DOMINATION AND MATRIX PROPERTIES IN TOURNAMENTS AND GENERALIZED TOURNAMENTS

by

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ABSTRACT

In this thesis we examine several matrix properties of tournament matrices. In particular we look at how domination and related parameters in tournaments affect these properties in the corresponding matrices. We begin by examining when a tournament can support an orthogonal matrix. To do so, we first look at the necessary condition of quadrangularity in tournaments. We classify tournaments with given properties which meet this necessary condition. We show how domination effects quadrangularity in regular tournaments and tournaments with certain minimum degrees. We also determine for exactly which orders a quadrangular tournament exists.

In chapter 3, we examine the stronger necessary condition for a digraph to support an orthogonal matrix of strong quadrangularity. We construct a class of tournaments which meet this condition. We also show that the 3-cycle is the unique tournament on ten or fewer vertices which supports an orthogonal matrix, and discuss a search conducted for a tournament matrix which supports an orthogonal matrix. In the following chapter we look at which tournament matrices are fully indecomposable and which are separable, as these properties are related to a necessary condition for a $(0,1)$-matrix to support an orthogonal matrix. We use our classification to derive a number of matrix and graph theoretic corollaries. As the domination graphs of tournaments play a central role in our classification of separable tournament matrices, we classify the domination graphs of complete paired comparison digraphs in chapter 5.

Finally, we examine the Boolean, non-negative integer and real ranks of tournament matrices. We find new bounds for the Boolean rank using results on dominating sets and related parameters. We determine, for the first time, a class of tournament matrices in which the real rank is less than the Boolean rank. We also determine a set theoretic dual to the problem of finding the Boolean row rank of a tournament matrix, and give some results and conjectures related to the problem of finding the minimum Boolean rank over all tournament matrices of order n.

This abstract accurately represents the content of the candidate's thesis. I recommend its publication.

Signed ______________________

J. Richard Lundgren

DEDICATION

For Melissa, for my family, and for Albion.

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Figure

1. Introduction

1.1 Background

The two main structures of interest in this thesis are tournaments and tournament matrices. Tournaments are a famous class of digraphs which originally arose from the model of a round robin competition in which each player plays against every other. Since this simple beginning, tournaments have been shown to have a great deal of interesting combinatorial structure, and the tournament matrix has proven to have many interesting algebraic properties. In this thesis we look at various problems which bring together these two aspects of tournaments and tournament matrices. In chapter 5, we will also study a related structure, the complete paired comparison digraph (sometimes called a generalized tournament).

In chapters 2, 3, and 4, we study three properties which arise from the question of orthogonal support, and how these properties effect tournaments and tournament matrices. In chapter 6, we take a look at various matrix rank problems in tournament matrices. We find that domination plays a role in each of these problems. Particularly, the domination graphs of tournaments can be applied to the problems in chapter 2, and will give us our characterization in chapter 4. Because these play such an important role in some of these problems, especially in chapter 4, we characterize the domination graphs of complete paired comparison digraphs in chapter 5.

The first problem addressed in this thesis is that of orthogonal support. This problem comes from an application to quantum physics and quantum computing. In physics the problem arises in modeling discrete quantum walks, and in computing it comes from writing quantum algorithms. In both cases, one sets up the model, much like a Markov chain, by repeatedly applying an orthogonal (or unitary) transition matrix to a state vector. The state vector gives the original state of the object we wish to model, and after a select number of multiplications by the transition matrix we return the probable outcomes for our object. Underlying the transition matrix is a directed graph which maps out the movements of our object from state to state. We are interested in knowing if this underlying digraph can be a tournament. We give an example of a Markov model below.

Suppose six school children are playing a game of catch. The digraph in Figure 1.1 shows the children as vertices, and the probability that child i will throw the ball to child j is given by the weight on arc (i, j). If we begin by giving the ball to child 1, then our model has the transition matrix

$$M = \begin{pmatrix} .6 & .4 & 0 & 0 & 0 & 0 \\ 0 & 0 & .3 & .7 & 0 & 0 \\ 0 & 0 & 0 & 0 & .4 & .6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and state vector $\mathbf{v} = (1\ 0\ 0\ 0\ 0\ 0)^\top$. We then repeatedly multiply $\mathbf{v}$ by M to determine possible outcomes. For example, if we wish to know the probability child 4 will end up with the ball after 12 throws, then we simply look at the 4^{th} entry of $M^{12}\mathbf{v}$. For more on Markov chains, the reader is referred to [55].

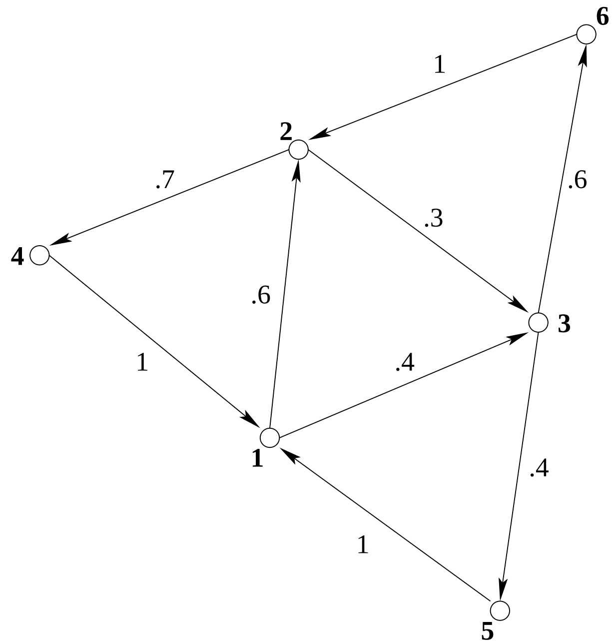

Figure 1.1: An example of a Markov model

The nature of decoherence in quantum physics requires the transition matrix in the quantum version of these models to be orthogonal. We want to know when a tournament can be used in these models, and hence when a tournament is the digraph underlying an orthogonal matrix. We have three necessary conditions for a directed graph to be the digraph of an orthogonal matrix. In chapter 2 we study the first of our necessary conditions, quadrangularity. In chapter 3 we study the stronger condition of strong quadrangularity, and discuss some of our searches for tournaments which are the digraphs of orthogonal matrices. In chapter 4 we see that an orthogonal matrix needs to be written in a particular way, and see exactly when a tournament matrix can be written in this way. In chapter 4 we also characterize exactly which tournaments can be used in a doubly stochastic Markov model.

In chapter 5 we look at the domination graphs of complete paired comparison digraphs. This is one extension of the study of the domination graphs of tournaments. The domination graphs of tournaments were introduced by Merz, Fisher, Lundgren, and Reid, in [44], and have been characterized in a series of papers (see [31], [43], [44], [45], [46]). Domination graphs have been generalized to tournaments in which ties are allowed, and proper subdigraphs of tournaments by Factor and Factor in [24] and [25]. Another generalization, called k-domination has also been studied by McKenna, Morton and Sneddon in [41]. We give a generalization of domination graphs for the context of complete paired comparison digraphs, and essentially characterize this class of graphs.

In chapter 6 we examine some problems involving ranks of tournament matrices. We will look at relationships between the Boolean, non-negative integer, real, term, and minimal rank of tournament matrices. We see that, as domination plays a role in the questions addressed in chapters 2, 3, and 4, it also acts in this context by giving us lower bounds for these ranks. We answer a question of Siewert [58], by constructing an infinite class of singular tournament matrices with full Boolean rank. We examine a dual to the Boolean row rank of a tournament matrix which is an extension of Schütte's problem, and also examine a dual to the problem of finding the Boolean rank of a tournament matrix.

1.2 Graph and digraph basics

In this section we review the background and notation for graphs and directed graphs necessary for this thesis. Recall first that a *graph* G is a set of vertices $V(G)$, together with a set $E(G)$ of unordered pairs of vertices, called *edges.* If G is a graph such that no edge occurs more than once, and G has no edge of the form $[u, u]$ for some $u \in V(G)$, then G is typically referred to as a *simple graph.* In this thesis we have no need to distinguish between graphs and simple graphs, so we shall assume that all graphs are simple. We also have no need for infinite graphs, and so shall assume the vertex and edges sets of our graphs are finite. We typically represent a graph pictorially by representing the vertices with dots, and connecting two dots with a line segment if there is an edge joining the two vertices in the graph. For example, the graph $G = (V(G), E(G))$ with $V(G) = \{1, 2, 3, 4, 5, 6, 7\}$ and $E(G) = \{[1, 2], [1, 3], [3, 4], [5, 6]\}$ is shown in Figure 1.2. We now give some of the basic definitions necessary to study the structure of graphs.

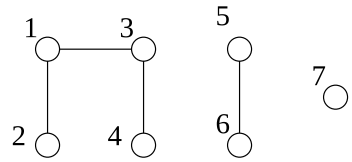

Figure 1.2: A drawing of a graph

Let G be a graph. If $[u, v] \in E(G)$, then we say that u and v are *adjacent* in G. A vertex u and edge e of G are said to be *incident* if $e = [u, v]$ for some vertex v. For a given $v \in V(G)$, we define the *degree of* v, $d(v)$, to be the number of edges incident to v. Since we are assuming our graphs are simple, we can equivalently define the degree of v as the number of vertices adjacent to v.

To study the structure of graphs it is sometimes useful to study related graphs, such as subgraphs and the complement of a graph. A *subgraph* of a graph G, is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Given a set $S \subseteq V(G)$, the *subgraph induced on* S*,* $G[S]$, is the graph with $V(G[S]) = S$, and $[u, v] \in E(G[S])$ if and only if $u, v \in S$ and $[u, v] \in E(G)$. A graph H is called an *induced subgraph* of G if $H = G[S]$ for some $S \subseteq V(G)$. The *complement* of a graph G is the graph $\overline{G}$, with $V(\overline{G}) = V(G)$ and $[u, v] \in E(\overline{G})$ if and only if $[u, v] \notin E(G)$.

A *path* between vertices u and v in a graph G (sometimes called a u, v-*path*) is an alternating sequence of vertices and edges of G, beginning with u and ending with v, such that no vertex or edge is repeated, and each vertex is incident with the edge immediately preceding and succeeding it in the sequence. For simplicity, we typically only list the vertices of a path. The *length of a path* is defined to be the number of edges in the path. The *distance* between vertices u and v is defined to be the length of a shortest u, v-path. Also, a graph G is said to be *connected* if there exists a u, v-path for all distinct $u, v \in V(G)$. A graph G which is not connected is called *disconnected,* and the maximally connected subgraphs of G are called the *connected components* or *components* of G. A component of order 1 is referred to as an *isolated vertex.*

Cycles and cliques are important graph structures that come up frequently in chapter 5. A *cycle* in a graph G is an alternating sequence of vertices and edges of G, beginning and ending with the same vertex, such that no edge is repeated, only the first vertex is repeated, and each vertex is incident to the edge immediately preceding and succeeding it. If all the vertices and edges of G are contained in a cycle, we call G a cycle. The *length of a cycle* is the number of edges occurring in the cycle, and a cycle of length n is called an n-cycle, typically denoted by C_n. If the length of a cycle is odd or even, then we refer to the cycle as an *odd cycle* or *even cycle* respectively. A *complete graph,* or *clique,* on n vertices, written K_n, is a graph with n vertices, such that $[u, v] \in E(K_n)$ for all distinct $u, v \in V(K_n)$. Typically the term clique is reserved for a complete subgraph of some graph.

Another important class of graphs which come up throughout the thesis are the bipartite graphs. A graph B is said to be *bipartite* if $V(B)$ can be partitioned into two sets X and Y, called a *bipartition,* such that no two vertices of X are adjacent and no two vertices of Y are adjacent. A well known result on bipartite graphs, which proves useful in chapter 5, is that a graph is bipartite if and only if it contains no odd cycles. A *complete bipartite graph,* or *biclique,* is a bipartite graph B with bipartition $X \cup Y$ so that $[x, y] \in E(B)$ for all $x \in X$ and $y \in Y$. If $|X| = r$ and $|Y| = s$, then we sometimes denote the biclique on $X \cup Y$ by $K_{r,s}$. As with clique, the term biclique is typically reserved for a complete bipartite subgraph of some graph. Bicliques play an important role in chapter 6 because of their relationship with rank 1 matrices. We now turn our attention to directed graphs.

A *directed graph,* or *digraph,* D is a set of vertices $V(D)$, together with a set $A(D)$ of ordered pairs of vertices, called *arcs.* An *orientation of a graph* G is a digraph D obtained from G by letting $V(D) = V(G)$ and for each $[u, v] \in E(G)$, making exactly one of (u, v) or (v, u) an arc of D. Directed graphs are typically drawn in a manner analogous to graphs, in that each vertex is represented by a dot, and we draw an arrow from vertex u to vertex v if (u, v) is an arc of of the digraph. Figure 1.3 shows a drawing of the digraph $D = (V(D), A(D))$ where

$V(D) = \{1,2,3,4,5,6\}$ and $A(D) = \{(2,3),(3,4),(5,5),(4,1),(4,3),(5,6)\}$. We now give some of the necessary definitions and notation to study digraphs in this thesis.

