non-negative integers c and d that yield

$$\nu(c\,\mathrm{K}_5 + d\,G(12,2)) = n \quad \text{and} \quad \alpha(c\,\mathrm{K}_5 + d\,G(12,2)) = r. \tag{6.155}$$

The required coefficients are obtained by solving

$$\begin{pmatrix} 1 & 2 \\ 5 & 12 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} r \\ n \end{pmatrix} \tag{6.156}$$

to obtain

$$c = 6r - n, \quad d = \frac{1}{2}(n - 5r). \tag{6.157}$$

Notice that this calculation yields integer coefficients exactly when r and n have the same parity. ∎

7. K_7-Free Graphs

In this chapter, we establish the size bound

$$\mathrm{e}(G) \geq 14\nu(G) - 69\alpha(G) \tag{7.1}$$

for members of $\mathcal{G}_7(7)$ that additionally satisfy $\delta(G) \leq 6$. We then use this information to obtain an independence ratio bound for such graphs. We begin by introducing notation for the class of graphs under consideration.

Definition 7.1 *Assume that $D \geq 1$. Let $0 \leq d < D$, and assume that $m \geq 2$. Define $\mathcal{G}_{d,D}(m) = \{G \in \mathcal{G}_D(m) \mid \delta(G) \leq d\}$.*

Therefore we are considering graphs that belong to $\mathcal{G}_{6,7}(7)$. We next introduce a variation of the notion of Ramsey-type numbers that includes a bound on minimum degree, and follow with the statement and proof of the associated theorem.

Definition 7.2 *Assume that $D \geq 1$, $0 \leq d < D$ and $m, n \geq 2$. Define $R_{d,D}(m, n) = \min\{r \mid G \in \mathcal{G}_{d,D}(m)$ and $\nu(G) = r$ imply $\alpha(G) \geq n\}$.*

Theorem 7.3

1. *Let $G \in \mathcal{G}_{6,7}(7)$. Then $\mathrm{e}(G) \geq 14\nu(G) - 69\alpha(G)$.*

2. *Let $r, n \in \mathbb{N}$ such that $\frac{2}{13} \leq \frac{r}{n} \leq \frac{1}{6}$. Then there exists $G \in \mathcal{G}_{6,7}(7)$ such that $\mathrm{e}(G) = 14\nu(G) - 69\alpha(G)$.*

3. *Let $G \in \mathcal{G}_{6,7}(7)$. Then $\mu(G) > \frac{21}{138}$.*

4. Let $n \in \mathbb{N}$. *Then* $R_{6,7}(7,n) \leq \left\lfloor \frac{138}{21}(n-1) \right\rfloor + 1$.

5. Let $n \in \mathbb{N}$. *Then*

(a) If $n = 2k$, *then* $R_{6,7}(7,n) \geq 13(k-1) + 7$.

(b) If $n = 2k+1$, *then* $R_{6,7}(7,n) \geq 13k + 1$.

Proof:

We establish (1) by induction on $\nu(G)$. As in the preceding chapters, the empty graph provides the base for the induction, and we assume that $G \in \mathcal{G}_{6,7}(7)$ such that $\nu(G) > 0$ and that the bound holds for members of $\mathcal{G}_{6,7}(7)$ of smaller order. The proof is partitioned into a hierarchy of cases and subcases with $\delta(G)$ providing the coarsest level of partitioning. In each case, we assume that $x \in \mathrm{MinDeg}(G)$.

Case I $(\delta(G) \leq 5.)$

Evaluation 7.4.2 and Lemma 3.12 yield the bound when $\delta(G) = 5$, and then an application of Lemma 3.18 establishes it for smaller values of $\delta(G)$.

Parameter Evaluation 7.4

1. $\zeta(6,1) = 14 \cdot 6 - 69 = 15$.

2. $\psi(5,6,1) = \binom{6}{2} - \zeta(6,1) = 0$.

Case II $(\delta(G) = 6)$

The following evaluations are required for this case.

Parameter Evaluation 7.5

1. $\zeta(7,1) = 14 \cdot 7 - 69 = 29$.

2. $\psi(6,7,1) = \binom{7}{2} - \zeta(7,1) = -8.$

3. $\rho(6,1) = 8.$

4. $\rho(6,3) = 10.$

5. $\rho(6,4) = 11.$

6. $\zeta(12,2) = 14 \cdot 12 - 69 \cdot 2 = 30.$

7. $\psi(6,12,2) = \frac{6 \cdot 12}{2} - \zeta(12,2) = 6.$

8. $\zeta(6,13,2) = 14 \cdot 13 - 69 \cdot 2 = 44.$

9. $\lambda(6,13,2) = 2\zeta(13,2) - 6 \cdot 13 - 1 = 9.$

10. $\zeta(6,14,2) = 14 \cdot 14 - 69 \cdot 2 = 58.$

11. $\lambda(6,14,2) = 2\zeta(14,2) - 6 \cdot 14 - 1 = 31.$

12. $\eta(6,2) = 12.$

13. $\zeta(18,3) = 14 \cdot 18 - 69 \cdot 3 = 45.$

14. $\psi(6,18,3) = \frac{6 \cdot 18}{2} - \zeta(18,3) = 9.$

15. $\zeta(6,19,3) = 14 \cdot 19 - 69 \cdot 3 = 59.$

16. $\lambda(6,19,3) = 2\zeta(19,3) - 6 \cdot 19 - 1 = 3.$

17. $\eta(6,3) = 18.$

18. $\zeta(25,4) = 14 \cdot 25 - 69 \cdot 4 = 74.$

19. $\psi(6,25,4) = \frac{6 \cdot 25}{2} - \zeta(25,4) = 1.$

20. $\eta(6,4) = 25.$

21. $\zeta(32,5) = 14 \cdot 32 - 69 \cdot 5 = 103.$

22. $\lambda(6,32,5) = 2\zeta(32,5) - 6 \cdot 32 - 1 = 13.$

23. $\zeta(38,6) = 14 \cdot 38 - 69 \cdot 6 = 118.$

24. $\lambda(6,38,6) = 2\zeta(38,6) - 6 \cdot 38 - 1 = 7.$

We partition the proof of the current case into three subcases as follows: Since G is K_7-free, we must have $\alpha(F(x)) \geq 2$. Strict inequality yields one subcase, and, when equality holds, we consider subcases based upon whether or not $\mathrm{N}(x)$ contains a disjoint pair of independent sets, each of size two.

Case II.A $\quad (\alpha F(x) \geq 3\ )$

Assume that $U \in \mathrm{Indep}(F(x))$ such that $|U| = 3$. Assumption 3.25 and Evaluation 7.5.4 imply

$$|\mathrm{N}[U]| \leq |\mathrm{N}[x]| + \rho(6,3) = 17. \tag{7.2}$$

Apply Evaluation 7.5.17 and Lemma 3.21.

Case II.B $\quad (\alpha(F(X)) = 2 \text{ and } \{p_0, q_0\}, \{p_1, q_1\} \in \mathrm{Indep}(F(x))\ )$

Now assume that $\alpha(F(x)) = 2$, and $\mathrm{N}(x)$ contains a disjoint pair of independent sets $\{p_0, q_0\}$ and $\{p_1, q_1\}$. It follows from the defining property of this case that

$$\bar{\mathrm{e}}(\langle p_0, q_0, p_1, q_1 \rangle) \leq 4, \tag{7.3}$$

and, by symmetry, we may assume that

$$|\mathrm{Inc}\,(\mathrm{N}\,[x])\, p_0, q_0| \leq |\mathrm{Inc}\,(\mathrm{N}\,[x])\, p_1, q_1|\,. \tag{7.4}$$

Therefore Assumption 3.25 and Evaluation 7.5.5 yield

$$|\mathrm{N}\,[p_0, q_0]| \leq |\mathrm{N}\,[x]| + \left\lfloor \frac{\rho(6,4)}{2} \right\rfloor = 12. \tag{7.5}$$

Now Evaluation 7.5.12 and Lemma 3.21 serve to complete the proof for this case.

Case II.C $\quad (\alpha(F(X)) = 2 \text{ and } F(x) = \mathrm{K}_7 - \mathrm{E}\,(\,\mathrm{K}_{1,k}))$

It only remains to assume that $F(x)$ is a clique from which the edges of a star have been excluded. We utilize the following notation for this case.

Notation 7.6

1. *We let* p *denote the center of the star.*
2. $Q = \{q_i\}^{i \in |Q|}$ *denotes the points.*
3. $\mathrm{M}\,[p; x] = Y = \{y_j\}^{j \in |Y|}$.
4. $\mathrm{M}\,[q_i; x] = Z_i = \left\{z^i_j\right\}^{j \in |Z_i|},\ i \in |Q|$.
5. $i < j \implies |Z_i| \leq |Z_j|$.
6. $Y \cup Z_0 = W = \{w_l\}^{l \in |W|}$.

Degree constraints immediately yield bounds for the number of edges pendant to $\mathrm{N}\,[x]$ at p and Q as follows:

$$\mathrm{d}(u) \geq \mathrm{d}(x), \ \ \forall\, u \in \mathrm{N}\,[x] \quad \text{and} \quad \delta(G) = \Delta(G) - 1 \tag{7.6}$$

imply

$$|Q| \leq |\mathrm{Pend}\,(\mathrm{N}\,[x]\,;p)| \leq |Q| + 1 \tag{7.7}$$

and

$$1 \leq |\mathrm{Pend}\,(\mathrm{N}\,[x]\,;q_i)| \leq 2, \quad \forall\, i \in |Q|\,. \tag{7.8}$$

Next, $\delta(G) = 6$ and $p \notin Q$ imply

$$|Q| \leq 5. \tag{7.9}$$

It follows from (7.7) and (7.8) that

$$|\mathrm{M}\,[p, q_0; x]| \leq |\mathrm{Pend}\,(\mathrm{N}\,[x]\,;p, q_0)| \leq 8, \tag{7.10}$$

and the second inequality can be an equality only if

$$|Q| = 5, \quad |\mathrm{Pend}\,(\mathrm{N}\,[x]\,;p)| = 6 \quad \text{and} \quad |\mathrm{Pend}\,(\mathrm{N}\,[x]\,;q_0)| = 2. \tag{7.11}$$

In this case, upon taking Notation 7.6.5 into consideration, we obtain

$$|\mathrm{Inc}\,(\mathrm{N}\,[x])| \geq 32 > \zeta(6,1), \tag{7.12}$$

and the bound follows by Lemma 3.9. Figure 7.1 illustrates this configuration.

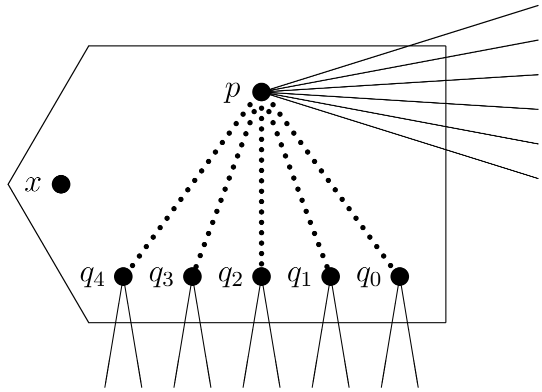
We may now assume that the second inequality in (7.10) is strict. Furthermore, Evaluation 7.5.12 and Assumption 3.22 yield

$$|\mathrm{N}\,[p_0, q_0]| \geq 13. \tag{7.13}$$

Therefore, we may assume that

$$6 \leq |\mathrm{M}\,[p, q_0; x]| \leq 7. \tag{7.14}$$

This inequality yields the next partition into subcases.



This figure illustrates the only configuration of $\mathrm{N}[x]$ for which $|\mathrm{M}[p, q_0; x]| = 8$. There are 32 edges incident with $\mathrm{N}[x]$, but $\zeta(7, 1) = 29$.

Figure 7.1: $|\mathrm{M}[p, q_0; x]| = 8$

Case II.C.1 $(|\mathrm{M}[p, q_0; x]| = 6)$

In this case,

$$|\mathrm{N}[p, q_0]| = 13, \tag{7.15}$$

and the bound follows by Lemma 3.27 and Evaluation 7.5.9 upon showing that

$$\mathrm{DegEx}(\mathrm{N}[p, q_0]) + |\mathrm{Pend}(\mathrm{N}[p, q_0])| \geq 9. \tag{7.16}$$

The following partition of this subcase is based upon whether or not W, as defined in Notation 7.6.6, induces a clique.