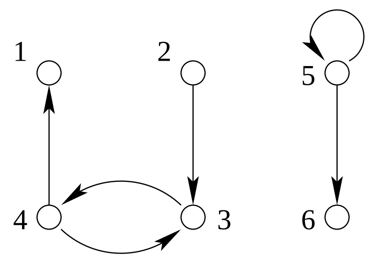

Figure 1.3: A drawing of a digraph

If $(u,v) \in A(D)$, then we typically say that u "beats" v or that u "dominates" v and write this as $u \to v$. We define the *outset,* $O_D(v)$, of a vertex v in D to be the set of all vertices that v dominates. That is,

$$O_D(v) = \{u \in V(D) : (v,u) \in A(D)\}.$$

Similarly, the *inset,* $I_D(v)$, of a vertex v in D is the set of all vertices which dominate v. So,

$$I_D(v) = \{u \in V(D) : (u,v) \in A(D)\}.$$

The *closed outset* and *closed inset* of a vertex v are defined by $O_D[v] = O_D(v) \cup \{v\}$ and $I_D[v] = I(v) \cup \{v\}$ respectively. If $S \subseteq V(D)$, then we define the *outset of* S as the set of all vertices dominated by some vertex in S, written $O_D(S)$. Equivalently, we can define the outset of S by $O_D(S) = \cup_{v \in S} O_D(v)$. We denote by $O_D[S]$, the *closed outset of* S*,* defined by $O_D[S] = O_D(S) \cup S$. The *inset of S* and *closed inset of S* are defined analogously as $I_D(S) = \cup_{v \in S} I(v)$ and $I_D[S] = I(S) \cup S$ respectively. When the context of which digraph we are referring to is clear we will drop the subscript.

The *out-degree* of a vertex v in a digraph D, written $d_D^+(v)$, is defined to be $|O_D(v)|$. Similarly, the *in-degree* of v is $d_D^-(v) = |I_D(v)|$. Again, when the context is clear, we will drop the subscript. The *minimum out-degree* of a digraph D, denoted $\delta^+(D)$, is defined to be $\delta^+(D) = \min\{d^+(v) : v \in V(D)\}$. Similarly, the *minimum in-degree* of D is $\delta^-(D) = \min\{d^-(v) : v \in V(D)\}$. We define the *maximum out-degree* and *maximum in-degree* of a digraph D to be the values $\Delta^+(D) = \max\{d^+(v) : v \in V(D)\}$ and $\Delta^-(D) = \max\{d^-(v) : v \in V(D)\}$ respectively.

Let D be a digraph. A *subdigraph* of D is a digraph D' such that $V(D') \subseteq V(D)$ and $A(D') \subseteq A(D)$. If $S \subseteq V(D)$, then the *subdigraph induced on S* is the digraph D' with $V(D') = S$ and $(u, v) \in A(D')$ if and only if $u, v \in S$ and $(u, v) \in A(D)$. A digraph D' is called an *induced subdigraph* of D if there exists some $S \subseteq V(D)$ which induces D'.

A *directed path* in a digraph D is an alternating sequence of vertices and arcs of D, beginning and ending with vertices, so that no vertex or arc is repeated, and if (u, v) is an arc in the path, the vertex immediately preceding it must be u and the vertex immediately succeeding it must be v. A directed path which begins at a vertex u and ends at a vertex v is called a *directed u, v-path.* Similar to the undirected case, the *length of a directed path* is defined to be the number of arcs in the path. If D is a directed graph so that there is a directed u, v-path for every distinct $u, v \in V(D)$, then we say D is *strongly connected.* If D is not strongly connected, then the maximal strongly connected subdigraphs of D are called the *strong components* of D.

A *directed cycle* in a digraph D is an alternating sequence of vertices and arcs of D which begin and end with the same vertex, repeat no vertices or edges, and if (u, v) is an arc in the cycle, then the vertex immediately preceding it must be u and the vertex immediately succeeding it must be v. Similar to undirected cycles, the length of a directed cycle is the number of arcs in the cycle. A directed cycle of length n is referred to as a directed n-cycle. Also, as before, an n-cycle is called a *directed odd cycle* or *directed even cycle* if n is odd or even respectively. If it is clear that we are dealing with a directed context, we will simply refer to directed cycles and directed paths as cycles and paths.

To study the structure of graphs and digraphs, it is useful to have a concept of isomorphism in graphs and digraphs. An *isomorphism of graphs* G and H is a bijective function $\phi : V(G) \to V(H)$ such that $[u, v] \in E(G)$ if and only if $[\phi(u), \phi(v)] \in E(H)$. That is, a graph isomorphism is an adjacency preserving bijection. We say that two graphs are *isomorphic* if there exists a graph isomorphism between them. Similarly, we define an *isomorphism of digraphs* D and D' to be a bijection $\phi : V(D) \to V(D')$ such that $(u, v) \in A(D)$ if and only if $(\phi(u), \phi(v)) \in A(D')$. That is, a digraph isomorphism is a dominance preserving bijection. An isomorphism from a graph to itself (digraph to itself) is called a *graph automorphism* (*digraph automorphism*). We now look at the class of digraphs studied most in this thesis.

1.3 Tournament basics

In this section we review some of the basic terminology and properties of tournaments. A *tournament* T is a complete asymmetric digraph. That is, for every two distinct $u, v \in V(T)$, either $(u, v) \in A(T)$ or $(v, u) \in A(T)$, but not both, and $(v, v) \notin A(T)$ for all $v \in V(T)$. A tournament with n vertices, is called an *n-tournament.* Equivalently, some define an n-tournament as an orientation of the complete graph K_n. An example of a 5-tournament is shown in Figure 1.4.

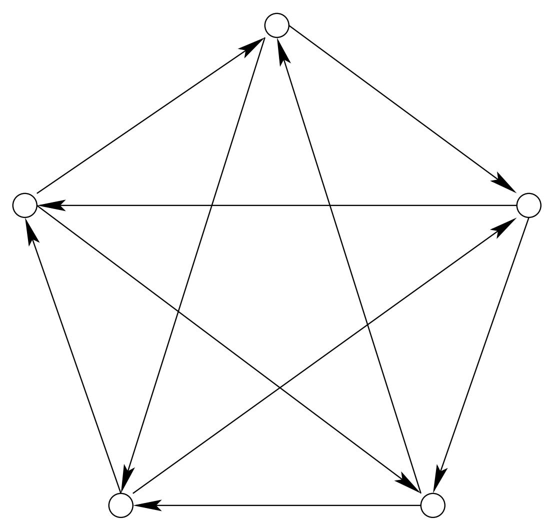

Figure 1.4: A 5-tournament

In studying tournament structure, we often want to look at subtournaments and the dual of a tournament. Given a tournament T, and $S \subseteq V(T)$, a *subtournament* of T is the tournament $T[S]$, with $V(T[S]) = S$, and $(u, v) \in A(T[S])$ if and only if $u, v \in S$ and $(u, v) \in A(T)$. That is, $T[S]$ is the subdigraph induced on S. Also, for a vertex $v \in V(T)$ or a set of vertices S, we define the subtournaments $T - v$ and $T - S$ to be the tournaments $T[V(T) - \{v\}]$ and $T[V(T) - S]$

respectively. The *dual* of T is the tournament T^r, with $V(T^r) = V(T)$ and $(u, v) \in A(T^r)$ if and only if $(v, u) \in A(T)$. We define the dual of a digraph in the same way, and use the same notation.

One of the most famous results about tournaments is Landau's Theorem. The list of out-degrees of the vertices of a tournament is called its *score sequence.* In 1953 Landau characterized which lists of n positive integers could be the score sequence of some tournament.

Theorem 1.1 (Landau's Theorem) *Let $s_1, s_2, \ldots, s_n$ be a non-decreasing sequence of positive integers. This list is the score sequence for some n-tournament if and only if*

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2}$$

for $1 \leq k \leq n-1$ and

$$\sum_{i=1}^{n} s_i = \binom{n}{2}.$$

The necessity in Landau's Theorem is easy to see. Any n-tournament must have an arc between every two vertices, and hence $\binom{n}{2}$ arcs. Further, every subtournament of an n-tournament on k vertices will have $\binom{k}{2}$ arcs. The argument follows by counting arcs in subtournaments in this way, and by counting them as the sum of the out-degrees of a subset of vertices. The sufficiency in Landau's Theorem has been proved in many ways, by many people. It has been proven by using classic combinatorial methods, using majorization of vectors, using posets, using network flows, and even using Hall's Theorem. Some of the many authors of these proofs include Landau, Ryser, Reid, Thomassen, Fulkerson, and Bang and Sharp. For a full proof of Landau's Theorem the reader is referred to [50].

Another classic theorem on tournaments involves Hamiltonian paths. Given a digraph D, a *Hamiltonian path* is a path which contains every vertex of D. By taking a path of maximum length in a tournament, one can show, via the complete asymmetry of the tournament, that there is no vertex not on the path and it is hence Hamiltonian. We discuss an analogous result about cycles in strong tournaments below. We now state this theorem.

Theorem 1.2 *Every tournament contains a Hamiltonian path.*

In this thesis, we are often concerned with when one set in a tournament completely dominates another. Let T be a tournament and $S, S' \subseteq V(T)$. If for every $u \in S$ and $v \in S'$, $u \to v$, then we write $S \Rightarrow S'$. In the case where $S = \{u\}$ ($S' = \{v\}$), we write $u \Rightarrow S'$ ($S \Rightarrow v$) rather than $\{u\} \Rightarrow S'$ ($S \Rightarrow \{v\}$). A vertex $s \in V(T)$ such that $s \Rightarrow V(T) - \{s\}$ is called a *transmitter.* A vertex $t \in V(T)$ such that $V(T) - \{t\} \Rightarrow t$ is called a *receiver.* The concept of one set completely dominating another plays an important part in our study of tournaments and tournament matrices, as it corresponds to large blocks of ones in the adjacency matrix. Tournaments with transmitters and receivers also play a part as special cases of some of our results. In particular, when studying tournaments which are not strongly connected, tournaments with transmitters or receivers are important special cases.

As with digraphs, we define a tournament T to be *strongly connected* if for any two vertices $u, v \in V(T)$ there is a directed path in T from u to v and another from v to u, and a *strong component* of a tournament to be a maximal strongly connected subtournament. We sometimes refer to a strongly connected tournament as a *strong tournament.* We note that a single vertex is trivially strongly connected, and that a 2-tournament has exactly one arc, and hence cannot be strongly connected. So, the smallest non-trivial strong component which a tournament could have is the 3-cycle. Strong tournaments have been characterized in terms of their score sequences by replacing the inequalities in Landau's Theorem with strict inequalities. The following two theorems are well known, useful results on the structure of tournaments.

Theorem 1.3 *Let T be a strong n-tournament, and let $v \in V(T)$. Then v is contained on a cycle of length k for $k = 3, \ldots, n$.*

Theorem 1.4 *Let T be a tournament which is not strongly connected. Then $V(T)$ can be partitioned by the strong components of T, and these components can be labeled $T_1, T_2, \ldots, T_m$ such that $T_i \Rightarrow T_j$ if and only if $i < j$.*

The strong components T_1 and T_m in the previous theorem are called the *initial* and *terminal strong components* of T respectively. The above result has also been stated as "The condensation of a tournament is transitive." A *transitive tournament* is one in which for all $u, v, w \in V(T)$, if $u \to v$ and $v \to w$, then $u \to w$. There are several equivalencies for the definition of a transitive tournament. The two others we will most frequently use are that the vertices can be labeled $v_1, v_2, \ldots, v_n$ such that $v_i \to v_j$ if and only if $i < j$, and that $0, 1, 2, \ldots, n-1$ is the score sequence of T. We also note that the transitive tournament on n vertices is unique up to isomorphism.

The condensation of a tournament T is typically defined to be the tournament T' obtained by representing each strong component of T with a vertex of

T', and letting $u \to v$ if and only if for the corresponding strong components T_u and T_v we have that $T_u \Rightarrow T_v$. Theorem 1.4 above, with this definition, may also be stated as "The condensation of a tournament is always a transitive tournament." In this thesis we will adopt the following, more versatile definition of condensation. Let T be a tournament, and suppose we can partition the vertices of T as $S_1, S_2, \ldots, S_m$ such that for all $i \neq j$ either $S_i \Rightarrow S_j$ or $S_j \Rightarrow S_i$. We define the *condensation* of T, with respect to this partition, to be the tournament T' with $V(T) = \{v_1, v_2, \ldots, v_m\}$ with $v_i \to v_j$ in T' if and only if $S_i \Rightarrow S_j$ in T.