Case II.C.1.a $(\langle W \rangle = \mathrm{K}_6)$

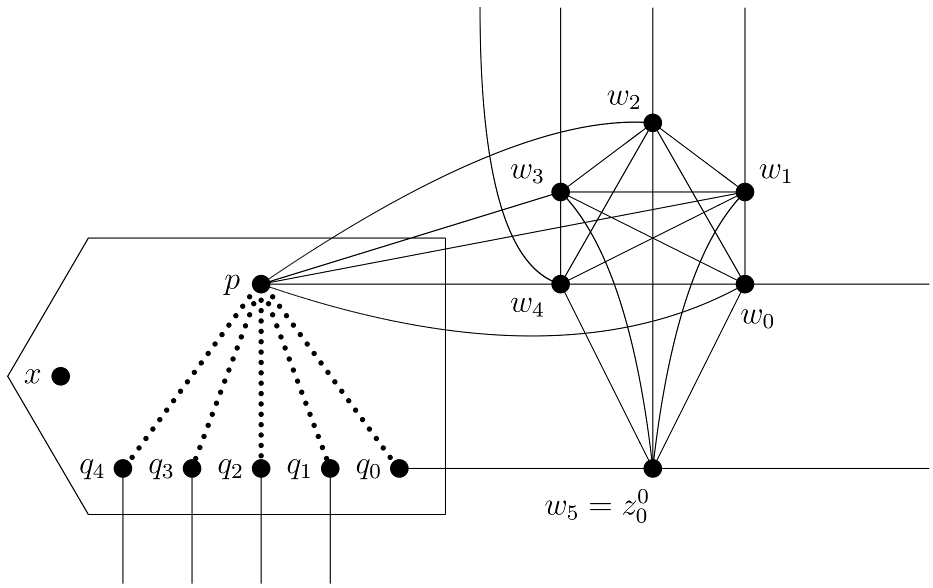
Since G is K_7-free, we may assume that

$$p \not\leftrightarrow z_0^0. \tag{7.17}$$

Now $Y \subseteq \mathrm{N}(z_0^0)$, and $|Y| \geq 4$. Therefore

$$\left|\mathrm{N}\left[p, z_0^0\right]\right| \leq 12 = \eta(6, 2), \tag{7.18}$$

and the theorem follows by Lemma 3.21. Figure 7.2 illustrates this case.



In this example $z_0^0 = w_5$, $Y = \{w_i\}^{i \in 5}$ and $|\mathrm{N}[p, z_0^0]| \leq 11$. It is also possible that $|Y| = 4$, and $|Z_0| = 2$, in which case $|\mathrm{N}[p, z_0^0]| \leq 12$. It is necessary that $W \not\subseteq \mathrm{N}(p)$, for otherwise we would have $\langle \{p\} \cup W \rangle = \mathrm{K}_7$.

Figure 7.2: $|\mathrm{M}[p, q_0; x]| = 6, \quad \langle W \rangle = \mathrm{K}_6$

Case II.C.1.b $\quad (\langle W \rangle \neq \mathrm{K}_6))$

The reader may wish to refer to Figures 7.3 and 7.4 for a pair of illustrations of this case. By symmetry we may assume that

$$w_0 \not\leftrightarrow w_1. \tag{7.19}$$

Notice that, if $w_l \not\leftrightarrow w_k$, then

$$\{x, w_l, w_k\} \in \operatorname{Indep}(G). \tag{7.20}$$

Furthermore,

$$\mathrm{N}[x] \subseteq \mathrm{N}[p, q_0], \tag{7.21}$$

so that Assumption 3.22 and Evaluation 7.5.17 imply

$$|\operatorname{Pend}(\mathrm{N}[p, q_0]; w_l, w_k)| \geq 6. \tag{7.22}$$

If W contains a disjoint pair of independent sets, each of size two, then two applications of (7.22) yield

$$|\operatorname{Pend}(\mathrm{N}[p, q_0]; W)| \geq 12, \tag{7.23}$$

and (7.16) is satisfied. Similarly, if W contains a set of three independent vertices $U = \{u_0, u_1, u_2\}$, then

$$\{x, u_0, u_1, u_2\} \in \operatorname{Indep}(G). \tag{7.24}$$

In this case Assumption 3.22 and Evaluation 7.5.20 yield

$$|\operatorname{Pend}(\mathrm{N}[p, q_0]; U)| \geq 13, \tag{7.25}$$

and we again obtain (7.16). We may therefore assume that W induces K_6 with the edges of a star excluded. We let w_0 denote the center of this star, n the number of points and, by symmetry, we furthermore let

$$\widehat{W} = \{w_l\}_{l=1}^{n} \tag{7.26}$$

denote the points. Now (7.22) yields

$$|\mathrm{Pend}\,(\mathrm{N}\,[p, q_0]\,; w_0, w_l)| \geq 6, \quad 1 \leq l \leq n. \tag{7.27}$$

Let

$$|\mathrm{Pend}\,(\mathrm{N}\,[p, q_0]\,; w_0)| = k. \tag{7.28}$$

Then, by (7.22),

$$\mathrm{Pend}\,(\mathrm{N}\,[p, q_0]\,; w_l) \geq 6 - k, \quad 1 \leq l \leq n. \tag{7.29}$$

Since w_0 has a neighbor in $\{p, q_0\}$, it follows that

$$|\mathrm{N}\,[w_0] \cap W| \leq 7 - (k + 1). \tag{7.30}$$

Upon noting that W contains five vertices distinct from w_0, we obtain

$$n \geq 5 - (7 - (k + 1)) = k - 1. \tag{7.31}$$

Now (7.28), (7.29) and (7.31) yield

$$|\mathrm{Pend}\,(\mathrm{N}\,[p, q_0]\,; W)| \geq k + (k - 1)(6 - k) = -k^2 + 8k - 6. \tag{7.32}$$

If $1 \leq l \leq n$, then w_l has a neighbor in $\mathrm{N}\,[x]$, and, since it is a point of the star, it has four neighbors in W. Therefore

$$k \geq 4. \tag{7.33}$$

We obtain 10, 9 and 6 as valuations for (7.32) as k increases from 4 to 6. Since inequality (7.16) is satisfied in the first two cases, we assume that

$$k = 6. \tag{7.34}$$

It immediately follows that

$$\operatorname{DegEx}(w_0) = 1, \quad n = 5 \quad \text{and} \quad \left\langle \widehat{W} \right\rangle = \mathrm{K}_5. \tag{7.35}$$

We consider the structure of of $\langle W \rangle$. If $|Z_0| = 2$, then Notation 7.6.5 implies

$$\operatorname{DegEx}(Q) \geq |Q| \geq 3. \tag{7.36}$$

Now (7.34) and (7.36) imply that (7.16) is satisfied, so we assume that

$$Z_0 = \left\{ z_0^0 \right\}, \tag{7.37}$$

and consequently,

$$|Y| \geq 5. \tag{7.38}$$

We now consider subcses based upon whether or not $z_0^0 = w_0$.

Case II.C.1.b.i $\quad (z_0^0 \neq w_0)$

Figure 7.3 illustrates this configuration. We have

$$z_0^0 \in \widehat{W}, \tag{7.39}$$

and therefore

$$\left| \mathrm{N}\left(z_0^0\right) \cap Y \right| \geq 4. \tag{7.40}$$

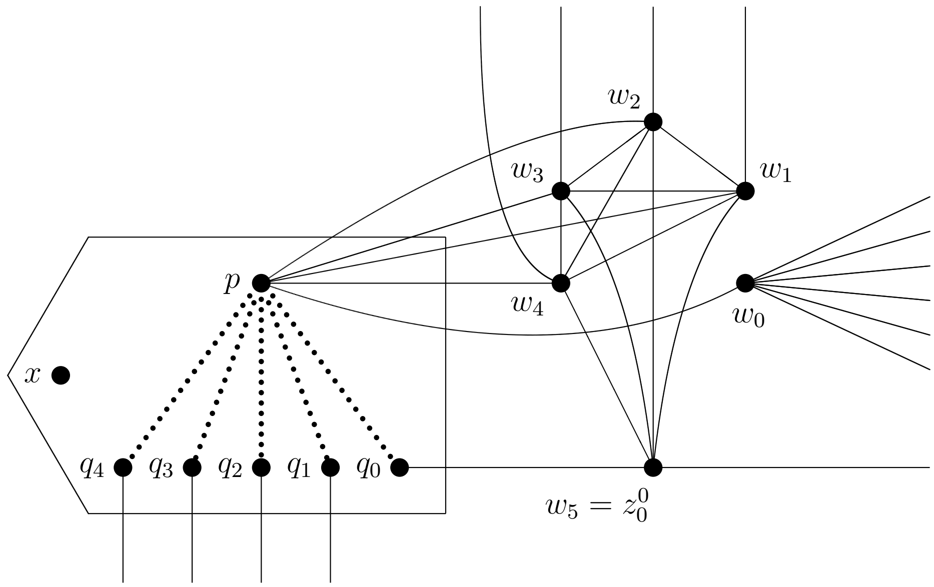
This implies

$$\left| \mathrm{N}\left[p, z_0^0\right] \right| \leq 12 = \eta(6, 2), \tag{7.41}$$

and now the theorem follows by Lemma 3.21 and Evaluation 7.5.12.

Case II.C.1.b.ii $\quad (z_0^0 = w_0)$

Figure 7.4 depicts one instance of the following argument.



$z_0^0 \neq w_0$. By symmetry, $z_0^0 = w_5$ and $Y = \{w_i\}^{i\in 5}$. Now $z_0^0 \leftrightarrow w_l$, $1 \leq l \leq 4$, and $|\mathrm{N}\,[p, z_0^0]| = 12$.

Figure 7.3: $\langle W\rangle \neq \mathrm{K}_6, \quad w_0 \neq z_0^0$

First notice that (7.7) and (7.38) imply

$$|Q| \geq 4. \tag{7.42}$$

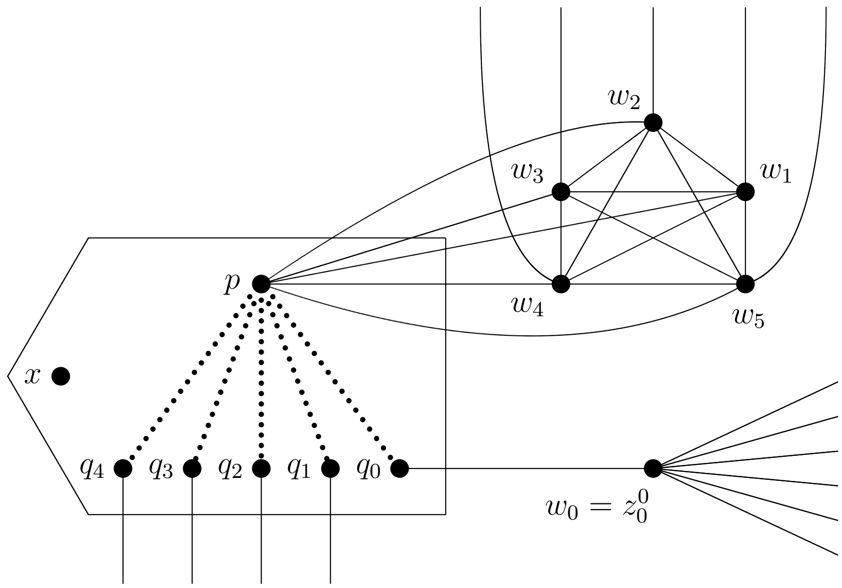
It follows from (7.34) that all of the neighbors of z_0^0 have been accounted for. Therefore

$$z_0^0 \not\leftrightarrow p, \tag{7.43}$$

and

$$z_0^0 \not\leftrightarrow q_i, \quad 1 \leq i \leq 3. \tag{7.44}$$

Notice that (7.43) implies that equality must hold in (7.38). Let $i \in \{1.2.3\}$.



$z_0^0 = w_0$ and it has no neighbors in $\{q_i\}_{i=1}^3$. Now the six pendant edges incident with z_0^0 along with three pendant edges incident with Q yield $\lambda(6,13,2)$ pendant edges.

Figure 7.4: $\langle W \rangle \neq \mathrm{K}_6, \quad w_0 = z_0^0$

We may assume that

$$Z_i \not\subseteq Y, \quad \forall\, i \in |P|, \tag{7.45}$$

for otherwise we would have

$$|\mathrm{N}\,[p, q_i]| \leq \eta(6,2). \tag{7.46}$$

Therefore

$$|\mathrm{Pend}\,(\mathrm{N}\,[p, q_0]\,; Q)| \geq 3. \tag{7.47}$$

Upon taking (7.34) into consideration, we see that inequality (7.16) is again satisfied.