In the study of tournaments, the extreme examples of many problems tend to be transitive tournaments and regular tournaments. We discussed the transitive tournaments above. A *regular tournament* is one in which every vertex has the same out-degree. Consequently, every vertex in a regular tournament has the same in-degree. Thus, for every regular n-tournament, n is odd and the out-degree and in-degree of every vertex is $\frac{n-1}{2}$. While there are no regular n-tournaments for n even, we do have an analogous tournament. A *near regular tournament* is a tournament in which the out-degrees of any two vertices differs by at most one. By some counting arguments, one can show that if T is a near regular n-tournament, then n is even, exactly half of the vertices have out-degree $\frac{n}{2}$ and exactly half of the vertices have out-degree $\frac{n}{2} - 1$.

Up to isomorphism, there is only one transitive n-tournament, however there can be many non-isomorphic regular n-tournaments. To better study regular tournaments we often turn to the class of rotational tournaments. Rotational tournaments have a nice structure which makes them somewhat easier to work with, and while not all regular tournaments are rotational, there are typically a large number of rotational n-tournaments for a given n. This makes rotational tournaments a nice class of regular tournaments to start with when studying questions about regular tournaments. We give the construction for rotational tournaments below.

Choose an odd positive integer n, and let S be a subset of $\{1, 2, \ldots, n-1\}$ of order $\frac{n-1}{2}$ such that $i \in S$ if and only if $-i \pmod n$ is not in S. Construct a digraph T on the vertices $\{0, 1, 2, \ldots, n-1\}$ with $i \to j$ in T if and only if $j - i \pmod n$ is in S. Since x is in S if and only if $-x \pmod n$ is not, we have that for any two distinct $i, j \in V(T)$ exactly one of $j - i \pmod n$ or $i - j \pmod n$ is in S. So we have that $i \to j$ or $j \to i$, but not both. Further, since $0 \notin S$, T has no arcs of the form (i, i). Thus T is a tournament. We call T a *rotational tournament* with *symbol* S.

To see that rotational tournaments are regular choose any vertex i. Then, $O(i) = \{i + j \pmod n : j \in S\}$, and so each vertex has the same out-degree. Another important property of rotational tournaments is that they are vertex transitive. A *vertex transitive* tournament T is one in which for every two vertices u, v in T, there exists an automorphism of T that maps u to v. Given a rotational tournament T with symbol S, pick $i, j \in V(T)$, and let $k \equiv j - i \pmod n$. Consider the function ϕ, defined by $\phi : a \mapsto a + k \pmod n$. It is easy to see that ϕ is a bijection. Further, if $a \to b$ in T, then $b - a \pmod n$ is in S, and $\phi(b) - \phi(a) \equiv b + k - (a + k) \equiv b - a \pmod n$ which is in S, so $\phi(a) \to \phi(b)$, and ϕ is an automorphism of T. Finally, $\phi(i) = i + (j - i) = j$, so ϕ is an automorphism which maps i to j, and since i and j were chosen arbitrarily, T is vertex transitive.

An important class of rotational tournaments which come up frequently in this thesis are the quadratic residue tournaments. Recall from number theory that an integer a is called a quadratic residue modulo n if there exists some x such that $a \equiv x^2 \pmod{n}$. For any given odd prime p, we know that there are $\frac{p-1}{2}$ quadratic residues modulo p. Further, given a prime p such that $p \equiv 3 \pmod{4}$, we know that i is a quadratic residue if and only if $-i$ is not. (For a proof of this, and more on quadratic residues see [49].) So, given a prime $p \equiv 3 \pmod{4}$, the quadratic residues modulo p form a symbol for a rotational p-tournament. We call this tournament the *quadratic residue tournament* of order p, written QR_p. An important, and well known property of quadratic residue tournaments is that they are arc-transitive. An *arc-transitive* tournament T is one in which for any two arcs (i, j) and (h, k) in T, there exists an automorphism of T which maps i to h and j to k. We now state and prove this property for completeness.

Theorem 1.5 *Quadratic residue tournaments are arc-transitive.*

Proof: Let $p \equiv 3 \pmod{4}$ be a prime, and choose arc (i, j) and (h, k) in QR_p. Let $a = (k-h)(j-i)^{-1}$ and $b = (hj - ki)(j-i)^{-1}$. Note a is a quadratic residue since $k-h$ and $j-i$ are quadratic residues, and the set of quadratic residues modulo p form a group under multiplication. Define the function f by $f : x \mapsto ax + b \pmod{p}$. Since $V(QR_p) = \mathbb{Z}_p$ is a field, f is a bijection from $V(QR_p)$ to $V(QR_p)$. To see that f is an automorphism, pick $(x, y) \in A(QR_p)$. Then $(y - x)$ is a quadratic residue modulo p, and

$$f(y) - f(x) = ay + b - (ax + b) = ay - ax = a(y - x).$$

Since a and $(y-x)$ are quadratic residues, $a(y-x)$ is a quadratic residue, and so $f(x) \to f(y)$. So, f preserves dominance and is hence and automorphism. Also,

$$\begin{aligned} f(i) &= ai + b \\ &= ((k-h)(j-i)^{-1})i + (hj - ki)(j-i)^{-1} \\ &= (ki - hi + hj - ki)(j-i)^{-1} \\ &= h(j-i)(j-i)^{-1} \\ &= h, \end{aligned}$$

and

$$\begin{aligned} f(j) &= aj + b \\ &= ((k-h)(j-i)^{-1})j + (hj - ki)(j-i)^{-1} \\ &= (kj - hj + hj - ki)(j-i)^{-1} \\ &= k(j-i)(j-i)^{-1} \\ &= k \end{aligned}$$

Thus, QR_p is arc-transitive. ■

The arc-transitivity of quadratic residue tournaments tells us that the tournament induced on $O(u) \cap O(v)$ is the same for any two distinct vertices u, v in QR_p. In particular, this tells us that $|O(u) \cap O(v)|$ is the same for all distinct $u, v \in V(QR_p)$. Tournaments with this property are called doubly regular tournaments. That is, a tournament T is called a *doubly regular tournament* if for all distinct $u, v \in V(T)$, $|O(u) \cap O(v)| = k$, for some given k.

A doubly regular tournament with $|O(u) \cap O(v)| = k$ for all distinct u, v is also a regular tournament on $4k + 3$ vertices with $d^+(v) = 2k + 1$ for all vertices v. Doubly regular tournaments have many nice properties. In particular, they form extremal cases for many problems in tournaments. While it is not certain, they also appear to be good candidates for extremal cases in some of the problems posed in this thesis. For instance, it is our current belief that if a tournament is going to be the digraph of an orthogonal matrix, the smallest non-trivial example will be doubly regular. (In fact, the only known example is the 3-cycle, which is doubly regular with $k = 0$.) We also believe that doubly regular tournaments will be the optimal choices when extending tournaments to create larger tournaments with small Boolean rank. For more on doubly regular tournaments, the reader is referred to Reid and Brown, [10].

1.4 Domination basics

Given a digraph D, and a set $S \subseteq V(D)$, we say that S is a *dominating set* in D if $O[S] = V(D)$. In this section we discuss some of the background on domination in digraphs and tournaments, and how it relates to the problems studied in this thesis. Domination in tournaments has been studied by many people. The original problem appears to be a problem of Schütte, in which he asks, "For a given k, what is the least n so that there exists an n-tournament T such that for all $S \subseteq V(T)$ with $|S| = k$, there exists a vertex $v \in V(T)$ with $v \Rightarrow S$?" Taking the asymmetry of tournaments into account, one can see that if there is no such vertex for a set S, then S is a dominating set. The order of a smallest dominating set in D is called the *domination number* of D, written $\gamma(D)$. So, answering Schütte's problem is equivalent to finding a smallest n-tournament T with $\gamma(T) > k$.

A property related to domination that we will discuss in chapter 6 is irredundance. An *irredundant set* in a digraph D is a set S such that for all $u \in S$, there exists a vertex $v \in V(D)$ such that $v \in O[u] - O(S)$. The size of a smallest maximal irredundant set in a digraph D is called the *lower irredundance number* of D, written $ir(D)$. The size of a largest irredundant set is called the *upper irredundance number,* written $IR(D)$. Similarly, the size of a largest minimal dominating set is called the *upper domination number,* written $\Gamma(D)$. A recent treatment and collection of results on domination and irredundance in tournaments can be found in Hedetniemi, Hedetniemi, McRae, and Reid [34]. The following useful result of basic bounds on γ, Γ, ir and IR is taken from their paper.

Theorem 1.6 [34] *For every tournament T,*

(i) $ir(T) \leq \gamma(T) \leq \Delta^+(T)$;

(ii) $ir(T) \leq \gamma(T) \leq \Gamma(T) \leq IR(T)$;

(iii) $\gamma(T) \leq n - \Delta^+(T)$;

(iv) $\gamma(T) \leq \delta^-(T) + 1$.

As mentioned in the previous section, doubly regular tournaments form extremal examples for several problems, and appear to be extremal examples for some of the problems in this thesis. This also appears to be true when looking for tournaments with large domination numbers. For instance, the smallest tournament with domination number 2 is the 3-cycle. The smallest tournament such that every set of 2 vertices is dominated by a third, and hence the smallest tournament with domination number 3, is QR_7 (note this is the only doubly regular 7-tournament and is hence the unique 7-tournament with domination number 3). It can also be verified that the smallest tournament with domination number 4 is QR_{19} (see [34]). Some of these examples prove quite useful when we begin relating domination in tournaments to properties in the corresponding tournament matrices.

One way in which we will relate domination to matrix properties, especially in chapters 2 and 4 is by studying the structure of dominant pairs in a tournament. A *dominant pair* in a tournament is a dominating set of order 2. Given a tournament T, the *domination graph* of T is the graph $\text{dom}(T)$ on the same vertices as T with $[u, v] \in E(\text{dom}(T))$ if and only if $\{u, v\}$ form a dominant pair in T. As mentioned, the domination graphs of tournaments were introduced by Merz, Lundgren, Reid, and Fisher in [44], and have since been characterized in a series of papers.

Our study of the domination graphs of tournaments comes from their relation to the competition graphs of tournaments. Given a digraph D, the *competition graph* of D is the graph $\text{comp}(D)$ on the same vertices as D with $[u, v] \in E(\text{comp}(D))$ if and only if $u \neq v$ and there exists some $w \in V(D)$ such that $u \to w$ and $v \to w$. Recall from the asymmetry of tournaments, that two vertices will dominate a tournament if and only if there does not exist a vertex which dominates them both. This is the main idea behind the following theorem of Merz et. al..

Theorem 1.7 [44] *The complement of the competition graph of a tournament is the domination graph of its dual.*

If D is a digraph, and M its adjacency matrix, then for $i \neq j$, the i, j entry of $MM^\top$ will be non-zero if and only if row i and row j share a common non-zero entry. That is, the i, j entry of $MM^\top$ will be non-zero if and only if vertex i and vertex j compete in D. So, the non-zero, off diagonal entries of the adjacency matrix of comp(D) are in direct correspondence with those of $MM^\top$. This relationship together with the above result is the main correspondence between domination in tournaments and the matrix properties discussed in chapter 2.

We use Merz et. al.'s characterization of the domination graphs of tournaments in chapter 4, along with a characterization of separable matrices in terms of the competition graphs of the corresponding digraphs, to characterize exactly which tournament matrices are separable. In chapter 5 we use this characterization as a jumping off point for our characterization of the domination graphs of complete paired comparison digraphs. We state the necessary conditions of the characterization here, but first recall the following definitions.

Recall that a *tree* is a connected acyclic graph. A *caterpillar* is a tree such that the removal of all pendant vertices results in a path. Each caterpillar has a path of maximum length called a *spine.* A *spiked cycle* is a graph such that the removal of all pendant vertices results in a cycle. If the cycle in a spiked cycle is odd, then we call it a *spiked odd cycle.* Note, a spiked cycle need not have pendants. So, cycles form a subclass of spiked cycles, and odd cycles form a subclass of spiked odd cycles.