Case II.C.2 $\quad (|\mathrm{M}\,[p, q_0; x]\,| = 7)$

We continue to use Notation 7.6. Now, because

$$|\mathrm{N}\,[p, q_0]| = 14, \tag{7.48}$$

the theorem follows if it can be shown that

$$|\mathrm{Inc}\,(\mathrm{N}\,[p, q_0])| \geq \zeta(14, 2) = 58. \tag{7.49}$$

This condition follows upon establishing

$$\mathrm{DegEx}\,(\mathrm{N}\,[x]) + |\mathrm{Pend}\,(\mathrm{N}\,[x])| \geq \lambda(6, 14, 2) = 31, \tag{7.50}$$

The proof of (7.50) requires a detailed investigation of possible structures for $F(p, q_0)$. Since $\mathrm{d}(x) = 6$ and $p \notin Q$, we have

$$|Q| \leq 5, \tag{7.51}$$

and therefore,

$$|\mathrm{M}\,[p; x]| \leq |\mathrm{Pend}\,(\mathrm{N}\,[x]\,; p)| \leq 6. \tag{7.52}$$

If $|z_0^0| = 2$, then Notation 7.6.5 requires that

$$|Z_i| = 2, \quad \forall\ i \in |Q|\,. \tag{7.53}$$

This forces

$$|\mathrm{Inc}\,(\mathrm{N}\,[x])| \geq 29 = \zeta(6, 1), \tag{7.54}$$

and the result follows by Lemma 3.9. Therefore we may assume that

$$|Z_0| = 1. \tag{7.55}$$

This necessitates that equality hold in (7.51) and across (7.52). Given these conditions, in order to attain $|\mathrm{Inc}\,(\mathrm{N}\,[x])| < \zeta(6,1)$, it is necessary that

$$|Z_i| = 1, \quad \forall\, i \in 4. \tag{7.56}$$

Clearly, either $\mathrm{d}(y_j) = 7, \forall\, j \in 6$ or some y_i, say y_0, has degree six. This observation yields the next partition into subcases. If $\mathrm{d}(y_0) = 6$, then

$$\mathrm{d}(x) = 6 \quad \text{and} \quad p \in \mathrm{N}\,(x) \cap \mathrm{N}\,(y_0) \tag{7.57}$$

imply

$$|\mathrm{N}\,[x, y_0]| \leq 13, \tag{7.58}$$

and we exploit the small value of $\lambda(6, 13, 2)$. If $\mathrm{d}(y_j) = 7,\ \forall\, j \in 6$, then we are forced to deal with closed neighborhoods of size 14. In this case, the relatively large value specified in (7.50) becomes an issue, and we take advantage of the fact that $\mathrm{DegEx}\,(Y \cup \{p\}) = 7$. Figures 7.5 and 7.6 illustrate the case $\mathrm{d}(y_0) = 6$ and Figure 7.8 illustrates the situation where $\mathrm{d}(y_j) = 7,\ \forall\, j \in 6$. These figures also serve to clarify Notation 7.7.

Case II.C.2.a $\quad (\,\mathrm{d}(y_0) = 6)$

Consider the structure of $\mathrm{N}\,[x, y_0]$. First $\mathrm{d}(x) = \mathrm{d}(y_0) = 6$ and $p \leftrightarrow x, y_0$ imply

$$|\mathrm{N}\,[x, y_0]| \leq 13, \tag{7.59}$$

while Assumption 3.22 and Evaluation 7.5.12 imply that equality must hold. Therefore the bound follows by Evaluation 7.5.9 and Lemma 3.27 upon showing that

$$\mathrm{DegEx}\,(\mathrm{N}\,[x, y_0]) + |\mathrm{Pend}\,(\mathrm{N}\,[x, y_0])| \geq 9. \tag{7.60}$$

To this end, notice that, if $\{u, v, w\} \subseteq \mathrm{N}[x, y_0]$ such that $\{u, v, w\} \in \mathrm{Indep}(G)$, then Assumption 3.22 and Evaluation 7.5.17 imply

$$|\mathrm{Pend}(\mathrm{N}[x, y_0]; u, v, w)| \geq 6. \tag{7.61}$$

Figures 7.5 and 7.6 illustrate the two subcases of the current case.

We begin by introducing notation for members of $\mathrm{N}(y_0) \setminus p$.

Notation 7.7

1. $R = \{r_l\}^{l \in 5} = \mathrm{N}(y_0) \setminus \{p\}$.

2. $|R \setminus \mathrm{N}(p)| = n$.

3. $\widehat{R} = \{r_l \in R \mid r_l \not\leftrightarrow p\} = \{r_l\}^{l \in n}$.

Notice that (3) asserts that the members of R that are not neighbors of p are assigned the smallest indices.

By construction,

$$p \leftrightarrow x, y_0 \tag{7.62}$$

and

$$\mathrm{N}(p) \cap Q = \emptyset. \tag{7.63}$$

It remains to account for five additional edges incident with p. Each such edge must be either incident with R, or pendant to $\mathrm{N}[x, y_0]$. Since $|R| = 5$, it follows that

$$|\mathrm{Pend}(\mathrm{N}[x, y_0]; p)| = \left|\widehat{R}\right|. \tag{7.64}$$

It is clear that R either does or does not induce a clique. We next consider these two subcases.

Case II.C.2.a.i $(\langle R\rangle = \mathrm{K}_5)$

Figure 7.5 illustrates this case. First notice that

$$\widehat{R} \neq \emptyset, \tag{7.65}$$

for otherwise we would have $\langle R \cup \{p, y_0\}\rangle = \mathrm{K}_7$. Now Notation 7.7.3 yields

$$r_0 \not\leftrightarrow p. \tag{7.66}$$

Assumption 3.22 and Evaluation 7.5.12 imply

$$|\mathrm{N}\,[p, r_0]| \geq 13, \tag{7.67}$$

and this is possible only if

$$|\mathrm{N}\,(p) \cap \mathrm{N}\,(r_0)| \leq 3. \tag{7.68}$$

Since $y_0 \leftrightarrow p, r_0$, and $\langle R\rangle = \mathrm{K}_5$, it is necessary that

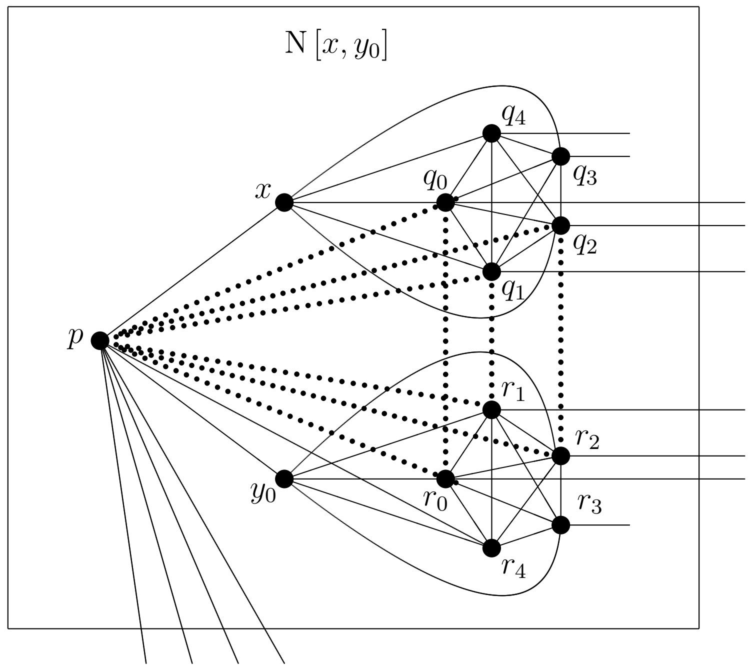
$$\left|\widehat{R}\right| \geq 3. \tag{7.69}$$

Each member of R is adjacent to y_0, and has four neighbors in R. Therefore $\Delta(G) = 7$ implies

$$|\mathrm{N}\,(r_l) \cap Q| \leq 2, \quad \forall\, l \in 3. \tag{7.70}$$

Now r_0 must have three non-neighbors in Q, so we match it with one of them. Next r_1 has at least two non-neighbors in Q that have not been matched to r_0 and we match it with one of them. Finally, we match r_2 with an unmatched non-neighbor. By symmetry, we may assume that

$$r_l \not\leftrightarrow q_l, \quad \forall\, l \in 3. \tag{7.71}$$



$|\mathrm{Pend}(\mathrm{N}[x, y_0]; p)| = 4$, and there are two additional pendant edges at each pair r_l, q_l, $l \in 3$, yielding $\mathrm{Pend}(\mathrm{N}[x, y_0]) \geq 10$. the partial edges at q_3, q_4 and r_3 are included in order to satisfy the minimum degree constraint, but it is immaterial whether or not they are pendant to $\mathrm{N}[x, y_0]$.

Figure 7.5: $\langle R \rangle = \mathrm{K}_5$

Given this matching, the definition of $\widehat{R}$ and (7.63), we obtain

$$\{p, r_l, q_l\} \in \mathrm{Indep}(G), \quad \forall\, l \in 3. \tag{7.72}$$

Inequality (7.61) therefore implies

$$|\mathrm{Pend}(\mathrm{N}[x, y_0]; p, r_l, q_l)| \geq 6, \quad \forall\, l \in 3. \tag{7.73}$$

Let

$$|\mathrm{Pend}(\mathrm{N}[x, y_0]; p)| = k. \tag{7.74}$$

We have

$$3 \leq k \leq 5, \tag{7.75}$$

with equations (7.64) and (7.69) yielding the first inequality, and the fact that $p \leftrightarrow x, y_0$ yielding the second. It follows from (7.73) and (7.74) that

$$|\mathrm{Pend}\,(\mathrm{N}\,[x, y_0])| \geq k + 3(6 - k) = 18 - 2k. \tag{7.76}$$

We obtain 12, 10 and 8 for valuations of (7.76) as k assumes the values 3, 4 and 5, respectively. Inequality (7.16) is clearly satisfied in the first two cases; it is satisfied in the third because $\mathrm{d}(p) = 7$.

Case II.C.2.a.ii $(\langle R \rangle \neq \mathrm{K}_5)$

Now we are concerned with the problem of finding independent sets of the form $\{x, r_l, r_k\}$. Given such a set, since all neighbors of x reside within $\mathrm{N}\,[x, y_0]$, it follows from (7.61) that

$$|\mathrm{Pend}\,(\mathrm{N}\,[x, y_0]\,; r_l, r_k)| \geq 6. \tag{7.77}$$

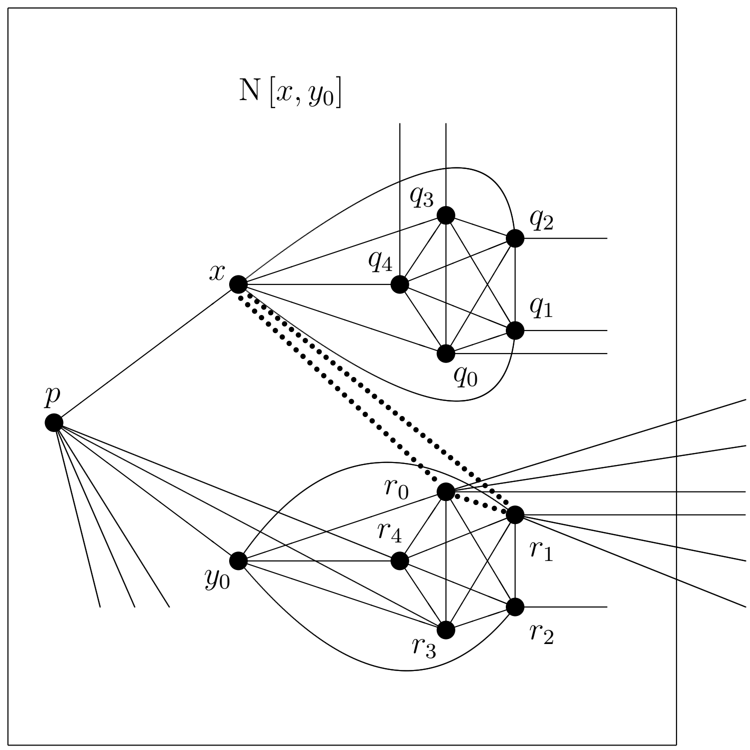
As with W, in Case II.C.1.b, we may assume that R induces a clique (In this case K_5), from which the edges of a star have been excluded. We let r_0 denote the center of the star and we let

$$R^* = \{r_l\}_{l=1}^{n} \tag{7.78}$$

denote the points. Figure 7.6 illustrates this configuration.