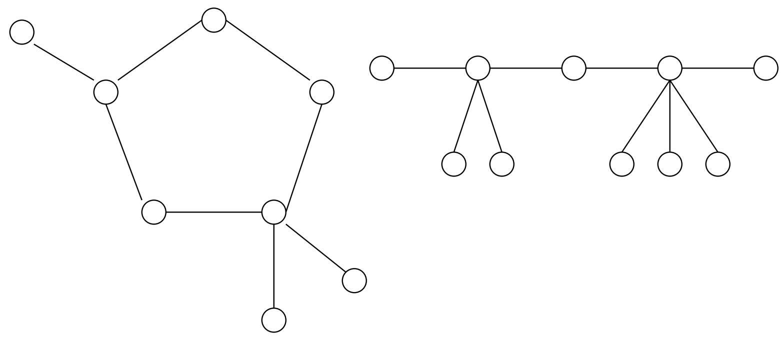

Figure 1.5: An example of a spiked cycle and a caterpillar

Theorem 1.8 [44] *The domination graph of a tournament is either a caterpillar or a spiked odd cycle.*

The sufficient conditions for the characterization of domination graphs of tournaments vary depending on connectedness restrictions and can be found in [31], [43], [45], and [46]. We now turn to some necessary background on $(0,1)$-matrices and matrix theory.

1.5 Matrix basics

In this section we cover some basic definitions and background on $(0,1)$-matrices. First, we introduce the notation we shall use for discussing matrices and vectors. Let M be a matrix. We denote the i,j entry of M by $M_{i,j}$. We use $M_{i\bullet}$ to represent the i^{th} row of M and M_i to represent the i^{th} column of M. If $\mathbf{v}$ is a vector, then we let v_i denote the i^{th} entry of the vector.

A $(0,1)$-*matrix* is a matrix whose every entry is a 0 or 1. We concern ourselves with square $(n \times n)$ $(0,1)$-matrices in this thesis. Any square $(0,1)$-matrix can be considered the adjacency matrix of some digraph. Let D be a digraph with $V(D) = \{1, 2, \dots, n\}$. The *adjacency matrix* of D is the $n \times n$ $(0,1)$-matrix M defined by

$$M_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in A(D), \\ 0 & \text{if } (i,j) \notin A(D). \end{cases}$$

In chapters 2, 3, and parts of chapters 4 and 6 we are interested in the patterns of matrices. Given a matrix A, with arbitrary entries, the *pattern* of A is the $(0,1)$-matrix M obtained from A by letting

$$M_{i,j} = \begin{cases} 1 & \text{if } A_{i,j} \neq 0, \\ 0 & \text{if } A_{i,j} = 0. \end{cases}$$

This has also been called a non-zero pattern by some authors. The digraph D whose adjacency matrix is the pattern of a matrix A is called the *digraph of* A. If M is the pattern of a matrix A, or D the digraph of A, then we sometimes say that M and D *support* A.

Just as tournaments are an important class of digraphs to study, tournament matrices give rise to several interesting problems in the study of $(0,1)$-matrices. A *tournament matrix* is the adjacency matrix of a tournament. By the complete asymmetry of tournaments, one could also define a tournament matrix to be a square $(0,1)$-matrix M with 0s on the diagonal and $M_{i,j} = 1$ if and only if $M_{j,i} = 0$. A third equivalent definition of tournament matrices is a square $(0,1)$-matrix M such that $M + M^\top = J - I$, where J is a matrix of all 1s, and I the identity matrix.

Some other important classes of matrices which come up are the zero matrices, the class of matrices $J_{m\times n}$, and the permutation matrices. An $m \times n$ *zero* *matrix,* $O_{m,n}$, is an $m \times n$ matrix in which every entry is a 0. If $m = n$, we will simply write O_n, or if the dimensions are clear from context, O. We will write a vector of all zeros as $\mathbf{0}$. The $m \times n$ matrix $J_{m,n}$ is an $m \times n$ matrix of ones. Again, if $m = n$, we will simply write J_n, or J if the dimensions are clear from context. We write a vector of all ones as $\mathbf{1}$. Given a bijection ϕ from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$, a *permutation matrix* is a matrix P with $P_{i,\phi(i)} = 1$ for $i = 1, \ldots, n$, and $P_{i,j} = 0$ otherwise. A well known property of permutation matrices is that left multiplying a matrix A by a permutation matrix $P^\top$ will reorder the rows of A according to the bijection corresponding to P, and right multiplying A by P will reorder the columns according to the bijection corresponding to P.

If M is the adjacency matrix of some digraph D, and P a permutation matrix, then left and right multiplying M by $P^\top$ and P respectively, results in the adjacency matrix of a digraph isomorphic to D. Left and right multiplying M by permutation matrices P and Q results in a *permutation equivalent matrix.* Permutation equivalent matrices do not necessarily correspond to isomorphic digraphs, but are important for some matrix properties. We study the permutation equivalence of tournaments in chapter 4.

We denote the standard Euclidean inner product of two vectors $\mathbf{x}$ and $\mathbf{y}$ by $\langle \mathbf{x}, \mathbf{y} \rangle$. That is, for column vectors $\mathbf{x}$ and $\mathbf{y}$, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{y}$. We say two vectors $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal* if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. An $n \times n$ matrix M is said to be orthogonal if $MM^\top = M^\top M = I$. So, if M is orthogonal, $\langle M_{i\bullet}, M_{j\bullet} \rangle = 0$ and $\langle M_i, M_j \rangle = 0$ for all $i \neq j$, and $\sum_{j=1}^n M_{i,j}^2 = \sum_{i=1}^n M_{i,j}^2 = 1$, for any i, j.

Two less traditional matrix concepts which arise in chapter 6 are complements of matrices, and the idea of one matrix being less than another. Given a

$(0,1)$-matrix, M, we define the *complement* of M to be the matrix M^c, where $M^c_{i,j} = 1$ if and only if $M_{i,j} = 0$. Given two matrices A and M, we say that $A \leq M$ if they have the same dimensions and for all i, j, $A_{i,j} \leq M_{i,j}$. Equivalently, for vectors $\mathbf{x}$ and $\mathbf{y}$, we say $\mathbf{x} \leq \mathbf{y}$ if $\mathbf{x}_i \leq \mathbf{y}_i$ for all i. We define these since $(0,1)$-matrices and vectors tie in closely with sets when we begin dealing with Boolean arithmetic in chapter 6. Also, for a set S and subset W of S, we will also write the complement of W in S as W^c.

2. Quadrangular Tournaments

2.1 Definitions and background

Recall that an $n \times n$ matrix M is defined to be *orthogonal* if $MM^\top = M^\top M = I$, where I is the $n \times n$ identity matrix. In this chapter we introduce a basic necessary condition for a digraph to support an orthogonal matrix, and study the tournaments which meet this condition. Our condition is derived from combinatorial orthogonality, which originally appeared in a matrix context in [5]. Given two n-vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ with entries from any field, we say that x and y are *combinatorially orthogonal* if $|\{i : x_i y_i \neq 0\}| \neq 1$. That is, they do not share exactly 1 non-zero entry. We say a matrix M is *combinatorially orthogonal* if every two rows are combinatorially orthogonal and every two columns are combinatorially orthogonal.

To see that this is in fact a necessary condition for a matrix to be orthogonal, let M be a matrix with rows $M_{i\bullet}$ and $M_{j\bullet}$ which are not combinatorially orthogonal (a similar argument holds for columns). Then $M_{i\bullet}$ and $M_{j\bullet}$ have exactly one non-zero entry in common, and so $\langle M_{i\bullet}, M_{j\bullet}\rangle$ is just the product of these two entries and hence, not zero. As any two rows (or columns) of an orthogonal matrix must be orthogonal, we see that combinatorial orthogonality is a necessary condition for a matrix to be orthogonal. Further, since combinatorial orthogonality only concerns itself with whether an entry is non-zero or not, combinatorial orthogonality is a necessary condition for a $(0,1)$-matrix to be the pattern of an orthogonal matrix.

A digraph D is called *quadrangular* if for all distinct $u, v \in V(D)$, $|O(u) \cap O(v)| \neq 1$ and $|I(u) \cap I(v)| \neq 1$. If we only require the restriction on the outsets or insets separately, we say D is *out-quadrangular* or *in-quadrangular* respectively. Figure 2.1 shows an example of a quadrangular digraph. Note that

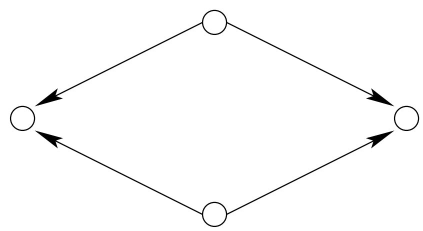

Figure 2.1: A Quadrangular Digraph

if A is the adjacency matrix of a digraph, and $A_{i\bullet}$ and $A_{j\bullet}$ two rows of A which correspond to vertices u and v respectively, then $\langle A_{i\bullet}, A_{j\bullet}\rangle = |O(u) \cap O(v)|$. Similarly, the inner product of two columns is equal to the size of the intersection of the insets of the corresponding vertices. So, D is a quadrangular digraph, if and only if its adjacency matrix is a combinatorially orthogonal matrix. Thus, quadrangularity is a necessary condition for a digraph to support an orthogonal matrix.

Many of the results in this chapter show that, in certain classes of tournaments, there is a relationship between quadrangularity and the domination number of the tournament or a subtournament. For example, Theorems 2.1, 2.8, and 2.19 all exhibit this relationship. We begin by examining tournaments which are not strongly connected. We see that in this case quadrangularity of the tournament can be reduced to the question of quadrangularity in a strong subtournament, or a bound on the domination number of a subtournament. We then look at cases where we have restrictions on the out-degrees and in-degrees of the tournament. We finish the chapter by characterizing the orders for which a quadrangular tournament exists.

2.2 Tournaments which are not strongly connected

Our first result of this section classifies quadrangular tournaments with a transmitter and a receiver by a bound on the domination number of a subtournament.

Theorem 2.1 *Let T be a tournament on 3 or more vertices with a transmitter s and receiver t. Then T is quadrangular if and only if both $\gamma(T-\{s,t\}) > 2$ and $\gamma((T-\{s,t\})^r) > 2$.*

Proof: Let T be a tournament with a transmitter s and receiver t. Suppose that both $\gamma(T-\{s,t\}) > 2$ and $\gamma((T-\{s,t\})^r) > 2$. Then, $E(\mathrm{dom}(T-\{s,t\})) = E(\mathrm{dom}((T-\{s,t\})^r)) = \emptyset$. Thus the competition graphs of both $T-\{s,t\}$ and $(T-\{s,t\})^r$ are complete. That is, for all $x, y \in V(T)-\{s,t\}$ there exist $w, z \in V(T)$ such that $w \to x$, $w \to y$, $x \to z$ and $y \to z$. Pick distinct $u, v \in V(T)$. We consider three cases.

Case 1: Suppose $u, v \notin \{s,t\}$. Then, as noted before, there exist vertices $w, z \in V(T-\{s,t\})$ so that $z \in O(u)\cap O(v)$ and $w \in I(u)\cap I(v)$. Also, $s \in O(u)\cap O(v)$ and $t \in I(u)\cap I(v)$. So, $|O(u)\cap O(v)| \geq 2$ and $|I(u)\cap I(v)| \geq 2$.

Case 2: Now assume that one of u, or v is t, say $u = t$. Since $O(t) = \emptyset$, $O(t)\cap O(v) = \emptyset$, so $|O(t)\cap O(v)| = 0$. Also, $I(t) = V(T)-\{t\}$, so $I(t)\cap I(v) = I(v)$. If $v = s$, then $I(v) = \emptyset$, and $|I(t)\cap I(v)| = 0$. So, suppose $v \neq s$. Since

$\gamma(T - \{s,t\}) \geq 3$, Theorem 1.6 gives us that $2 \leq \gamma(T - \{s,t\}) \leq \delta^-(T - \{s,t\})$. Thus, $d^-(v) \geq 2$, and since $I(v) \subseteq I(t)$, $|I(t) \cap I(v)| = d^-(v) \geq 2$ as desired.

Case 3: Now, assume that one of u, v is s, say $u = s$. Since $I(s) = \emptyset$, $I(s) \cap I(v) = \emptyset$, so $|I(s) \cap I(v)| = 0$. Also, since $O(s) = V(T) - \{s\}$, $O(s) \cap O(v) = O(v)$. The case where $v = t$ is covered in case 2, so assume $v \neq t$. Since $\gamma((T - \{s,t\})^r) \geq 3$, Theorem 1.6 gives us that $2 \leq \gamma((T - \{s,t\})^r) \leq \delta^-((T - \{s,t\})^r) = \delta^+(T - \{s,t\})$. Thus, $d^+(v) \geq 2$, and since s is a transmitter, $O(v) \subseteq O(s)$, so $|O(s) \cap O(v)| = d^+(v) \geq 2$.