Clearly, if $r_l \in R^*$, then

$$|\mathrm{N}\,(r_l) \cap R| = 3. \tag{7.79}$$



$\{x, r_0, r_1\} \in \mathrm{Indep}\,(G)$. This necessitates $\mathrm{Pend}\,(\mathrm{N}\,[x, y_0]\,; r_0, r_1) \geq 6$. These six edges are represented by the lines extending leftwards out of the box. $|R^*| = 1$ and $|\mathrm{Pend}\,(\mathrm{N}\,[x, y_0]\,; r_l)| = 3$, $l \in 2$ requires $\mathrm{d}(r_l) = 7$, $l \in 2$. In Addition, $\mathrm{d}(p) = 7$. Therefore (7.60) is satisfied.

Figure 7.6: $\langle R \rangle \neq \mathrm{K}_5$

Each member of R is also adjacent to y_0, so that

$$|\mathrm{Pend}\,(\mathrm{N}\,[x, y_0]\,; r_l)| \leq 3, \tag{7.80}$$

with equality possible only if $\mathrm{d}(r_l) = 7$. Let

$$|\mathrm{Pend}\,(\mathrm{N}\,[x, y_0]\,; r_0)| = k. \tag{7.81}$$

Then

$$3 \leq k \leq 6. \tag{7.82}$$

with (7.61) and (7.80) yielding the lower bound. We next obtain a lower bound for $|R^*|$. Since $r_0 \leftrightarrow y_0$, we have

$$\mathrm{d}_R(r_0) \leq 7 - (k+1), \tag{7.83}$$

and therefore

$$|R^*| \geq 4 - (7 - (k+1)) = k - 2, \tag{7.84}$$

with equality possible only if $\mathrm{d}(r_0) = 7$. In this event we have

$$\mathrm{DegEx}\,(r_0) = 1. \tag{7.85}$$

Next, if $r_l \in R^*$, then (7.61) implies

$$|\mathrm{Pend}\,(\mathrm{N}\,[x, y_0]\,; r_l)| \geq 6 - k. \tag{7.86}$$

If $\mathrm{d}(r_0) = 6$, then $k < 6$, and it follows that inequality holds in (7.84). Now (7.77), (7.28) and inequality in (7.84) yield

$$|\mathrm{Pend}\,(\mathrm{N}\,[x, y_0]\,; R)| \geq k + (k-1)(6-k) \tag{7.87}$$

$$= -k^2 + 8k - 6. \tag{7.88}$$

This expression assumes the values 9, 10 and 9 as k varies from 3 to 5, and (7.60) is satisfied in each case.

If $\mathrm{d}(r_0) = 7$, then

$$|\mathrm{Pend}\,(\mathrm{N}\,[x, y_0]\,; R)| \geq k + (k-2)(6-k) \tag{7.89}$$

$$= -k^2 + 9k - 12, \tag{7.90}$$

and we obtain 6, 8, 8 and 6 for lower bounds as k ranges from 3 to 6. Since

$$\mathrm{DegEx}\,(p, r_0) = 2, \tag{7.91}$$

the bound follows if $k \in \{4, 5\}$, and one additional vertex of degree seven or one additional pendant edge suffices to establish it if $k \in \{3, 6\}$. Let $k = 3$. In this case, the bound follows unless

$$|R^*| = 1, \tag{7.92}$$

so assume that this is the case. Now we have

$$|\mathrm{Pend}\,(\mathrm{N}\,[x, y_0]\,; r_1)| = 3. \tag{7.93}$$

This is possible only if

$$\mathrm{d}(r_1) = 7, \tag{7.94}$$

and we obtain an additional vertex of maximum degree. If $k = 6$, then we must have

$$p \not\leftrightarrow r_0. \tag{7.95}$$

This implies

$$|\mathrm{Pend}\,(\mathrm{N}\,[x, y_0]\,; p)| > 0, \tag{7.96}$$

and we have found an additional pendant edge.

Case II.C.2.b $\quad (\,\mathrm{d}(y_i) = 7, \quad \forall\, i \in 6)$

Now the theorem follows if

$$\mathrm{DegEx}\left(\mathrm{N}\left[p, q_0\right]\right) + \left|\mathrm{Pend}\left(\mathrm{N}\left[p.q_0\right]\right)\right| \geq \lambda(6, 14, 2) = 31. \tag{7.97}$$

The large degree excess for this case plays an important role in the following proof. In particular, notice that

$$\mathrm{DegEx}\left(Y \cup \{p\}\right) = 7. \tag{7.98}$$

We begin by showing that the bound follows easily if $\exists\, j \in 5$ such that either

$$\mathrm{d}(z_0^j) = 6, \tag{7.99}$$

or

$$\left|\mathrm{N}\left(z_0^j\right) \cap Q\right| \geq 2. \tag{7.100}$$

Figure 7.7 illustrates this argument.

If (7.99) and (7.100) both hold, then

$$\left|\mathrm{N}\left[x, z_0^j\right]\right| \leq 12 = \eta(6, 2), \tag{7.101}$$

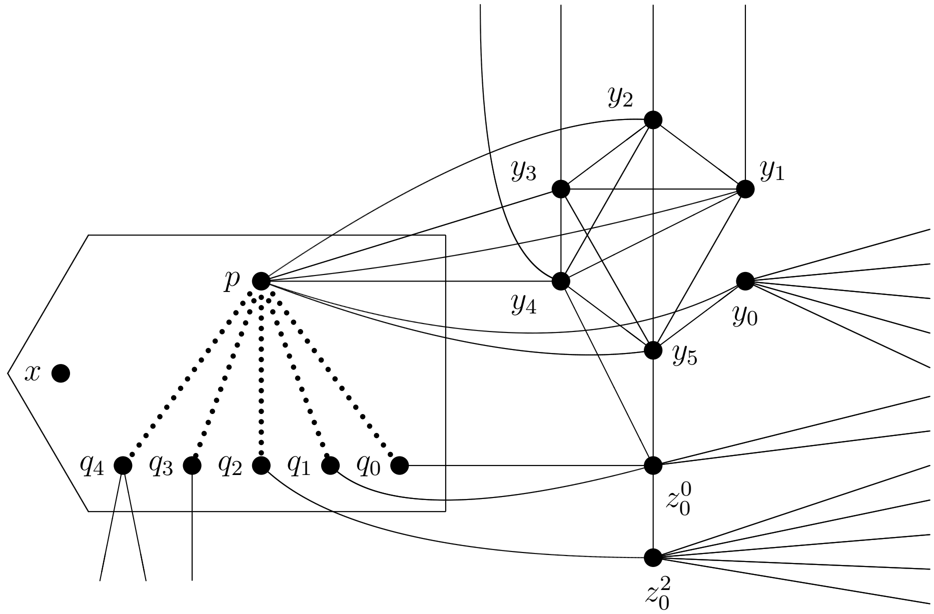
and the result follows by Lemma 3.21. Therefore we may assume that exactly one of (7.99) or (7.100) holds. In either case $\mathrm{d}(x) = 6$ and $Q \subseteq \mathrm{N}\left[x\right]$ imply

$$\left|\mathrm{N}\left[x, z_0^j\right]\right| \leq 13, \tag{7.102}$$

and Evaluation 7.5.9 and Assumption 3.28 imply that equality must hold.

Let $i \in 6$. Then either

$$py_i \in \mathrm{Pend}\left(\mathrm{N}\left[x, z_0^j\right]; p\right), \tag{7.103}$$



In this example $\mathrm{d}(z_0^0) = 7$ and $|\mathrm{N}\,(z_0^0) \cap Q| = 2$, so that $|\mathrm{N}\,[x, z_0^0]| = 13$. $d(p) = 7$, and each y_1 is either the endpoint of an edge pendant to $\mathrm{N}\,[x, z_0^0]$, or it contributes one to the degree excess. Pendant edges at q_1 and q_2 would suffice to establish the theorem, but $z_0^2 \leftrightarrow z_0^0$ and we obtain $|\mathrm{N}\,[q_2, z_0^0]| \leq 12 = \eta(6, 2)$.

Figure 7.7: $|\mathrm{N}\,[q_2, z_0^0]| \leq 12$

or

$$y_i \leftrightarrow z_0^j. \tag{7.104}$$

Therefore

$$\mathrm{DegEx}\left(Y \cap \mathrm{N}\left(x, z_0^j\right)\right) + \left|\mathrm{Pend}\left(\mathrm{N}\left[x, z_0^j\right]; p\right)\right| \geq 6. \tag{7.105}$$

It is also clear that

$$\mathrm{d}(p) = 7. \tag{7.106}$$

In order to ensure that inequality (7.101) does not hold, we assume that

$$\left|\mathrm{N}\left[z_0^j\right] \cap Q\right| \leq 2. \tag{7.107}$$

with equality possible only if $\mathrm{d}(z_0^j) = 7$. Therefore, there exist $k, l \in 4$ such that

$$q_l, q_k \not\leftrightarrow z_0^j. \tag{7.108}$$

Notice that $k, l \neq 4$ implies

$$\mathrm{d}(q_l),\ \mathrm{d}(q_k) = 6. \tag{7.109}$$

Now assume that

$$q_l z_0^l, q_k z_0^k \in \mathrm{Pend}\left(\mathrm{N}\left[x, z_0^j\right]\right). \tag{7.110}$$

Upon taking (7.105) and (7.106) into account, we obtain

$$\mathrm{DegEx}\left(\mathrm{N}\left[x, z_0^j\right]\right) + \left|\mathrm{Pend}\left(\mathrm{N}\left[x, z_0^j\right]\right)\right| \geq 9 = \lambda(6, 13, 2), \tag{7.111}$$

in contradiction to Assumption 3.28. Therefore, by symmetry, we assume that

$$z_0^l \leftrightarrow z_0^j. \tag{7.112}$$

By construction,

$$z_0^l \leftrightarrow q_l. \tag{7.113}$$

Therefore, if $\mathrm{d}(z_0^j) = 6$, then

$$\left|\mathrm{N}\left[q_l, z_0^j\right]\right| \leq 12 = \eta(6, 2), \tag{7.114}$$

and the result again follow, while, if $\mathrm{d}(z_0^j) = 7$, then z_0^j and q_l have an additional common neighbor in Q so that inequality (7.114) still holds. The only remaining possibility is specified by the following:

Assumption 7.8 *Let $j \in 5$. Then*

1. $\mathrm{d}(z_0^j) = 7$.

2. $\left|\mathrm{N}\left[z_0^j\right] \cap Q\right| = 1$.

It follows immediately that

$$\mathrm{DegEx}\left(z_0^j\right) = 1, \quad \forall\, j \in 5, \tag{7.115}$$

and

$$|\mathrm{Pend}\left(\mathrm{N}\left[p, q_0\right]; Q\right)| \geq 4. \tag{7.116}$$

Since $\zeta(7,1) = 29$, we have $\mathrm{d}(q_4) = 7$, and hence, $|Z_4| = 2$. In this case, the possibility that

$$Z_4 \cap Y \neq \emptyset \tag{7.117}$$

remains. We next show that the bound follows if this condition holds.

Assume that

$$Z_4 = \left\{z_0^4, z_1^4\right\}, \quad \text{and} \quad z_0^4 \in Y. \tag{7.118}$$

Since $\mathrm{d}(x) = 6$ and $p, q_4 \in \mathrm{N}\left(x\right) \cap \mathrm{N}\left(z_0^4\right)$, we have

$$\left|\mathrm{N}\left[x, z_0^4\right]\right| \leq 13, \tag{7.119}$$

while Evaluation 7.5.12 and Assumption 3.22 assert that equality must hold.

Each member of Y other than z_0^4 must either be a neighbor of z_0^4 or the endpoint of an edge pendant to $\mathrm{N}\left[x, z_0^4\right]$ at p. Therefore

$$\mathrm{DegEx}\left(Y \cap \mathrm{N}\left[x, z_0^4\right]\right) + \left|\mathrm{Pend}\left(\mathrm{N}\left[x, z_0^4\right]; p\right)\right| \geq 5. \tag{7.120}$$

If $i \in 4$, then either

$$z_0^i \leftrightarrow z_0^4, \tag{7.121}$$

or

$$q_i z_0^i \in \operatorname{Pend}\left(\mathrm{N}\left[x, z_0^4\right]; Q\right). \tag{7.122}$$

Therefore, it follows that

$$\operatorname{DegEx}\left(\left\{z_0^i\right\}^{i\in 4} \cap \mathrm{N}\left[x, z_0^4\right]\right) + \left|\operatorname{Pend}\left(\mathrm{N}\left[x, z_0^4\right]; Q\right)\right| \geq 4. \tag{7.123}$$

Now combine (7.120) and (7.123) to obtain

$$\operatorname{DegEx}\left(\mathrm{N}\left[x, z_0^4\right]\right) + \left|\operatorname{Pend}\left(\mathrm{N}\left[x, z_0^4\right]\right)\right| \geq 9 = \lambda(6, 13, 2). \tag{7.124}$$

Now, should it occur that $\mathrm{d}(q_4) = 7$, we may assume that,

$$q_4 z_k^4 \in \operatorname{Pend}\left(\mathrm{N}\left[p, q_0\right]\right), \quad \forall\, k \in 2. \tag{7.125}$$

Notation 7.6.6 yields

$$\mathrm{M}\left[p, q_0; x\right] = Y \cup Z_0 = W = \left\{w_l\right\}^{l\in 7}. \tag{7.126}$$

We now consider the structure of $\langle W \rangle$. If it can be shown that

$$\operatorname{DegEx}\left(\mathrm{N}\left[p, q_0\right]\right) + \left|\operatorname{Pend}\left(\mathrm{N}\left[p, q_0\right]\right)\right| \geq 31, \tag{7.127}$$

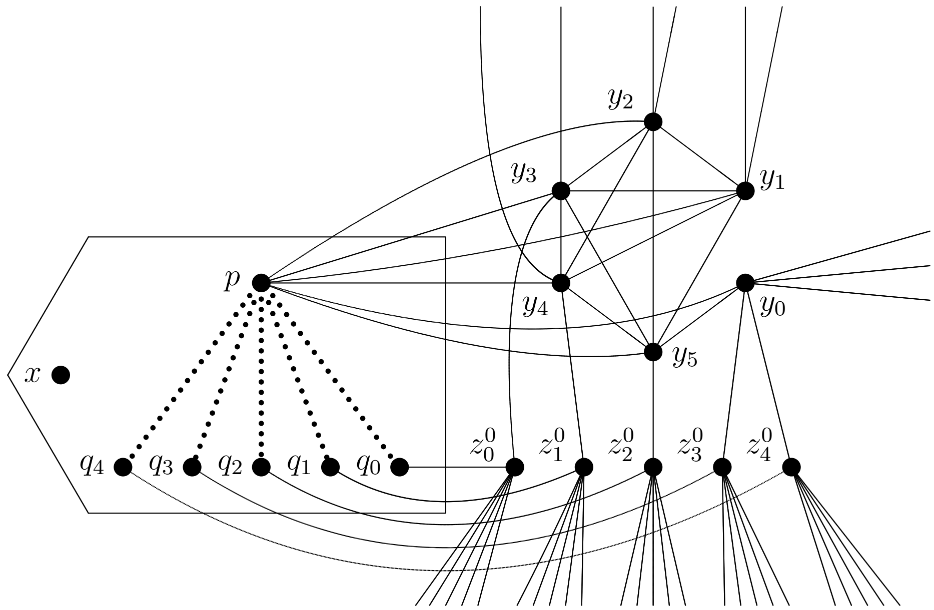
then the bound follows by Evaluation 7.5.11 and Lemma 3.27. It has already been shown that

$$\operatorname{DegEx}\left(Y \cup \left\{p, z_0^0\right\}\right) = 8 \quad \text{and} \quad \left|\operatorname{Pend}\left(\mathrm{N}\left[p, q_0\right]; Q\right)\right| \geq 4. \tag{7.128}$$

Therefore the bound follows upon showing that

$$\left|\operatorname{Pend}\left(\mathrm{N}\left[p, q_0\right]; W\right)\right| \geq 19. \tag{7.129}$$

The reader is referred to Figure 7.8 for the following argument.



$x, y_5 \not\leftrightarrow z_0^0, z_0^1, z_0^3, z_0^4$ and (7.160) imply $U = \{x, y_5, z_0^0, z_0^1, z_0^3, z_0^4\} \in \text{Indep}(G)$. $\text{d}(x) = 6$, $\text{d}(y_5) = 7$ and $\text{N}(x) \cap \text{N}(y_5) = \{p\}$ imply $|\text{N}[x, y_5]| = 14$. Each z_0^i, $i \in \{0, 1, 3, 4\}$ has a pair of neighbors that have already been counted, one each in Q and Y. Therefore $|\text{N}[U]| \leq 38$. Finally, $\{p\} \cup Y \subseteq \text{N}[x, y_5]$ implies $\text{DegEx}(\text{N}[x, y_5]) \geq 7 = \lambda(6, 38, 6)$.

Figure 7.8: $|\text{N}[U]| = 38$

Since each member of W has degree equal to seven, and a unique neighbor in $\text{N}[x]$, it remains to account for an increase of 42 in the degree sum of W. Each edge internal to W contributes two to this value, and each pendant edge contributes one. Therefore (7.129) is satisfied unless

$$\text{e}(\langle W \rangle) \geq 12, \tag{7.130}$$

so assume that this inequality holds. If $\text{d}(q_4) = 7$, then (7.125) establishes the existence of an additional edge pendant to $\text{N}[p, q_0]$, and we may assume that inequality (7.130) is strict in that case.

If there exist $w_k, w_l \in W$ such that $w_k \not\leftrightarrow w_l$, then

$$\{x, w_k, w_l\} \in \mathrm{Indep}\,(G)\,. \tag{7.131}$$

Since

$$\mathrm{DegEx}\,(p, w_k, w_l) = 3 = \lambda(6, 19, 3), \tag{7.132}$$

it follows by Assumption 3.28 that

$$|\mathrm{N}\,[x, w_k, w_l]| \geq 20, \tag{7.133}$$

and now $|\mathrm{N}\,[p, q_0]| = 14$, implies

$$|\mathrm{Pend}\,(\mathrm{N}\,[p, q_0]\,; w_k, w_l)| \geq 6. \tag{7.134}$$

Next, since G is K_7-free, we may assume that

$$y_0 \not\leftrightarrow y_1, \tag{7.135}$$

and, by symmetry,

$$|\mathrm{Pend}\,(\mathrm{N}\,[p, q_0]\,; y_0)| \geq |\mathrm{Pend}\,(\mathrm{N}\,[p, q_0]\,; y_j)|\,, \quad 1 \leq j \leq 5. \tag{7.136}$$

Now assume that

$$|\mathrm{N}\,(y_0) \cap \mathrm{N}\,(y_1)| \geq 3. \tag{7.137}$$

Then, because $\mathrm{d}(x) = 6$ and $p \in \mathrm{N}\,[x] \cap \mathrm{N}\,[y_0, y_1]$, we obtain

$$|\mathrm{N}\,[x, y_0, y_1]| \leq 19. \tag{7.138}$$

But then

$$\mathrm{DegEx}\,(p, y_0, y_1) = 3 \tag{7.139}$$

yields a contradiction to Assumption 3.28. Furthermore, since p is a common neighbor of y_0 and y_1, it is necessary that y_0 and y_1 have no more than one common neighbor in Y, and this implies

$$\mathrm{e}\left(\langle Y\rangle\right) \leq 11. \tag{7.140}$$

Any additional edges internal to W must have z_0^0 as an endpoint, and, in order to satisfy (7.130), there must be at least one such edge. First assume that

$$\left|\mathrm{N}\left(z_0^0\right) \cap Y\right| \geq 2. \tag{7.141}$$

This implies

$$\left|\mathrm{N}\left[p, z_0^0\right]\right| \leq 14. \tag{7.142}$$

Let $j \in \{1,2,3\}$. It follows from $\mathrm{d}(q_j) = 6$, $x \in \mathrm{N}[p] \cap \mathrm{N}[q_j]$ and $q_0 \in \mathrm{N}[z_0^0] \cap \mathrm{N}[q_j]$ that

$$\left|\mathrm{N}\left[z_0^0, p, q_j\right]\right| \leq 19. \tag{7.143}$$

Now

$$\mathrm{DegEx}\left(\mathrm{N}\left[p\right]\right) \geq 3 = \lambda(6,19,3) \tag{7.144}$$

yields a contradiction to Assumption 3.28. Therefore

$$\left|\mathrm{N}\left(z_0^0\right) \cap Y\right| = 1, \tag{7.145}$$

and (7.130) implies that equality must hold in (7.140). This means that the bound follows if $\mathrm{d}(q_4) = 7$. This justifies the following:

$$\mathrm{d}(q_i) = 6, \ \ \forall\, i \in 5. \tag{7.146}$$

Now, all of the q_i are indistinguishable and we may apply the preceding argument to each of them to obtain

$$\left|\mathrm{N}\left(z_0^i\right) \cap Y\right| = 1, \quad \forall\, i \in 5. \tag{7.147}$$

It follows from (7.140) that there are exactly four red edges internal to Y. We next consider possible configurations of them. If Y contains an independent set U such that $|U| = 3$, then

$$(\{x\} \cup U) \in \mathrm{Indep}\,(G)\,, \tag{7.148}$$

and Evaluation 7.5.20 and Assumption 3.22 imply

$$|\mathrm{Pend}\,(\mathrm{N}\,[p, q_0]\,; U)| \geq 12. \tag{7.149}$$

But, since $\delta(G) = \Delta(G) - 1$, this forces the existence of three additional red edges internal to Y, and this is impossible. Next, if there exists a disjoint pair of red edges internal to Y, say y_0y_1 and y_2y_3, then each pair along with x yields an independent set, and we have

$$|\mathrm{Pend}\,(\mathrm{N}\,[p, q_0]\,; y_0, y_1)|\,, |\mathrm{Pend}\,(\mathrm{N}\,[p, q_0]\,; y_2, y_3)| \geq 6, \tag{7.150}$$

and this necessitates the existence of two additional red edges incident with each of $\{y_0, y_1\}$ and $\{y_2, y_3\}$. Now $\overline{\mathrm{e}}\,(\langle Y\rangle) = 4$ provides for the existence of only two additional red edges. By symmetry, we may assume that $y_0 \not\leftrightarrow y_2$, and $y_1 \not\leftrightarrow y_3$. But then we have

$$p, y_4, y_5 \in \mathrm{N}\,[y_0] \cap \mathrm{N}\,[y_1] \quad \text{and} \quad p \in \mathrm{N}\,[x]\,. \tag{7.151}$$

Therefore

$$|\mathrm{N}\,[x, y_0, y_1]| \leq 19. \tag{7.152}$$

Now, since

$$\mathrm{DegEx}\,(p, y_0, y_1) = 3 = \lambda(6, 19, 3), \tag{7.153}$$

an application of Lemma 3.27 yields (1).

The only remaining configuration is specified in the following:

Assumption 7.9

1. $y_0 \not\leftrightarrow y_i,\ 1 \leq i \leq 4$.

2. $y_0 \leftrightarrow y_5$.

3. $\left\langle \{y_i\}_{i=1}^{5} \right\rangle = \mathrm{K}_5$.

This configuration is illustrated in Figure 7.8.

Next, we define

$$Z = \left\{ z_0^i \right\}^{i \in 5}, \tag{7.154}$$

and show that the theorem follows unless Z is independent. Assume that $\exists\ i, j \in 5$ such that

$$z_0^i \leftrightarrow z_0^j. \tag{7.155}$$

It follows from (7.146) and

$$q_j, z_0^i \in \mathrm{N}\,(q_i) \cap \mathrm{N}\left(z_0^j\right), \tag{7.156}$$

that

$$\left|\mathrm{N}\left[q_i, z_0^j\right]\right| \leq 13. \tag{7.157}$$

Clearly $p, q_i \leftrightarrow x$, and (7.147) implies that z_0^j has a neighbor in $\mathrm{N}\,(p)$. Therefore

$$\left|\mathrm{N}\left[p, q_i, z_0^j\right]\right| \leq 19, \tag{7.158}$$

and

$$\operatorname{DegEx}\left(\mathrm{N}\left[p\right]\right) > 3 = \lambda(6,19,3) \tag{7.159}$$

again yields a contradiction to Assumption 3.28. This implies

$$Z \in \operatorname{Indep}\left(G\right). \tag{7.160}$$

Assumption 7.9 asserts that we have accounted for all but one neighbor of y_5. Therefore there exists $Z^* \subseteq Z$ such that

$$|Z^*| = 4, \tag{7.161}$$

and

$$z_i^0 \in Z^* \implies z_i^0 \not\leftrightarrow y_5. \tag{7.162}$$

Now define

$$U = \{x, y_5\} \cup Z^*. \tag{7.163}$$

It is clear that x and y_5 have no neighbors in Z^*, so that (7.160) implies

$$U \in \operatorname{Indep}\left(\langle \mathrm{N}\left[p, q_0\right]\rangle\right). \tag{7.164}$$

Since

$$\mathrm{d}(p) = 7 \quad \text{and} \quad \mathrm{d}(y_i) = 7, \quad \forall\, i \in 6, \tag{7.165}$$

we obtain

$$\operatorname{DegEx}\left(\mathrm{N}\left[U\right]\right) \geq 7 = \lambda(6,38,6). \tag{7.166}$$

Therefore the theorem will follow by Lemma 3.27 upon showing that

$$|\mathrm{N}\left[U\right]| \leq 38. \tag{7.167}$$

First notice that

$$\mathrm{d}(x) = 6, \quad \mathrm{d}(y_5) = 7 \quad \text{and} \quad \mathrm{N}\,[x] \cap \mathrm{N}\,[y_5] = \{p\} \tag{7.168}$$

imply

$$|\mathrm{N}\,[x, y_5]| = 14. \tag{7.169}$$

Each member of Z^* has a neighbor in Q that has already been counted as a neighbor of x, and a neighbor in Y that has already been counted as a neighbor of y_5. Now (7.167) follows from (7.169), (7.161), and

$$\left|\mathrm{N}\left[z_0^j\right]\right| = 8, \quad \forall\, z_0^j \in Z^*. \tag{7.170}$$

This completes the proof of (1).