Conversely assume that T is a quadrangular tournament with both a transmitter s and receiver t. If $u, v \in V(T) - \{s,t\}$, then $s \in O(u) \cap O(v)$, and $t \in I(u) \cap I(v)$. So, since T is quadrangular, $|O(u) \cap O(v)| \geq 1$ and $|I(u) \cap I(v)| \geq 1$, there must exist vertices w, z in $T - \{s,t\}$ such that $z \in O(u) \cap O(v)$ and $w \in I(u) \cap I(v)$. Since w beats u and v they cannot be a dominant pair in $T - \{s,t\}$ and since u and v beat z they cannot be a dominant pair in $(T - \{s,t\})^r$. Thus, $E(\text{dom}(T - \{s,t\})) = E(\text{dom}((T - \{s,t\})^r)) = \emptyset$. Equivalently, $\gamma(T - \{s,t\}) > 2$ and $\gamma((T - \{s,t\})^r) > 2$. This completes our proof. ■

From Theorem 1.6, and Theorem 2.1 we get the following corollary which allows for a quick check to see if a tournament with both a transmitter and a receiver is quadrangular.

Corollary 2.2 *Let T be a tournament on 3 or more vertices with a transmitter s and receiver t. If T is quadrangular, then $\delta^+(T - \{s,t\}) \geq 2$ and $\delta^-(T - \{s,t\}) \geq 2$.*

The following results characterize quadrangular tournaments in the case where the tournament has a transmitter or receiver, but not both, by a bound on the domination number of a subtournament.

Theorem 2.3 *Let T be a tournament with a transmitter s and no receiver. Then T is quadrangular if and only if, $\gamma(T-s) > 2$, $T-s$ is out-quadrangular, and $\delta^+(T-s) \geq 2$.*

Proof: First suppose that $\gamma(T-s) > 2$, $T-s$ is out-quadrangular, and $\delta^+(T-s) \geq 2$. Pick distinct $u, v \in V(T)$. First suppose that $u, v \in V(T) - \{s\}$. Since $\gamma(T-s) > 2$ there exists $x \in V(T) - \{s\}$ such that $x \to u$ and $x \to v$. So, $s, x \in I(u) \cap I(v)$ and so $|I(u) \cap I(v)| \geq 2$. Also, since $T-s$ is out-quadrangular, $|O(u) \cap O(v)| \neq 1$. Now, suppose that one of u, v is s, say $u = s$. Since $I(s) = \emptyset$, $|I(s) \cap I(v)| = 0$. Also, since $\delta^+(T-s) \geq 2$, $|O(s) \cap O(v)| = |O(v)| \geq 2$, Thus, $T-s$ is quadrangular as desired.

Now, assume that T is quadrangular. Since $O(s) = V(T) - \{s\}$, $|I(u) \cap I(v)| \geq 1$ for all distinct $u, v \in V(T) - \{s\}$. Since T is quadrangular this means we must have $|I(u) \cap I(v)| \geq 2$ for each $u, v \in V(T) - \{s\}$. Thus, for all $u, v \in V(T) - \{s\}$, there must exist some $x \in V(T) - \{s\}$ such that $x \to u$ and $x \to v$. So, $\gamma(T-s) > 2$. Since T has no receiver, $|O(v)| \geq 1$ for all $v \in V(T)$. Since $O(s) \cap O(v) = O(v)$ for all $v \in V(T) - \{s\}$, and T is quadrangular, we must then have that $d^+(v) = |O(v)| = |O(s) \cap O(v)| \geq 2$. Thus, $\delta^+(T-s) \geq 2$. Now, pick distinct $u, v \in V(T) - \{s\}$. Since T is quadrangular, $|O(u) \cap O(v)| \neq 1$, so $T-s$ is out-quadrangular. ∎

If T is a tournament with a receiver and no transmitter, then it is the dual of a tournament with a transmitter and no receiver. A tournament is quadrangular if and only if its dual is, so by Theorem 2.3, T is quadrangular if and only if $\gamma((T-t)^r) > 2$, $(T-t)^r$ is out-quadrangular and $\delta^+((T-t)^r) \geq 2$. Since $(T-t)^r$ being out-quadrangular is equivalent to $T-t$ being in-quadrangular, and $\delta^+((T-t)^r) = \delta^-(T-t)$ we get the following corollary.

Corollary 2.4 *Let T be a tournament with a receiver t and no transmitter. Then T is quadrangular if and only if $\gamma((T-t)^r) > 2$, $T-t$ is in-quadrangular, and $\delta^-(T-t) \geq 2$.*

Theorem 1.6 together with Theorem 2.3 and Corollary 2.4 give analogous results to Corollary 2.2. Namely that if T is a quadrangular tournament with a transmitter s and no receiver (receiver t and no transmitter), then $\delta^+(T-s) \geq 2$ and $\delta^-(T-s) \geq 2$ ($\delta^+(T-t) \geq 2$ and $\delta^-(T-t) \geq 2$). We finish this section by characterizing quadrangular tournaments with no transmitter or receiver which are not strongly connected in terms of properties of their initial and terminal strong components.

Theorem 2.5 *Let T be a tournament with no transmitter or receiver which is not strongly connected. Then T is quadrangular if and only if the initial strong component T_1 is in-quadrangular with $\delta^-(T_1) \geq 2$ and the terminal strong component T_m is out-quadrangular with $\delta^+(T_m) \geq 2$.*

Proof: Let T be a tournament with no transmitter or receiver, which is not strongly connected. Suppose that T_1 is in-quadrangular with $\delta^-(T_1) \geq 2$, and that T_m is out-quadrangular with $\delta^+(T_m) \geq 2$. Note that since T has no transmitter or receiver, T_1 and T_m must contain at least 3 vertices each. Pick distinct $u, v \in V(T)$. We consider 5 cases.

Case 1: Suppose that u and v are in neither T_1 nor T_m. Every vertex of T_1 beats every vertex in $T - T_1$ and every vertex of T_m is beaten by every vertex of $T - T_m$. So,

$$|O(u) \cap O(v)| \geq |V(T_m)| \geq 3 \text{ and } |I(u) \cap I(v)| \geq |V(T_1)| \geq 3.$$

Case 2: Suppose that both $u, v \in T_1$. Then, since T_1 is in-quadrangular, $|I(u) \cap I(v)| \neq 1$. Also, u and v beat every vertex in $T - T_1$, in particular, $V(T_m) \subseteq O(u) \cap O(v)$. Thus,

$$|O(u) \cap O(v)| \geq |V(T_m)| \geq 3.$$

Case 3: Suppose that both $u, v \in V(T_m)$. Then, since T_m is out-quadrangular, $|O(u) \cap O(v)| \neq 1$. Also, $|I(u) \cap I(v)| \geq |V(T_1)| \geq 3$.

Case 4: Suppose that $u \in V(T_1)$ and $v \notin V(T_1)$. Since $v \notin V(T_1)$ we know that $I(u) \subseteq V(T_1) \subseteq I(v)$ and so $I(u) \cap I(v) = I(u)$. So, since $\delta^-(T_1) \geq 2$, $|I(u) \cap I(v)| = |I(u)| = d^-(u) \geq 2$. Also, since $u \in V(T_1)$ and $v \notin V(T_1)$, we know that $O(v) \subseteq O(u)$. Thus, $O(u) \cap O(v) = O(v)$. If $v \notin V(T_m)$, then $V(T_m) \subseteq O(v)$, and so $|O(u) \cap O(v)| \geq |V(T_m)| \geq 3$. So, assume that $v \in V(T_m)$. Then $|O(u) \cap O(v)| = |O(v)| = d^+(v) \geq 2$.

Case 5: Suppose that $u \in V(T_m)$ and $v \notin V(T_m)$. Since $v \notin V(T_m)$, $O(u) \subseteq V(T_m) \subseteq O(v)$, and so $O(u) \cap O(v) = O(u)$. So, since $\delta^+(T_m) \geq 2$, $|O(u) \cap O(v)| = |O(v)| \geq 2$. Now, if $v \in V(T_1)$ then as in case 4, $|I(u) \cap I(v)| \geq 2$. So, assume that $v \notin V(T_1)$. Then, every vertex in T_1 beats both u and v, and so $|I(u) \cap I(v)| \geq |V(T_1)| \geq 3$.

Now, assume that T is quadrangular. Since T is quadrangular, if $u, v \in V(T_1)$, then $|I(u) \cap I(v)| \neq 1$, so T_1 is in-quadrangular. Also, since T is quadrangular, if $u, v \in V(T_m)$ then $|O(u) \cap O(v)| \neq 1$, and so T_m is out-quadrangular. Now, pick $u \in V(T_1)$ and $v \in V(T_m)$. Then, $O(u) \cap O(v) = O(v)$, and $I(u) \cap I(v) = I(u)$. Since T has no receiver, $d^+(v) \geq 1$, and so we must have that $d^+(v) = |O(v)| = |O(u) \cap O(v)| \geq 2$. Thus, $\delta^+(T_m) \geq 2$. Also, since T has no transmitter, $d^-(u) \geq 1$, and so $d^-(u) = |I(u)| = |I(u) \cap I(v)| \geq 2$. Thus, $\delta^-(T_1) \geq 2$. These are exactly the conditions from the theorem statement, and so the result follows. ■

In studying tournaments which are quadrangular, the previous results allow us to restrict our attention to strongly connected tournaments. Beyond this, as we will see in Chapter 4, a necessary condition for a tournament to be the digraph of an orthogonal matrix is to be strongly connected.

2.3 Tournaments with given minimum degrees

We now look at tournaments with a given minimum out-degree or in-degree. In the previous section we studied tournaments with minimum out-degree or in-degree 0. In Theorem 2.8 and Corollary 2.9 we characterize quadrangular tournaments with minimum in-degree 1 or minimum out-degree 1 respectively. The following results show that no tournament with a vertex of out-degree or in-degree 2 or 3 can be quadrangular. First we give some lemmas.

Lemma 2.6 *Let T be a quadrangular tournament with a vertex x of out-degree 1. Suppose $x \to y$,then $O(y) = V(T) - \{x, y\}$*

Proof: Suppose there exists a vertex v in $V(T) - \{x, y\}$ such that $v \to y$. Then, since $O(x) = \{y\}$, $|O(x) \cap O(v)| = |\{y\}| = 1$. This contradicts the quadrangularity of T. ■

Applying Lemma 2.6 to the dual of T we obtain the following lemma.

Lemma 2.7 *Let T be a quadrangular tournament with a vertex x of in-degree 1. Suppose $y \to x$, then $I(y) = V(T) - \{x, y\}$.*

Theorem 2.8 *Let T be a tournament on 4 or more vertices with a vertex x of out-degree 1, and suppose $x \to y$. Then, T is quadrangular if and only if*

1. $O(y) = V(T) - \{x, y\}$,
2. $\gamma(T - \{x, y\}) > 2$,
3. $\gamma((T - \{x, y\})^r) > 2$.

Proof: First, suppose that T is quadrangular. Then, by Lemma 2.6, $O(y) = V(T) - \{x, y\}$. Now, pick distinct vertices u and$v$ in $T - \{x, y\}$. Since $x \in O(u) \cap O(v)$ there must exist some other vertex w in $T - x$ for which $w \in O(u) \cap O(v)$. Since $O(y) = V(T) - \{x, y\}$, this vertex w must be in $T - \{x, y\}$. So, there exists $w \in V(T) - \{x, y\}$ such that $w \in O(u) \cap O(v)$. This is equivalent to saying $\gamma((T - \{x, y\})^r) > 2$. Also, $y \in I(u) \cap I(v)$. So, since T is quadrangular, there must exist a vertex z in $T - y$ such that $z \in I(u) \cap I(v)$. Since $O(x) = \{y\}$, this vertex must be in $T - \{x, y\}$. So, we must also have that $\gamma(T - \{x, y\}) > 2$. Now, if $v \in V(T) - \{x, y\}$ then $I(v) \cap I(x) = I(v)$. So, these conditions are necessary.

Now assume that T is a tournament with a vertex x such that $O(x) = y$, and $O(y) = V(T) - \{x, y\}$, $\gamma(T - \{x, y\}) > 2$, and $\gamma((T - \{x, y\})^r) > 2$. Pick distinct $u, v \in V(T)$. We show T is quadrangular using three cases.

Case 1: Suppose $u, v \in V(T) - \{x, y\}$. Then, $x \in O(u) \cap O(v)$, and since $\gamma((T - \{x, y\})^r) > 2$, there exits $w \in V(T) - \{x, y\}$ such that $w \in O(u) \cap O(v)$. Thus, $|O(u) \cap O(v)| > 1$. Also, $y \in I(u) \cap I(v)$, and since $\gamma(T - \{x, y\}) > 2$ there exists $z \in V(T) - \{x, y\}$ such that $z \in I(u) \cap I(v)$. So, $|I(u) \cap I(v)| > 1$.

Case 2: Suppose that $u = x$. Then $O(u) = \{y\}$ and since $y \notin O(v)$, $|O(u) \cap O(v)| = 0$. Now, $I(u) \cap I(v) = I(v) - \{y\}$ since $u = x$. By Theorem 1.6, $\delta^-(T - \{x, y\}) \geq \gamma(T - \{x, y\}) - 1 \geq 2$, and so $|I(u) \cap I(v)| = |I(v) - \{y\}| \geq 2$.