We next prove (2). The size bound specified in (1) was defined so as to hold for K_6 and the graph $G(13,2)$, depicted in Figure 7.9. We have

$$\nu(G(13,2)) = 13, \quad \alpha(G(13,2)) = 2 \quad \text{and} \quad \mathrm{e}\,(G(13,2)) = 44. \tag{7.171}$$

Furthermore this graph is the unique member of $\mathcal{G}_{6,7}\,(7)$ that possesses the specified order and independence number. The size bound coefficients

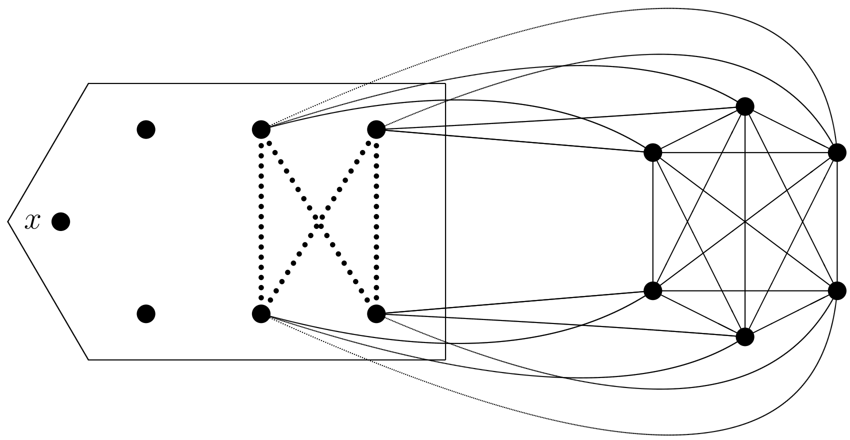
$$a = 14 \quad \text{and} \quad b = -69 \tag{7.172}$$

were obtained by solving the following linear system:

$$\begin{pmatrix} 6 & 1 \\ 13 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 15 \\ 44 \end{pmatrix}. \tag{7.173}$$

It is clear that any sum of the form

$$c\,\mathrm{K}_6 + d\,G(13,2) \tag{7.174}$$



This graph is the unique member of $\mathcal{G}_{6,7}\,(7)$ that has order 13 and independence 2.

Figure 7.9: $G(13,2)$

must also satisfy (1) with equality.

Now assume the hypothesis of (2). In order to find non-negative integers c and d such that

$$\nu(c\,\mathrm{K}_6 + d\,G(13,2)) = n \quad \text{and} \quad \alpha(c\,\mathrm{K}_6 + d\,G(13,2)) = r. \tag{7.175}$$

we solve

$$\begin{pmatrix} 1 & 2 \\ 6 & 13 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} r \\ n \end{pmatrix}, \tag{7.176}$$

and obtain

$$c = 13r - 2n \quad \text{and} \quad d = n - 6r. \tag{7.177}$$

Both c and d are integers, and they clearly satisfy

$$c \geq 0 \iff \frac{r}{n} \geq \frac{2}{13} \tag{7.178}$$

and

$$d \geq 0 \iff \frac{r}{n} \leq \frac{1}{6}. \tag{7.179}$$

Therefore we obtain a well defined graph.

We establish (3) by solving

$$\frac{7\nu(G)}{2} \geq \mathrm{e}\,(G) > 14\nu(G) - 69\alpha(G) \tag{7.180}$$

to obtain

$$\mu(G) \geq \frac{21}{138}, \quad \forall\, G \in \mathcal{G}_{6,7}\,(7)\,. \tag{7.181}$$

Now (3) immediately yields (4), and it remains to establish (5). Lower bounds for $R_{6,7}(7,n)$ are obtained from sums consisting of copies of $G(13,2)$ and possibly one copy of K_6 as follows: If $n = 2k$, then

$$\nu((k-1)\,G(13,2) + \mathrm{K}_6) = 13(k-1) + 6. \tag{7.182}$$

and

$$\alpha((k-1)\,G(13,2) + \mathrm{K}_6) = 2(k-1) + 1 = 2k - 1. \tag{7.183}$$

If $n = 2k+1$, then

$$\nu(k\,G(13,2)) = 13k. \tag{7.184}$$

and

$$\alpha(k\,G(13,2)) = 2k. \tag{7.185}$$

Now (7.3.5) has been established. ■

We have shown that $\frac{2}{13}$ is an attainable independence ratio and $\frac{21}{138}$ yields a lower bound for independence ratios. Since

$$\frac{2}{13} - \frac{21}{138} \approx 0.0017, \tag{7.186}$$

we see that the tight independence ratio bound resides in a narrow range. Intuition suggests that $\frac{2}{13}$ yields a tight bound, but this issue remains open. The issue of tightness of (1) for 7-regular graphs also remains open. Some partial results have been obtained, but a few difficult cases remain to be proved.

8. Butterflies

In the preceding chapters, we established size bounds and then derived independence ratio bounds from them, but in this chapter, we begin with a independence ratio bounds, and use tightness results to obtain candidates for size bounds. In Section 8.1, we present Fajtlowicz's size bound and briefly review his tightness results for the case $3m - 2\Delta = 5$. This information is used in Section 8.2 in order to define a class of prospective size bounds. Tightness properties for these bounds are also established in that section. Section 8.3 contains original results that pertain to Fajtlowicz's bound, in the case that $3m - 2\Delta = 4$. and independence is equal to two.

8.1 $3m - 2\Delta = 5$

In 1978, Fajtlowicz [4] showed that any K_m-free graph G of maximum degree Δ satisfies the independence ratio bound

$$\mu(G) \geq \frac{2}{m + \Delta}. \tag{8.1}$$

He furthermore showed that this bound cannot hold with equality if

$$3m - 2\Delta > 5, \tag{8.2}$$

and, for each m and Δ that satisfy

$$3m - 2\Delta = 5, \tag{8.3}$$

he exhibited a graph of independence two that does satisfy (8.1) with equality. He extended this result in 1984 [5], by showing that, given (8.3), independence two is necessary in order to attain equality, and, when attainable, it is achieved by a unique graph. He called these graphs butterflies because they have large chromatic numbers, and they are highly symmetric.

It is of interest that, whenever m and Δ are related as in (8.3), this tightness result immediately implies

$$R_{\Delta}(m, 3) = m + \Delta + 1, \tag{8.4}$$

but these numbers have never been published.

We now assume that m and Δ are related as in (8.3), and construct the unique butterfly for which (8.1) holds with equality.

Definition 8.1 *Let $k \geq -1$, and define*

1. $m = 2k + 5$.

2. $\Delta = 3k + 5$.

We have now determined all m and Δ for which (8.3) holds. For each k. Throughout this section, and the next, we assume that m and Δ follow via this definition.

Next, given $k \geq -1$, we construct the associated butterfly. We begin with the cyclic graph C_5, replace each vertex with the clique K_{k+2}, and include all edges between vertices belonging to neighboring cliques. We let B_m denote the resulting butterfly. Technically, butterflies are instances of structures that are referred to as both lexicographic products, and strong direct products of C_5 and K_{k+2}. In the case that $k = -1$, we again obtain C_5, and, should $k = 0$, the 5-regular graph $G(10, 2)$ depicted in Figure 1.2 results.

It is clear that

$$\nu(B_m) = 5(k+2) = m + \Delta, \tag{8.5}$$

and, since each pair of vertices belonging to non-neighboring cliques yields a maximum independent set, it follows that

$$\alpha(B_m) = 2. \tag{8.6}$$

Next, since maximal cliques are obtained by taking unions of neighboring cliques, we obtain

$$\omega(B_m) = 2(k+2) = m - 1, \tag{8.7}$$

and we see that B_m is K_m-free. If $v \in \mathrm{V}(B_m)$, then v has $k+1$ neighbors in the clique that contains it, and $k+2$ neighbors in each of the neighboring cliques. Therefore

$$\mathrm{d}(v) = 3k + 5 = \Delta. \tag{8.8}$$

We have established that each B_m is K_m-free, Δ-regular and it satisfies (8.3). In 1984, Fajtlowicz [5] showed that these are the unique graphs for which tightness is attainable when m and Δ are defined by Definition 8.1.

8.2 An Associated Size Bound

There exists a Turan bound that holds with equality for K_m-free graphs at a given independence ratio, so long as the independence ratio is assumed to be at least $\frac{1}{m-1}$, but this is not so for smaller independence ratios. We next introduce a collection of candidates for size bounds for $\mathcal{G}(m)$, and establish some tightness results for independence ratios smaller than $\frac{1}{m-1}$.

Definition 8.2 *Let $k \geq -1$ and define*

1. $a_m = \frac{7}{2}k + \frac{13}{2}$.

2. $b_m = -5(k+2)^2$.

We have defined a_m and b_m in terms of k and used m as the subscript of each. We shall also adhere to this practice in the next section, where m assumes even values. We now make the following:

Conjecture 8.3 *Let $k \geq -1$, and let m be defined as in Definition 8.2.1. Assume that $G \in \mathcal{G}(m)$. Then*

$$\mathrm{e}(G) \geq a_m \nu(G) - b_m \alpha(G), \quad \forall G \in \mathcal{G}(m). \tag{8.9}$$

In the course of writing this dissertation, this conjecture was shown to be true whenever $-1 \leq k \leq 5$. In fact, the two bounds presented in Chapter 4 were obtained by letting $k = -1, 0$. Table 8.1 contains information pertaining to values of k between -1 and 5.

The coefficients specified in Definition 8.2 are defined so as to hold for K_{m-1} and B_m as follows: Since B_m is Δ-regular, we have

$$\mathrm{e}(B_m) = \frac{\Delta(m+\Delta)}{2} = \frac{5}{2}(3k+5)(k+2). \tag{8.10}$$

Furthermore, it is clear that

$$\mathrm{e}(\mathrm{K}_{m-1}) = \binom{m-1}{2} = (k+2)(2k+3). \tag{8.11}$$

Therefore coefficients for the desired bound are obtained by solving

$$\begin{pmatrix} 2(k+2) & 1 \\ 5(k+2) & 2 \end{pmatrix} \begin{pmatrix} a_m \\ b_m \end{pmatrix} = \begin{pmatrix} (k+2)(2k+3) \\ \frac{5}{2}(3k+5)(k+2) \end{pmatrix}. \tag{8.12}$$

Table 8.1: $3m - 2\Delta = 5,\ -1 \le k \le 5$

k	m	Δ	ν	e	a_m	b_m
-1	3	2	5	5	3	5
0	5	5	10	25	$\frac{13}{2}$	-20
1	7	8	15	60	10	-45
2	9	11	20	110	$\frac{27}{2}$	-80
3	11	14	25	175	17	-125
4	13	17	30	255	$\frac{41}{2}$	-180
5	15	20	35	380	24	-245

We obtain the values specified in Definition 8.2.

Any sum consisting of copies of K_{m-1} and B_m must also satisfy the associated size bound with equality, and it is clear that the independence ratio of any such graph must lie between that of B_m and K_{m-1}. We next determine which orders and independence numbers are attainable by such sums.

Let $r, n > 0$ such that

$$\frac{2}{m+\Delta} \le \frac{r}{n} \le \frac{1}{m-1}. \tag{8.13}$$

We wish to find non-negative integers c and d that yield

$$\nu(c\,\mathrm{K}_{m-1} + dB_m) = n \quad \text{and} \quad \alpha(c\,\mathrm{K}_{m-1} + dB_m) = r. \tag{8.14}$$

Such coefficients must satisfy the following linear system:

$$\begin{pmatrix} 1 & 2 \\ 2(k+2) & 5(k+2) \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} r \\ n \end{pmatrix}. \tag{8.15}$$

We obtain

$$c = 5r - \frac{2n}{k+2} \quad \text{and} \quad d = \frac{n}{k+2} - 2r. \tag{8.16}$$

It is easily verified that

$$c \geq 0 \iff \frac{r}{n} \geq \frac{2}{5(k+2)} = \frac{2}{m+\Delta} \tag{8.17}$$

and

$$d \geq 0 \iff \frac{r}{n} \leq \frac{1}{2(k+2)} = \frac{1}{m-1}. \tag{8.18}$$

Therefore c and d are both non-negative exactly when $\frac{r}{n}$ lies within the desired range of independence ratios, but in order for

$$c\,\mathrm{K}_{m-1} + dB_m \tag{8.19}$$

to yield a well defined graph, it is additionally necessary and sufficient that n be a multiple of $k+2$. Notice that it is always possible to construct a graph with independence ratio equal to $\frac{r}{n}$ for which tightness holds, by assuming that it has order $n(k+2)$, and independence $r(k+2)$.