Case 3: Suppose that $u = y$. Then, $I(u) = \{x\}$ and since $x \notin I(v)$, $|I(u) \cap I(v)| = 0$. Now, $O(u) \cap O(v) = O(v) - \{x\}$ since $u = y$. By Theorem 1.6, $\delta^+(T - \{x, y\}) \geq \gamma((T - \{x, y\})^r) - 1 \geq 2$, and so $|O(u) \cap O(v)| = |O(v) - \{x\}| \geq 2$. Thus, T is quadrangular. ■

Applying Theorem 2.8 to the dual of T we obtain the following corollary.

Corollary 2.9 *Let T be a tournament with a vertex x with in-degree 1. Suppose $y \to x$. Then, T is quadrangular if and only if $I(y) = V(T) - \{x, y\}$, $\gamma(T - \{x, y\}) > 2$, and $\gamma((T - \{x, y\})^r) > 2$.*

Theorem 1.6 together with Theorem 2.8 and Corollary 2.9 gives us the following corollary.

Corollary 2.10 *Let T be a quadrangular tournament with $\delta^+(T) = \delta^-(T) = 1$. Let x and y be vertices of out-degree and in-degree 1, respectively. Then $x \to y$,*

$O(x) = I(y) = V(T) - \{x, y\}$, $\delta^+(T - \{x, y\}) \geq 2$ *and* $\delta^-(T - \{x, y\}) \geq 2$.

Theorem 2.11 *Let T be an out-quadrangular tournament and choose $v \in V(T)$. Then $T[O(v)]$ contains no vertices of out-degree 1.*

Proof: Let T be an out-quadrangular tournament, and choose a vertex $v \in V(T)$. If $x \in O_T(v)$, then $O_T(v) \cap O_T(x) = O_{T[O(v)]}(x)$ and so since T is out-quadrangular, $d^+_{T[O(v)]}(x) = |O_{T[O(v)]}(x)| = |O_T(v) \cap O_T(x)| \neq 1$. ■

Applying Theorem 2.11 to the dual of a tournament we get the following theorem.

Theorem 2.12 *Let T be an in-quadrangular tournament and choose $v \in V(T)$. Then $T[I(v)]$ contains no vertices of in-degree 1.*

The only tournaments on 2 or 3 vertices are the single arc, the 3-cycle and the transitive triple, each of which contain a vertex of out-degree 1 and a vertex of in-degree 1. Therefore, Theorems 2.11 and 2.12 give us the following four corollaries.

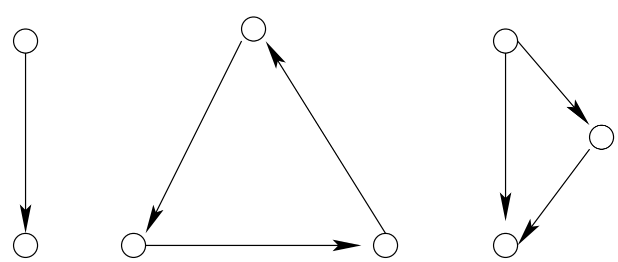

Figure 2.2: The 2-tournament, the 3-cycle, and the transitive triple.

Corollary 2.13 *If T is a quadrangular tournament, and $v \in V(T)$, then $d^+(v) \neq 2,3$ and $d^-(v) \neq 2,3$.*

Corollary 2.14 *If T is an out-quadrangular tournament with $\delta^+(T) \geq 2$, then $\delta^+(T) \geq 4$.*

Corollary 2.15 *If T is an in-quadrangular tournament with $\delta^-(T) \geq 2$, then $\delta^-(T) \geq 4$.*

Corollary 2.16 *If T is a quadrangular tournament with $\delta^+(T) \geq 2$ and $\delta^-(T) \geq 2$, then $\delta^+(T) \geq 4$ and $\delta^-(T) \geq 4$.*

We now turn our attention to regular and rotational tournaments.

2.4 Regular and rotational tournaments

In this section we look at regular tournaments and rotational tournaments, and how this affects quadrangularity. We see that regularity makes the job of determining whether or not a tournament is quadrangular a bit easier. We give a sufficient condition for regular tournaments to be quadrangular based on the domination number. We also restate the problem of whether or not a rotational tournament is quadrangular in a more number theoretic context.

Proposition 2.17 *Let T be a tournament on n vertices, then T is in-quadrangular if and only if for all distinct $u, v \in V(T)$, $|O[u] \cup O[v]| \neq n-1$.*

Proof: Note that since T is a tournament $I(x) = V(T) - O[x]$ for all $x \in V(T)$. Since T is in-quadrangular if and only if $|I(u) \cap I(v)| \neq 1$ for all distinct $u, v \in V(T)$, we have that T is in-quadrangular if and only if for all distinct $u, v \in V(T)$

$$\begin{aligned}1 &\neq |I(u) \cap I(v)| \\ &= |(V(T) - O[u]) \cap (V(T) - O[v])| \\ &= |V(T) - (O[u] \cup O[v])| \\ &= n - |O[u] \cup O[v]|.\end{aligned}$$

Thus, T is in-quadrangular if and only if $|O[u] \cup O[v]| \neq n - 1$ for all $u \neq v \in V(T)$. ■

From this proposition, we can restate the property of quadrangularity in tournaments as "for all distinct $u, v \in V(T)$, $|O(u) \cap O(v)| \neq 1$ and $|O[u] \cup O[v]| \neq n - 1$."

Theorem 2.18 *A regular tournament is quadrangular if and only if it is out-quadrangular or in-quadrangular.*

Proof: Let T be a regular tournament on $n = 2k + 1$ vertices. Note that for any two distinct vertices x and y in T, $|O[x] \cap O[y]| = |O(x) \cap O(y)| + 1$ since either $x \to y$ or $y \to x$. Thus for any two distinct $x, y \in V(T)$,

$$\begin{aligned}|O[x] \cup O[y]| &= 2k + 2 - |O[x] \cap O[y]| \\ &= n + 1 - |O[x] \cap O[y]| \\ &= n + 1 - 1 - |O(x) \cap O(y)| \\ &= n - |O(x) \cap O(y)|.\end{aligned}$$

Therefore, $|O[x] \cup O[y]| = n - 1$ if and only if $|O(x) \cap O(y)| = 1$. Thus, T is out-quadrangular if and only if it is in-quadrangular, and so it is quadrangular if and only if out-quadrangular or in-quadrangular. ■

The previous theorem can also be taken as a corollary to the linear algebra folk lore that regular tournament matrices are equivalent to the the class of tournament matrices which are normal. Note, a matrix M is *normal* if $MM^\top = M^\top M$. Graph theoretically, this is equivalent to saying that $|O(u) \cap O(v)| = |I(u) \cap I(v)|$ for any two vertices u, v. A proof of this result can be found in Appendix A.

Theorem 2.19 *If T is a regular tournament with $\gamma(T) \geq 4$, then T is out-quadrangular.*

Proof: Let T be a regular tournament on $2k+1$ vertices with $\gamma(T) \geq 4$. Assume, to the contrary ,that T is not out-quadrangular. Then there exist distinct $u, v \in V(T)$ such that $|O(u) \cap O(v)| = 1$. Let w be the single vertex in $O(u) \cap O(v)$, and without loss of generality assume that $u \to v$. So, $|O[u] \cup O[v]| = 2k$ since $O(u) \cap O(v) = \{w\}$ and $u \to v$. Thus, there is only one vertex in $T - \{u, v\}$ which is not dominated by u or v, call it x. Then every vertex in T is either one of u, v, x or dominated by one of u, v, x. Hence $\{u, v, x\}$ is a dominating set of order 3 in T. This contradicts our assumption that $\gamma(T) \geq 4$. Thus, T is out-quadrangular. ∎

Applying Theorem 2.18, we get the following corollary.

Corollary 2.20 *If T is a regular tournament with $\gamma(T) \geq 4$, then T is quadrangular.*

We denote by U_n the rotational tournament whose symbol is $\{1 \leq i \leq n-1 : i \text{ is odd}\}$. In [44], Merz, Fisher, Lundgren, and Reid show that if a tournament on n vertices has an n-cycle as its domination graph, then it is isomorphic to U_n. We use this fact to prove the following result, which helps us obtain our main result on rotational tournaments.

Theorem 2.21 *Let T be a rotational tournament on $n = 2k+1$ vertices. If T is not isomorphic to U_n, then for all distinct $u, v \in V(T)$, $O(u) \cap O(v) \neq \emptyset$.*

Proof: Let T be a rotational tournament with $V(T) = \{0, 1, \ldots, 2k\}$. Assume there exist distinct vertices $u, v \in V(T)$ such that $O(u) \cap O(v) = \emptyset$. By the vertex transitivity of rotational tournaments, we may assume $u = 0$. Now, since T is regular and $O(u) \cap O(v) = \emptyset$, $|O[u] \cup O[v]| = 2k+2-1-0 = 2k+1 = n$. Thus, u and v form a dominant pair. Since T is rotational, $O(u+i) \cap O(v+i) = \emptyset$ for all i. This says that $\{i, i+v \pmod{n}\}$ forms a dominant pair for all i. This implies the domination graph of T is a cycle. Thus T is isomorphic to U_n. ■

Pick $n \geq 5$. Then for $0, \frac{n-3}{2} \in V(U_n)$, $|O(0) \cap O(\frac{n-3}{2})| = 1$. So, U_n is not quadrangular for any $n \geq 5$, and we get the following corollary.

Corollary 2.22 *If T is an n-tournament, for $n \geq 5$, which is both rotational and quadrangular, then $O(u) \cap O(v) \neq \emptyset$ for all $u, v \in V(T)$.*

Theorem 2.23 *Let T be a rotational tournament on $n \geq 5$ vertices, with $V(T) = \{0, 1, \ldots, n-1\}$ and symbol S. Then, T is quadrangular if and only if for all integers m with $1 \leq m \leq \frac{n-1}{2}$ there exist distinct subsets $\{i, j\}, \{k, l\} \subseteq S$ such that $(i - j) \equiv (k - l) \equiv m \pmod{n}$.*

Proof: Pick $u \neq 0 \in V(T)$, and suppose S has the property stated in the theorem. We show that $|O(u) \cap O(0)| \neq 1$. Then for $x \in V(T)$ we can use the vertex transitivity of T to map x to 0, and as u is arbitrarily chosen, for any vertex $y \neq x$ we will have $|O(x) \cap O(y)| = |O(0) \cap O(u)| \neq 1$. So T will be out-quadrangular and hence quadrangular, since T is regular. Now, if $u \leq \frac{n-1}{2}$ there exist sets $\{i, j\}, \{k, l\} \subseteq S$ such that $(i - j) \equiv (k - l) \equiv u \pmod{n}$. So, $i - u \equiv j \pmod{n}$ and $k - u \equiv l \pmod{n}$ Thus, $j, l \in O(u)$. Further, $j, l \in O(0)$ since $j, l \in S$. Note $j \neq l$, for otherwise $i = k$, contradicting $\{i, j\}$ and $\{k, l\}$ being distinct sets. Thus, $|O(u) \cap O(0)| \geq 2$. If $u \geq \frac{n-1}{2}$ then $-u \leq \frac{n-1}{2}$ and so there exist sets $\{i, j\}, \{k, l\} \subseteq S$ such that $(i - j) \equiv (k - l) \equiv -u \pmod{n}$. So, $(j - i) \equiv (l - k) \equiv u \pmod{n}$, and the argument is the same. Thus T is quadrangular.

Now assume that T is quadrangular. Then by Corollary 2.22, $|O(u) \cap O(v)| \geq 1$ for all distinct $u, v \in V(T)$. Thus, for all distinct $u, v \in V(T)$, we must have that $|O(u) \cap O(v)| \geq 2$. In particular, for some $m \in V(T)$ such that $1 \leq m \leq \frac{n-1}{2}$, we must have that $|O(0) \cap O(m)| \geq 2$. Since $O(0) = S$, there must be at least 2 elements of S say j, l such that $j, l \in O(m)$. So, there must exist $i, k \in S$ such that $i - m \equiv j \pmod{n}$ and $k - m \equiv l \pmod{n}$. This makes $\{i, j\}$ and $\{k, l\}$ the sets stated in the theorem, and completes our proof. ■

This theorem lets us restate the existence question for quadrangular rotational n-tournaments, $n \geq 5$, as the following:

For which odd integers n does there exist a set of size $\frac{n-1}{2}$ such that if $i \in S$, $-i \pmod{n} \notin S$ and for all integers $1 \leq m \leq \frac{n-1}{2}$, there exist distinct sets $\{i, j\}, \{k, l\} \subseteq S$ such that $(i - j) \equiv (k - l) \equiv m \pmod{n}$?

The smallest such n is 11 with $S = \{1, 3, 4, 5, 9\}$. In fact, one can generalize this set and verify that for $n \equiv 3 \pmod 4$ the set

$$S = \left\{ i : 1 \leq i \leq n-2, i \text{ is odd }, i \neq \left(\frac{n+3}{2}\right) \right\} \cup \left\{ \left(\frac{n-3}{2}\right) \right\}$$

is the symbol for a quadrangular rotational tournament.

2.5 Known orders of quadrangular tournaments

In this section we determine for exactly which n there exists a quadrangular tournament on n vertices. As we see in the proof of Theorem 2.34, the results from sections 2.2 and 2.3 show us there are no quadrangular tournaments on $4, 5, 6, 7$ or 8 vertices. We give constructions for quadrangular tournaments on $9, 11, 12, 13,$ and 14 vertices and a general construction for quadrangular tournaments on 15 or more vertices. We also show that there is no quadrangular tournament on 10 vertices in a series of results.

Before beginning, recall that up to isomorphism, there are only four 4-tournaments. These are shown in Figure 2.3. Of these, the only tournament on 4 vertices with no vertex of out-degree 1 is a 3-cycle together with a receiver. Similarly, the only tournament on 4 vertices with no vertex of in-degree 1 is a 3-cycle with a transmitter. Thus, by Theorem 2.11, if a quadrangular tournament T has a vertex v of out-degree 4, $T[O(v)]$ must be a 3-cycle with a receiver, and if u has in-degree 4, $T[I(u)]$ must be a 3-cycle with a transmitter.

Theorem 2.24 *There does not exist a quadrangular near regular tournament of order* 10.

Proof: Suppose T is such a tournament and pick a vertex x with $d^+(x) = 5$. So $d^-(x) = 4$. Therefore $I(x)$ must induce a subtournament comprised of a 3-cycle, and a transmitter. Call this transmitter u. If a vertex y in $O(x)$ has

$O(y) = I(x)$, then $|O(y) \cap O(w)| = 1$ for all $w \neq u$ in $I(x)$. This contradicts T being quadrangular, so $O(y) \neq I(x)$ for any $y \in O(x)$. Since every vertex in $O(x)$ beats at most 3 vertices outside of $O(x)$, and since T is near regular we have that $\delta^+(T[O(x)]) \geq 1$. Thus, by Theorem 2.11, we have $\delta^+(T[O(x)]) \geq 2$. This means that $T[O(x)]$ must be a regular tournament on 5 vertices.

Consider the vertex u which forms the transmitter in $T[I(x)]$. Since $u \Rightarrow I[x] - \{u\}$, and T is near regular, u can beat at most one vertex in $O(x)$. If $u \rightarrow z$ for any $z \in O(x)$, then $|O(u) \cap O(x)| = |\{z\}| = 1$ which contradicts T being quadrangular. Thus, $z \rightarrow u$ for all $z \in O(x)$.

Since T is near regular, it has exactly 5 vertices of out-degree 5, one of which is x. So, there can be at most 4 vertices in $O(x)$ with out-degree 5. Thus, there exists some vertex in $O(x)$ with out-degree 4, call it v. Since $x \rightarrow v$, v beats 2 vertices in $O(x)$ and $v \rightarrow u$ there is exactly one vertex $r \in I(x) - \{u\}$ such that $v \rightarrow r$. Since $O(u) = I[x] - \{u\}$, we have $|O(v) \cap O(u)| = |\{r\}| = 1$. Therefore, T is not quadrangular, and such a tournament does not exist. ■

The following lemmas and theorem on domination are necessary in showing that a quadrangular 10-tournament does not exist.

Lemma 2.25 *If T is a tournament on 8 vertices with $\gamma(T) \geq 3$ and $\gamma(T^r) \geq 3$, then T is near regular. Further, if $d^-(x) = 3$, then $I(x)$ induces a 3-cycle, and if $d^+(y) = 3$, then $O(y)$ induces a 3-cycle.*

Proof: Let T be such a tournament. By Theorem 1.6, $\delta^-(T) \geq \gamma(T) - 1 = 3 - 1 = 2$. If T has a vertex b with $I(b) = \{u, v\}$, where $u \rightarrow v$, then $\{u, b\}$ forms a dominating set of size 2. So $d^-_T(x) \geq 3$ for all $x \in V(T)$. Similarly, $d^-_{T^r}(x) \geq 3$

for all $x \in V(T)$. Thus,

$$3 \leq d^-_{T^r}(x) = d^+_T(x) = 8 - 1 - d^-_T(x) \leq 7 - 3 = 4$$

for all $x \in V(T)$. That is $3 \leq d^+_T(x) \leq 4$ for all $x \in V(T)$, and T is near regular. Now, pick $x \in V(T)$ with $d^-(x) = 3$. If $I(x)$ induces a transitive triple with transmitter u, then $\{u, x\}$ would form a dominating set in T. Thus, $I(x)$ must induce a 3-cycle. By duality we have that $O(y)$ induces a 3-cycle for all y with $d^+(y) = 3$. ■

The 4 tournaments on 4 vertices are shown in Figure 2.3. Exactly one of these is strongly connected. We refer to this tournament as the *strong 4-tournament,* and note that it is also the only tournament on 4 vertices without a vertex of out-degree 3 or 0.

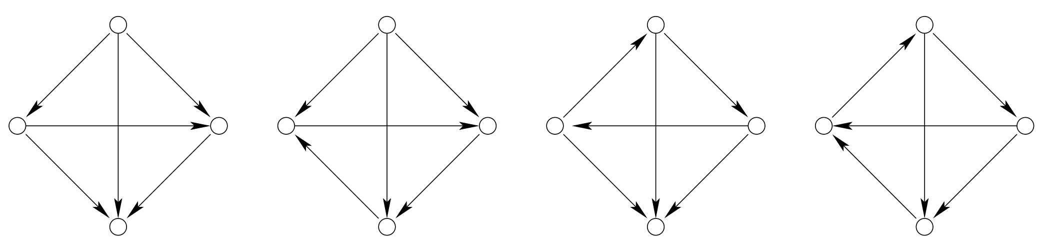

Figure 2.3: The four 4-tournaments with the strong 4-tournament on the far right.

Lemma 2.26 *Suppose T is a tournament on 8 vertices with $\gamma(T) \geq 3$ and $\gamma(T^r) \geq 3$. Then if $x \in V(T)$ with $d^+(x) = 4$, $O(x)$ induces the strong 4-tournament.*

Proof: By Lemma 2.25, T is near regular so pick $x \in V(T)$ with $d^+(x) = 4$, and let W be the subtournament induced on $O(x)$. If there exists $u \in V(W)$

with $d^+_W(u) = 0$, then since $d^+_T(u) \geq 3$, $u \Rightarrow I(x)$ and $\{u, x\}$ forms a dominating set in T. This contradicts $\gamma(T) \geq 3$, so no such u exists. Now assume there exists a vertex $v \in V(W)$ with $d^+_W(v) = 3$. If $d^+_T(v) = 4$, then $v \to y$ for some $y \in I(x)$. So, $I(v) = I[x] - \{y\}$. However, $I(v) = I[x] - \{y\}$ forms a transitive triple, a contradiction to Lemma 2.25. So $d^+_T(v) = 3$. Now, since $\delta^+(W) > 0$, the vertices of $W - v$ all have out-degree 1 in W. If some $z \in V(W) - v$ has $d^+_T(z) = 4$, then $z \Rightarrow I(x)$ and $\{x, z\}$ would form a dominating set of size 2. Therefore, all $z \in V(W)$ have $d^+_T(z) = 3$. Since T is near regular, this implies that every vertex of $I[x]$ must have out-degree 4. Further, since $d^+_T(v) = 3$, $O(v) \subseteq O(x)$ and so $I(x) \Rightarrow v$. So, each vertex of $I(x)$ dominates x, v and another vertex of $I(x)$. Thus, each vertex of $I(x)$ dominates a unique vertex of $O(x) - \{v\}$. Further each vertex of $O(x) - \{v\}$ has out-degree 3 in T and so must be dominated by a unique vertex of $I(x)$. Label the vertices of $I(x)$ as y_1, y_2, y_3 and the vertices of $O(x) - \{v\}$ as w_1, w_2, w_3 so that $y_i \to w_i$, and $w_i \to y_j$ for $i \neq j$. Since $I(x)$ and $O(x) - \{v\}$ form 3-cycles we may also assume that $y_1 \to y_2 \to y_3$, $y_3 \to y_1$, and $w_1 \to w_2 \to w_3$ and $w_3 \to w_1$. So, $O(w_1) = \{w_2, y_2, y_3\}$ which forms a transitive triple, a contradiction to Lemma 2.25. Hence, no such v exists and $1 \leq \delta^+(W) \leq \Delta^+(W) \leq 2$ and W is the strong 4-tournament. ■

Theorem 2.27 *Let T be a tournament on 8 vertices. Then $\gamma(T) \leq 2$ or $\gamma(T^r) \leq 2$.*

Proof: Suppose to the contrary that T is a tournament on 8 vertices with $\gamma(T) \geq 3$ and $\gamma(T^r) \geq 3$. By Lemma 2.25 we know that T is near regular. Let W be the subtournament of T induced on the vertices of out-degree 4. Choose

x in W with $d^-_W(x) \geq 2$. So pick $x \in V(T)$ with $d^+_T(x) = 4$ so that it dominates at most one vertex of out-degree 4. By Lemma 2.26, $O(x)$ induces the strong 4-tournament. By our choice of x, at least one of the vertices with out-degree 2 in $T[O(x)]$ has out-degree 3 in T. Call this vertex x_1. Label the vertices of $O(x_1) \cap O(x)$ as x_2 and x_3 so that $x_2 \to x_3$, and label the remaining vertex of $O(x)$ as x_0. Note since $T[O(x)]$ is the strong 4-tournament, we must have $x_3 \to x_0$ and $x_0 \to x_1$. Since $d^+_T(x_1) = 3$, x_1 must dominate exactly one vertex in $I(x)$, call it y_1. Recall $I(x)$ must induce a 3-cycle by Lemma 2.25, so we can label the remaining vertices of $I(x)$ as y_2 and y_3 so that $y_1 \to y_2 \to y_3$ and $y_3 \to y_1$. Note since $O(x_1) \cap I(x) = y_1$, $y_2 \to x_1$ and $y_3 \to x_1$. Also, by Lemma 2.25, $O(x_1)$ forms a 3-cycle, so $x_3 \to y_1$ and $y_1 \to x_2$.

Now, assume to the contrary that $y_1 \to x_0$. Then $O(y_1) = \{x_0, x_2, x, y_2\}$. Now, since $O(x_3) \cap O(x) = \{x_0\}$, $d^+_T(x_3) = 3$ or else $x_3 \Rightarrow I(x)$ and $\{x, x_3\}$ is a dominating set of size 2. So, x_3 dominates exactly one of y_2 or y_3. If $x_3 \to y_2$ then $y_3 \to x_3$ and since $y_3 \to x_1$, $\{y_1, y_3\}$ forms a dominating set of size 2. So, assume $x_3 \to y_3$ and $y_2 \to x_3$. Then $x, y_3, x_1, x_3 \in O(y_2)$, and $\{y_2, y_1\}$ forms a dominating set of size 2. Thus $x_0 \to y_1$.

If $x_3 \to y_2$, then $\{x_3, x\}$ forms a dominating set of size 2, a contradiction. So, $y_2 \to x_3$. If $x_3 \to y_3$, then $O(x_3) = \{y_1, y_3, x_0\}$. However, $y_3 \to y_1$ and $x_0 \to y_1$ so $O(x_3)$ forms a transitive triple, a contradiction to Lemma 2.25. Thus $y_3 \to x_3$. Since $d^+_T(y_3) \leq 4$ and $y_1, x, x_1, x_3 \in O(y_3)$, these are all the vertices in $O(y_3)$. So, $x_0 \to y_3$.