We have used information pertaining to a class independence ratio bounds, and defined a class of prospective size bounds. Now, by applying techniques developed in Chapter 2, it is possible to obtain a class of prospective independence ratio bounds. If $G \in \mathcal{G}_\Delta(m)$ such that

$$\mathrm{e}(G) \geq a_m \nu(G) + b_m \alpha(G), \tag{8.20}$$

then, by the same logic used in the proof of Lemma 2.2, we obtain

$$\mu(G) \geq \frac{2}{m+\Delta}. \tag{8.21}$$

This result suggests that the prospective size bounds defined in this section are closely related to Fajtlowicz's independence ratio bound.

8.3 $3m - 2\Delta = 4$

Fajtlowicz established tightness results for 8.1 when m and Δ are related by

$$3m - 2\Delta \geq 5. \tag{8.22}$$

In this section we consider the case

$$3m - 2\Delta = 4, \tag{8.23}$$

and show that, when m and Δ satisfy (8.23) and independence two is assumed, order $m+\Delta$ is possible only if $m = \Delta = 4$, while order $m+\Delta-1$ is always attainable. As in the preceding section, we use this information to define candidates for tight size bounds for K_m-free graphs. Notice that, in the current context, m must be even. We begin by defining m and Δ in terms of the variable k.

Definition 8.4 *Let $k \geq 0$ and define*

1. $m = 2k + 4$.
2. $\Delta = 3k + 4$.

Throughout this section we assume that m and Δ are obtained in this way.

Theorem 8.5 *Let $k \geq 0$. Assume that m and Δ are defined by 8.4.1 and 8.4.2, respectively. Then*

$$R_\Delta(m, 3) = \begin{cases} 9 & \text{if } m = 4 \\ m + \Delta & \text{otherwise} \end{cases}. \tag{8.24}$$

Proof: It follows from (8.1) that

$$R_{\Delta}(m,3) \leq m+\Delta+1. \tag{8.25}$$

The case $m = 4$ follows because the graph $G(8,2)$, found by ABT [1], and depicted in Figure 1.2 is critical. We now assume that $m \geq 6$, and therefore $k \geq 1$. To establish this case, we assume that 8.1 holds with equality and obtain a contradiction. The theorem then follows upon exhibiting a graph $G \in \mathcal{G}_{\Delta}(m)$ that satisfies $\nu(G) = m+\Delta-1$ and $\alpha(G) = 2$.

Assume that $G \in \mathcal{G}_{\Delta}(m)$ such that

$$\nu(G) = m+\Delta \quad \text{and} \quad \alpha(G) = 2. \tag{8.26}$$

Let $x \in \mathrm{MinDeg}(G)$ and consider the graphs $F(x)$ and $H(x)$, as defined in Definition 3.4. Since any independent subset of $\mathrm{V}(H(x))$ can be extended to an independent subset of $\mathrm{V}(G)$ by including x, we see that $H(x)$ must be a clique. Therefore it can have order no larger that $m-1$. Now order $m+\Delta$ is attainable only if

$$\mathrm{d}(x) = \Delta \quad \text{and} \quad H(x) = \mathrm{K}_{m-1}. \tag{8.27}$$

Notice that the first inequality implies that

$$G \text{ is } \Delta\text{-regular.} \tag{8.28}$$

Now, because G is K_m-free and $\Delta > m$, it is necessary that

$$\alpha(F(x)) = 2. \tag{8.29}$$

Let $u \in \mathrm{N}(x)$. It follows from (8.28) that

$$\overline{\mathrm{d}}_{\mathrm{N}[x]}(u) = |\mathrm{Pend}(\mathrm{N}[x];u)| = |\mathrm{M}[u;x]|. \tag{8.30}$$

If $p, q \in \mathrm{N}(x)$ such that $p \not\leftrightarrow q$, then it follows from $\alpha(G) = 2$ that

$$\mathrm{V}(H(x)) \subseteq \mathrm{M}[p, q; x], \tag{8.31}$$

while we must have

$$\mathrm{V}(H(x)) \nsubseteq \mathrm{M}[p; x], \mathrm{M}[q; x], \tag{8.32}$$

because G is K_m-free.

Let $v \in \mathrm{V}(H(x))$. Since $H(x) = \mathrm{K}_{2k+3}$, it is clear that v has exactly $k+2$ neighbors in $\mathrm{N}[x]$, and therefore

$$|\mathrm{Pend}(\mathrm{N}[x])| = (2k+3)(k+2) = 2k^2 + 7k + 6. \tag{8.33}$$

In order for G to have independence two, it is necessary for each $v \in \mathrm{V}(H(x)) \setminus \mathrm{M}[p; x]$ to be adjacent to each member of $\overline{\mathrm{N}}_{\mathrm{N}[x]}(p)$ Since v must have exactly $k+2$ neighbors in $\mathrm{N}[x]$, it follows from (8.30) that

$$|\mathrm{M}[p; x]| \leq k + 2. \tag{8.34}$$

Clearly, the analogous condition also holds for q. Upon applying (8.31), we obtain

$$k + 1 \leq |\mathrm{M}[p; x]|, |\mathrm{M}[q; x]| \leq k + 2, \tag{8.35}$$

and the first inequality cannot be an equality for both p and q. This implies

$$|\mathrm{M}[p; x] \cap \mathrm{M}[q; x]| \leq 1. \tag{8.36}$$

Since G is K_m-free, there exist $p_0, q_0 \in \mathrm{N}(x)$ such that $p_0 \not\leftrightarrow q_0$. We define

$$\overline{\mathrm{N}}_{\mathrm{N}(x)}(p_0) = Q = \{q_j\}^{j \in |Q|} \tag{8.37}$$

and

$$\overline{\mathrm{N}}_{\mathrm{N}(x)}\left(q_0\right) = P = \{p_i\}^{i \in |P|,} \tag{8.38}$$

and, by symmetry, we may assume that

$$|P| \leq |Q|\,. \tag{8.39}$$

It follows from $\alpha(F(x)) = 2$ that P and Q both induce cliques.

First assume that

$$|\mathrm{M}\left[p_0; x\right]| = |\mathrm{M}\left[q_0; x\right]| = k + 2. \tag{8.40}$$

In this case,

$$|P| = |Q| = k + 2 \tag{8.41}$$

and

$$|\mathrm{M}\left[p_0; x\right] \cap \mathrm{M}\left[q_0; x\right]| = 1. \tag{8.42}$$

It follows from the first inequality of (8.35) and (8.40) that

$$|\mathrm{Pend}\left(\mathrm{N}\left[x\right]; P \cup Q\right)| \geq (2k + 4)(k + 1) + 2 = 2k^2 + 6k + 6. \tag{8.43}$$

Upon taking (8.33) into consideration, we see that it is necessary to account for exactly k additional edges pendant to $\mathrm{N}\left[x\right]$. Since any member of $\mathrm{N}\left(x\right)$ that has a non-neighbor in that set must have at least $k+1$ neighbors in $\mathrm{V}\left(H(x)\right)$, it follows that each red edge of $F(x)$ must have one endpoint in P, and the other in Q.

Now define

$$S = \{u \in \mathrm{N}\left(x\right) \mid |\mathrm{Pend}\left(\mathrm{N}\left[x\right]; u\right)| = k + 1\} \tag{8.44}$$

and

$$T = \{u \in \mathrm{N}(x) \mid |\mathrm{Pend}(\mathrm{N}[x]; u)| = k+2\}. \tag{8.45}$$

It follows fron (8.33), that

$$|S|(k+1) + |T|(k+2) = 2k^2 + 7k + 6. \tag{8.46}$$

Therefore

$$|S| = |T| = k+2. \tag{8.47}$$

Since $p_0, q_0 \in T$, we may assume that

$$p_{k+1}, q_{k+1} \in S. \tag{8.48}$$

But this implies

$$T = \{q_j\}^{j \in k+1} \cup \{p_i\}^{i \in k+1}, \tag{8.49}$$

and therefore

$$|T| = 2k+2. \tag{8.50}$$

Now we have obtained a contradiction to (8.47), and we see that (8.40) cannot hold.

We may now assume that

$$\mathrm{M}[p; x] \cap \mathrm{M}[q; x] = \emptyset \tag{8.51}$$

whenever $p, q \in \mathrm{N}(x)$, such that $p \not\leftrightarrow q$. The constraint on clique size implies that there exist $p_0, q_0 \in \mathrm{N}(x)$ such that $p_0 \not\leftrightarrow q_0$. We again use the notation introduced in (8.37), (8.38) and assume (8.39). It now follows that

$$|P| = k+1 \quad \text{and} \quad |Q| = k+2. \tag{8.52}$$

Furthermore,

$$\mathrm{M}\left[q_j;x\right] = \mathrm{M}\left[q_0;x\right],\ 1 \le j \le k+1, \tag{8.53}$$

and

$$\mathrm{M}\left[p_i;x\right] = \mathrm{M}\left[p_0;x\right],\ 1 \le i \le k. \tag{8.54}$$

This implies

$$\left|\mathrm{Pend}\left(\mathrm{N}\left[x\right];P\right)\right| = \left|\mathrm{Pend}\left(\mathrm{N}\left[x\right];Q\right)\right| = (k+1)(k+2). \tag{8.55}$$

It follows from (8.33), that there must exist $k+2$ additional edges pendant to $\mathrm{N}\left[x\right]$; this requires the existence of an additional red edge. Also notice that

$$\langle \mathrm{N}\left[x\right] \setminus P\rangle = \mathrm{K}_m. \tag{8.56}$$

Therefore this red edge must have one endpoint in Q and the other outside of $P \cup Q$. Assume that q_1 is an endpoint of this edge. Δ-regularity implies that

$$\overline{\mathrm{d}}_{\mathrm{N}[x]}(q_1) = k+2, \tag{8.57}$$

and it is therefore necessary for q_1 to have an additional neighbor in $\mathrm{V}\left(H(x)\right)$. This neighbor must reside in $\mathrm{M}\left[p_0;x\right]$. But now we have obtained a contradiction to (8.51). Therefore

$$R_\Delta m, 3 \le m + \Delta. \tag{8.58}$$

It only remains to exhibit a graph G of order $m+\Delta-1$ that satisfies all of the constraints. We begin with the cyclic graph C_5, and assume it has vertex set $\{v_n\}^{n\in 5}$ and edge set $\{v_n v_{n+1}\}^{n\in 4} \cup \{v_0, v_4\}$. We replace each of v_0, v_2 and v_3 with the clique K_{k+1}, and we replace each of v_1 and v_4 with the clique K_{k+2}, so as to obtain the vertex set for G. It is clear that

$$\nu(G) = 5k+7 = m+\Delta-1. \tag{8.59}$$

We construct the edge set for G by retaining the edges of the cliques, and adding edges between each pair of vertices that belong to neighboring cliques. Now, any pair of vetices that belong to non-neighboring cliques yield a maximum independent set; we obtain

$$\alpha(G) = 2. \tag{8.60}$$

A maximum clique is obtained by forming the union of one of the cliques corresponding to vertex v_1 or v_4 and a neighboring clique to obtain

$$\omega(G) = 2k + 3 = m - 1. \tag{8.61}$$

Finally, let u belong to the clique corresponding to v_0. It is clear that u has k neighbors in that clique, and $k+2$ neighbors in each of the neighboring cliques. We now have

$$\mathrm{d}(u) = 4k + 4 = \Delta(G), \tag{8.62}$$

and it is easily verified that u is of maximum degree. Therefore G possesses the required structure. ∎

The graphs constructed in the preceding proof are structurally similar to butterflies, but they do not have quite as much symmetry. For each $m \geq 6$, we let B_m denote the resulting graph, and we let $B_4 = G(8, 2)$. As in the preceding section, we define a candidate for a size bound that holds for K_{m-1} and B_m. By summing the degree sums of the five cliques used in the construction of B_m, we obtain

$$\mathrm{d}(\mathrm{V}\,(B_m)) = (k+1)(3k+4) + 2(k+1)(3k+3) + 2(k+2)(3k+3) \tag{8.63}$$

$$= 15k^2 + 37k + 22. \tag{8.64}$$

Now, in order to obtain the desired bound, we solve

$$\begin{pmatrix} 2k+3 & 1 \\ 5k+7 & 2 \end{pmatrix} \begin{pmatrix} a_m \\ b_m \end{pmatrix} = \begin{pmatrix} 2k^2+5k+3 \\ \frac{1}{2}(15k^2+37k+22)) \end{pmatrix} \tag{8.65}$$

and obtain

$$a_m = \frac{7}{2}k + 5 \quad \text{and} \quad b_m = -5k^2 - \frac{31}{2}k - 12 \tag{8.66}$$

These results suggest the following:

Conjecture 8.6 *Let $k \geq -1$, and let m be defined as in Definition 8.4.1. Assume that $G \in \mathcal{G}(m)$. Then*

$$\mathrm{e}(G) \geq a_m \nu(G) - b_m \alpha(G), \quad \forall\, G \in \mathcal{G}(m).. \tag{8.67}$$

The graph B_m, has maximum degree $\Delta = 3k + 4$, while the preceding conjecture asserts that the associated size bound should hold for all members of $\mathcal{G}(m)$. This strong conclusion is supported by the following results: Fraughnaugh and Lokce [7] established the bound for the case $k = 0$, and the case $k = 1$ is presented in Section 6.2. Additionally, in the course of writing this dissertation, the cases $k = 2, 3, 4$ were also verified. In each case, the bound has been shown to hold without any restriction on maximum degree. Table 8.2 contains information pertaining to the cases $0 \leq k \leq 5$.

As in the preceding section, we establish a tightness result that holds for each m under consideration. Let $r, n > 0$ such that

$$\frac{2}{m + \Delta - 1} \leq \frac{r}{n} \leq \frac{1}{m-1}. \tag{8.68}$$

We consider the problem of finding non-negative integers c and d that yield

$$\nu(c\,\mathrm{K}_{m-1} + d\,B_m) = n \quad \text{and} \quad \alpha(c\,\mathrm{K}_{m-1} + d\,B_m) = r. \tag{8.69}$$

Table 8.2: $3m - 2\Delta = 4,\ -1 \leq k \leq 4$

k	m	Δ	ν	e	a_m	b_m
0	4	4	8	16	5	-12
1	6	7	12	37	$\frac{17}{2}$	$-\frac{65}{2}$
2	8	10	17	78	12	-63
3	10	13	22	134	$\frac{31}{2}$	$-\frac{207}{2}$
4	12	16	27	205	19	-154
5	14	19	32	291	$\frac{45}{2}$	$-\frac{429}{2}$

We simply solve the linear system

$$\begin{pmatrix} 1 & 2 \\ 2k+3 & 5k+7 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} r \\ n \end{pmatrix}, \tag{8.70}$$

and obtain

$$c = \frac{(5k+7)r - 2n}{k+1} \quad \text{and} \quad d = \frac{n - (2k+3)r}{k+1}. \tag{8.71}$$

It follows from

$$c \geq 0 \iff \frac{r}{n} \geq \frac{2}{5k+7} = \frac{2}{m + \Delta - 1} \tag{8.72}$$

and

$$d \geq 0 \iff \frac{r}{n} \leq \frac{1}{2k+3} = \frac{1}{m-1}. \tag{8.73}$$

that c and d have the appropriate signs, but in order to obtain integer coefficients, it is necessary that $(5k+7)r - 2n$ and $n - (2k+3)$ both be divisible by $k+1$.

Finally, we use the conjectured size bound to obtain a candidate for an independence ratio bound, and compare the result to initial independence ratio bound, as introduced by Fajtlowicz. If $G \in \mathcal{G}_\Delta(m)$ satisfies the bound stated in Conjecture 8.6, then we once again obtain

$$\mu(G) \geq \frac{2}{m + \Delta}. \tag{8.74}$$

This result is of interest because we obtain (8.1) even though a_m and b_m were defined so as to yield a tight bound for for a graph of order $m + \Delta - 1$. As in the preceding section, it appears that we have constructed candidates for size bounds that are closely related to Fajtlowicz's independence ratio bound.

9. Conclusions

Several size and independence ratio bounds have been established in this dissertation. In this chapter, we briefly summarize these results, discuss patterns that arose and suggest problems for future study. In particular, we are interested in the problem of defining coefficients for size bounds by expressions that yield infinite classes of bounds.

An infinite class of size bounds, each of which holds for all graphs, was introduced in Chapter 2. It was shown that, for each positive integer l, the expression

$$\mathrm{e}(G) \geq l\nu(G) - \binom{l+1}{2}\alpha(G) \tag{9.1}$$

yields a size bound that holds for any graph G, and, furthermore, holds with equality for the complement of the Turan graph $\mathrm{T}_{n,r}$ whenever

$$\frac{1}{l+1} \leq \frac{r}{n} \leq \frac{1}{l}. \tag{9.2}$$

These bounds provide the starting point for the study of size bounds. The problem of determining coefficients for size bounds becomes much more difficult when a forbidden clique constraint is applied. In fact, very little is known about tight bounds for K_m-free graphs when independence ratios are assumed to be smaller that $\frac{1}{m-1}$.

In Chapter 5, it was shown that, with one exception for the case $m = 5$, any connected member of $\mathcal{G}_m(m)$ satisfies the size bound

$$\mathrm{e}(G) \geq (2m-3)\nu(G) + \left(-\frac{3}{2}m^2 + \frac{7}{2}m - 2\right)\alpha(G) \tag{9.3}$$

and the independence ratio bound

$$\mu(G) \geq \frac{3m-6}{3m^2-7m+4}. \tag{9.4}$$

It follows from the work of ABT [1] and Fajtlowicz [4] that $\mu(G) \geq \frac{1}{m}$ for each $G \in \mathcal{G}_m\,(m)$. Staton [21] and Sandberg [20] improved upon these results for the cases $m = 3$ and $m = 6$, respectively. The independence ratio bounds stated in (9.4) dominate any published results for the cases $m = 5$ and $m \geq 7$.

Tight size bounds were established for members of $\mathcal{G}_6\,(6)$ and non-regular members of $\mathcal{G}_7\,(7)$, in Chapters 6 and 7, respectively. The first bound holds tightly for independence ratios between $\frac{2}{11}$ and $\frac{1}{5}$, and the second bound holds tightly for independence ratios between $\frac{2}{13}$ and $\frac{1}{6}$. It remains to extend the second bound so as to include 7-regular graphs. In this case, the bound has been shown to hold unless the closed neighborhood of each vertex induces one of four graphs. Both proofs utilized deeply nested subcase hierarchies, and required complex pendant edge counting arguments.

Such complex arguments were not required for the proofs of size bounds associated with butterflies. Two such bounds were established in Chapter 4, another in Chapter 6, and eight additional bounds were established in the course of conducting the research for this dissertation. It now appears that allowing maximum degree to be larger than the forbidden clique number provides for more readily established theorems.

While writing this dissertation, it became apparent that subtle relationships exist between coefficients of tight size bounds for K_m-free graphs, and certain tightness conditions associated with Fajtlowicz's independence ratio bound

$$\mu(G) \geq \frac{2}{m + \Delta(G)}. \tag{9.5}$$

The equalities

$$3m - 2\Delta = 5 \tag{9.6}$$

and

$$3m - 2\Delta = 4 \tag{9.7}$$

are of particular interest; issues pertaining to them were addressed in Chapter 8. If we let $k \geq -1$, then

$$m = 2k + 5 \quad \text{and} \quad \Delta = 3k + 5 \tag{9.8}$$

yield the values of m and Δ that satisfy (9.6), and the values for (9.7) follow by letting $k \geq 0$ and defining

$$m = 2k + 4 \quad \text{and} \quad \Delta = 3k + 4 \tag{9.9}$$

Notice that, in both cases, the maximum degree is asymptotically three halves as large as the forbidden clique number.

Fajtlowicz [4] showed that, when (9.6) holds, there exists a K_m-free, Δ-regular graph of independence two that satisfies (9.5) with equality, but equality cannot be attained when $3m - 2\Delta > 5$. In [5], he established that independence two is necessary in order for tightness to hold, and for each m, tightness is achieved for a unique graph; he called these graphs butterflies.

Fajtlowicz did not address the issue of tightness when $3m - 2\Delta < 5$, but it was shown in Chapter 8 that, when independence two and equality (9.7) are assumed. order $m + \Delta$ is possible only if $m = \Delta = 4$, but order $m + \Delta - 1$ is always attainable. These results immediately yielded a class of Ramsey-type numbers.

It is clear that (9.6) pertains to odd values of m and (9.7) pertains to even values. In Chapter 8, a candidate for a size bound was defined for each odd value of m so as to hold with equality for the associated butterfly and the clique K_{m-1}. The coefficients for this bound are defined by

$$a_m = \frac{7}{2}k + \frac{13}{2} \quad \text{and} \quad b_m = -5(k+2)^2. \tag{9.10}$$

It has been verified bu the author that tight size bounds for $\mathcal{G}(m)$ result when $3 \leq m \leq 15$; the proofs for the cases $m = 3, 5$ are presented in Chapter 4. Similarly, when m is even, the coefficients

$$a_m = \frac{7}{2}k + 5 \quad \text{and} \quad b_m = -5k^2 - \frac{31}{2}k - 12, \tag{9.11}$$

were obtained. Fraughnaugh and Locke [9] had previously established the resulting bound for the case $m = 4$, the case $m = 6$ is proved in Chapter 6, and the cases $m = 8, 10, 12$ have also been verified. Verification of these two classes bounds in full generality are obvious problems for further study. The forms of the conjectured coefficients clearly depend on the parity of m, and, while proofs of these conjectures have yet to be obtained, sufficiently many individual cases have been verified in order to establish that at least two classes of bounds must be considered.

Given $m \geq 3$, if we assume that the associated bound holds for $\mathcal{G}(m)$, then, upon applying the maximum degree constraint yielded by the corresponding value of Δ, and performing the same calculations that yielded Lemma 2.2, we again obtain the independence ratio bound (9.5). This result suggests that there exists a natural relationship between Fajtlowicz's independence ratio bound and the prospective size bounds associated with (9.10) and (9.11)

The preceding tightness results are associated with independence two. When we consider graphs of independence one, we are dealing with cliques. It is clear that K_{m-1} is K_m-free and $\Delta(\mathrm{K}_{m-1}) = m-2$. Therefore we associate the equality

$$m - \Delta = 2 \tag{9.12}$$

with independence one.

Now we consider larger independence numbers. Given independence r, it would be instructive to determine the relationships between m and Δ that yield optimal or nearly optimal independence ratios for graphs of that independence number. If it were possible to establish expressions for optimal values of x_r, y_r and z_r as they occur in

$$x_r m + y_r \Delta = z_r, \tag{9.13}$$

then a great deal of information pertaining to Ramsey-type numbers and classical Ramsey numbers would result.

There are no known results that suggest optimal relationships between m and Δ when $r \geq 3$, but a particular pattern has frequently arisen when constructing critical graphs of independence two. If we let $\alpha(G) = 2$, and assume that $x \in \mathrm{MinDeg}(G)$, then, using the notation presented in Definition 3.4, we partition G into the graphs $F(x)$ and $H(x)$. It has occurred in several cases that $F(x)$ was a clique from which the edges of a complete, balanced bipartite graph had been excluded and $H(x)$ was a clique. In order to obtain independence two in such a situation it is necessary that the endpoints of each red edge of $F(x)$ cover the vertices of $H(x)$. In particular, butterflies possess this structure.

The preceding observation suggests that, given independence r, it might be possible to split a critical graph G into a pair of graphs F and H, such that F is a clique from which the edges of an r-partite Turan graph have been removed, H is a clique, and independence r is maintained for G by letting the partite sets cover the vertices of H. It is reasonable to begin with the case $r = 3$, and, for some small values of m, try to construct critical graphs that conform to these constraints. It is clear that (9.12) yields an optimal relationship when independence one is assumed and it appears likely that there are two classes of bounds, depending on the parity of m, when independence is assumed to equal two. Should this condition be established, then it would be of interest to see if three different classes of size bounds, each dependent upon the value of (m mod 3), arise when independence three is assumed.

In any case, the problem of determining critical graphs for which the forbidden clique numbers and maximum degrees are small when independence three is assumed is a reasonable next step. Given such graphs, analogues of equalities (9.6) and (9.7) should follow easily. It would also be of interest to define coefficients for prospective size bounds that hold with equality for these critical graphs and butterflies and verify that size bounds do indeed result. Additionally, it would be instructive to apply the techniques described in Chapter 2 to these size bounds, so as to obtain independence ratio bounds, and then compare them to Fajtlowicz's bound.

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