If $x_0 \to y_2$ then $x_0 \Rightarrow I(x)$ and $\{x, x_0\}$ is a dominating set of size 2, so $y_2 \to x_0$. So, $x_0, y_3, x \in O(y_2)$ and $y_1, x_2, x_3 \in O(x_1)$, and so $\{y_2, x_1\}$ forms a dominating set of size 2. Therefore, such a tournament cannot exist. ∎

Theorem 2.28 *No 9-tournament T with $\delta^+(T) \geq 2$ is out-quadrangular.*

Proof: Assume, to the contrary, T is such a tournament. Since T is out-quadrangular, and $\delta^+(T) \geq 2$, by Corollary 2.14, $\delta^+(T) \geq 4$. Since T is a 9-tournament, this implies T must be regular. Pick a vertex $x \in V(T)$. Then $O(x)$ must induce a subtournament which is a 3-cycle together with a receiver. Call the receiver of this subtournament y. Since T is regular, $d^+(y) = 4$. Since $I(y) = O[x] - \{y\}$, this means $O(y) = I(x)$. So, $O(y) = I(x)$ must induce a subtournament which is a 3-cycle together with a receiver. Call this receiver z. Since $d^+(z) = 4$, $y \to z$ and $I(x) - \{z\} \Rightarrow z$, we have $O(z) = O[x] - \{y\}$. Now, $x \Rightarrow O(x) - \{y\}$ and $O(x) - \{y\}$ is a 3-cycle so $T[O(z)]$ must contain a vertex of out-degree 1. Hence, by Theorem 2.11, T is not out-quadrangular. Thus no such tournament exists. ∎

Corollary 2.29 *No 9-tournament T with $\delta^-(T) \geq 2$ is in-quadrangular.*

Proof: Let T be a tournament on 9 vertices with $\delta^-(T) \geq 2$. Then T^r is not out-quadrangular by Theorem 2.28. Thus T is not in-quadrangular. ∎

Corollary 2.30 *No quadrangular tournament of order 10 exists.*

Proof: By Corollaries 2.4 and 2.16, and by Theorems 2.1, 2.3 and 2.8, a quadrangular tournament T must satisfy one of the following.

1. $\delta^+(T) \geq 4$, and hence T is near regular.

2. T has a transmitter s and receiver t such that $\gamma(T - \{s, t\}) > 2$ and $\gamma((T - \{s, t\})^r) > 2$.

3. T contains an arc (x, y) such that $O(y) = I(x) = V(T) - \{x, y\}$ and $\gamma(T - \{x, y\}) > 2$ and $\gamma((T - \{x, y\})^r) > 2$.

4. T has a transmitter s and $T - s$ is out-quadrangular with $\delta^+(T - s) \geq 2$.

5. T has a receiver t and $T - t$ is in-quadrangular with $\delta^-(T - t) \geq 2$.

Note, Theorem 2.24 implies that case 1 is impossible. If 2 or 3 were satisfied, then there would be a tournament on 8 vertices such that it and its dual have domination number at least 3, which contradicts Theorem 2.27. If 4 were satisfied, then $T - s$ would be of order 9 and out-quadrangular, a contradiction to Theorem 2.28. Similarly, 5 contradicts Corollary 2.29. Thus, no quadrangular tournament on 10 vertices exists. ■

Theorem 2.31 *There exist quadrangular tournaments of order* 11, 12 *and* 13.

Proof: Consider the quadratic residue tournament of order 11, QR_{11}. Recall QR_{11} is a doubly regular tournament, so for all $u, v \in V(QR_{11})$, $|O(u) \cap O(v)| = |I(u) \cap I(v)| = \frac{11-3}{4} = 2$. Thus, QR_{11} is quadrangular. Further, this implies that for any two vertices $u, v \in V(QR_{11})$ there exists a vertex which dominates both u and v, so $\gamma(QR_{11}) > 2$. Also, since QR_{11} is regular, $\delta^+(QR_{11}) = 5 \geq 2$. Let W be the tournament formed by adding a transmitter to QR_{11}. Then by Theorem 2.3, W is quadrangular.

Now, let T be the rotational tournament on 13 vertices with symbol $S = \{1, 2, 3, 5, 6, 9\}$. The following table gives the subsets of S which satisfy

Theorem 2.23. Thus, T is quadrangular.

m	subsets
1	$\{2,1\},\{3,2\}$
2	$\{3,1\},\{5,3\}$
3	$\{5,2\},\{6,3\}$
4	$\{6,2\},\{9,5\}$
5	$\{6,1\},\{1,9\}$
6	$\{9,3\},\{2,9\}$

■

Theorem 2.32 *There exists a quadrangular tournament of order* 14.

Proof: Construct T of order 14 in the following way. Start with a set V of 14 distinct vertices. Partition V into 7 sets of order 2 labeled $V_0, V_1, V_2, \ldots, V_6$. Place arcs in T so that each V_i induces the 2-tournament, and $V_i \Rightarrow V_j$ if and only if $j - i \pmod 7$ is one of $1, 2, 4$. We show that the resulting 14-tournament, T, is quadrangular.

Note that the condensation of T on $V_0, \ldots, V_6$ is just the quadratic residue tournament on 7 vertices, QR_7. Now, QR_7 is doubly regular, so $|O(x) \cap O(y)| = \frac{7-3}{4} = 1$ for all $x, y \in V(QR_7)$. Thus, if $u, v \in V(T)$ such that $u \in V_i$, $v \in V_j$ for $i \neq j$, $|O(u) \cap O(v)| = 2$. Further, since QR_7 is regular of degree 3, if $u, v \in V(T)$ with $u, v \in V_i$ then $|O(u) \cap O(v)| = 6$. Thus, $|O(u) \cap O(v)| \neq 1$ for all $u, v \in V(T)$, and so T is out-quadrangular. Now, since QR_7 and the 2-tournament are isomorphic to their duals, a similar argument shows that T is in-quadrangular and hence quadrangular. ■

Theorem 2.33 *If $n \geq 15$, then there exists a quadrangular tournament on n vertices.*

Proof: Pick $n \geq 15$. Let $a_1, a_2, a_3, \ldots, a_l$ be a sequence of at least 3 integers such that $a_i \geq 5$ for each i, and $\sum_{i=1}^{l} a_i = n$. Pick l regular or near regular tournaments $T_1, T_2, \ldots, T_l$ such that $|V(T_i)| = a_i$ for each i. Let T' be a tournament with $V(T') = \{1, 2, 3, \ldots l\}$ such that T' has no transmitter or receiver. Construct the tournament T on n vertices as follows. Start with a set V of n vertices, and partition V into sets $S_1, S_2, \ldots, S_l$ of size $a_1, a_2, \ldots, a_l$ respectively. Place arcs in each S_i to form T_i. Now, add arcs such that $S_i \Rightarrow S_j$ if and only if $i \to j$ in T'. We claim that the resulting tournament, T, is quadrangular.

Pick $u, v \in V(T)$. We consider two possibilities. First, suppose that $u, v \in S_i$ for some i. By choice of T', $i \to j$ for some j. Thus

$$|O(u) \cap O(v)| \geq |S_j| = a_j \geq 5 > 1.$$

Now, suppose that $u \in S_i$ and $v \in S_j$ for $i \neq j$. Since T' is a tournament either $i \to j$ or $j \to i$. Without loss of generality, assume that $i \to j$. Then

$$|O(u) \cap O(v)| \geq |O(v) \cap S_j| \geq \frac{|S_j| - 1}{2} \geq 2 > 1.$$

This shows that T is out-quadrangular. The proof that T is in-quadrangular is similar. Thus, T is a quadrangular tournament of order $n \geq 15$. ∎

Observe that if T' in the construction is strong, then T is strong. Further, if $a_i = k$ for all i and T' is regular, then T is regular or near regular depending on if k is odd or even. We now characterize those n for which there exist a quadrangular tournament of order n.

Theorem 2.34 *There exists a quadrangular tournament of order n if and only if $n = 1, 2, 3, 9$ or $n \geq 11$.*

Proof: Note that the single vertex, the single arc, and the 3-cycle are all quadrangular. Now, recall that the smallest tournament with domination number 3 is QR_7. Further, QR_7 is isomorphic to its dual, so $\gamma(QR_7^r) = 3$. This fact together with Theorems 2.8 and 2.1 tell us that the smallest quadrangular tournament, T, on $n \geq 4$ vertices with $\delta^+(T) = \delta^-(T) = 0$ or $\delta^+(T) = 1$ or $\delta^-(T) = 1$ has order 9.

Theorem 2.3 and Corollary 2.4 together with the fact that QR_7 is the smallest tournament with domination number 3 imply that a quadrangular tournament with just a transmitter or receiver must have at least 8 vertices. However, QR_7 is the only tournament on 7 vertices with domination number 3 and since $|O(x) \cap O(y)| = |I(x) \cap I(y)| = 1$ for all $x \neq y \in V(QR_7)$, QR_7 is neither out-quadrangular nor in-quadrangular. So, QR_7 together with a transmitter or receiver, but not both, is not quadrangular, and hence any quadrangular tournament with a transmitter or receiver, but not both, must have order 9 or higher.

Corollary 2.16 states that if $\delta^+(T) \geq 2$ and $\delta^-(T) \geq 2$, then $\delta^+(T) \geq 4$ and $\delta^-(T) \geq 4$. The smallest tournament which meets these requirements is a regular tournament on 9 vertices. Thus, there are no quadrangular tournaments of order $4, 5, 6, 7$ or 8. The result now follows from Corollary 2.30 and Theorems 2.31, 2.32 and 2.33. ■

As it turns out, not only do there exist quadrangular n-tournaments for

$n = 9$ and $n \geq 11$, but quadrangularity is a common (asymptotic) property in tournaments. We finish the chapter with the following probabilistic result.

Theorem 2.35 *Almost all tournaments are quadrangular.*

Proof: Let $P(n)$ denote the probability that a random tournament on n vertices contains a pair of distinct vertices x and y so that $|O(x) \cap O(y)| = 1$. We now give an over-count for the number of labeled tournaments on n vertices which contain such a pair, and show $P(n) \to 0$ as $n \to \infty$.

There are $\binom{n}{2}$ ways to pick distinct vertices x and y, and the arc between them can be oriented so that $x \to y$ or $y \to x$. There are $n-2$ vertices which can play the role of z where $\{z\} = O(x) \cap O(y)$. For each $w \notin \{x, y, z\}$ there are 3 ways to orient the arcs from x and y to w, namely $w \Rightarrow \{x, y\}$, $w \to x$ and $y \to w$, or $w \to y$ and $x \to w$. Also, there are $n-3$ such w. The arcs between all other vertices are arbitrary. So there are $2^{\binom{n-2}{2}}$ ways to finish the tournament. When orienting the remaining arcs we may double count some of these tournaments, so all together there are at most

$$2\binom{n}{2}(n-2)3^{n-3}2^{\binom{n-2}{2}}$$

tournaments containing such a pair of vertices. Now, there are $2^{\binom{n}{2}}$ total labeled tournaments so,

$$\begin{aligned}
0 \le P(n) &\le \frac{2\binom{n}{2}(n-2)3^{(n-3)}2^{\binom{n-2}{2}}}{2^{\binom{n}{2}}} \\
&= \frac{n(n-1)(n-2)3^{(n-3)}2^{\binom{n-2}{2}}}{2^{\binom{n-2}{2}+n-2+n-1}} \\
&= \frac{n(n-1)(n-2)3^{n-3}}{2^{2n-3}} \\
&= \frac{n(n-1)(n-2)3^{n-3}}{2^{2(n-3)}2^{3}} \\
&= \frac{n(n-1)(n-2)}{8}\left(\frac{3}{4}\right)^{n-3} \\
&= \frac{\frac{1}{8}n(n-1)(n-2)}{(\frac{4}{3})^{n-3}}.
\end{aligned}$$

Since this value tends to 0 as n tends to ∞, it must be that $P(n) \to 0$ as $n \to \infty$.

From duality we have that the probability that vertices x and y exists such that $|I(x) \cap I(y)| = 1$ also tends to 0 as n tends to ∞. Thus, the probability that a tournament is not quadrangular tends to 0 as n tends to ∞. That is, almost all tournaments are quadrangular. ∎

3. Strongly Quadrangluar Tournaments

3.1 Definitions and background

In this chapter we look at a stronger necessary condition for a digraph to support an orthogonal matrix. Let D be a digraph. Let $S \subseteq V(D)$ such that for all $u \in S$, there exists $v \in S$ such that $O(u) \cap O(v) \neq \emptyset$, and let $S' \subseteq V(D)$ such that for all $u \in S'$, there exists $v \in S'$ such that $I(u) \cap I(v) \neq \emptyset$. We say that D is *strongly quadrangular* if for all such sets S and S',

(i) $$\left| \bigcup_{u,v \in S} (O(u) \cap O(v)) \right| \geq |S|,$$

(ii) $$\left| \bigcup_{u,v \in S'} (I(u) \cap I(v)) \right| \geq |S'|.$$

In [56], Severini showed that strong quadrangularity is a necessary condition for a digraph to support an orthogonal matrix, where the entries in the matrix come from a field of characteristic 0 (ex. the rational numbers, the real numbers or the complex numbers). We give a proof of this result here for completeness.

Theorem 3.1 *If D is the digraph of an orthogonal matrix over a field of characteristic 0, then D is strongly quadrangular.*

Proof: Let $\mathbb{F}$ be a field of characteristic 0. Let D be a digraph which supports an orthogonal matrix U over $\mathbb{F}$, and assume, to the contrary, that D is not strongly quadrangular. So, there exists a set $S \subseteq V(D)$ so that for all $u \in S$, there exists $v \in S$ such that $O(u) \cap O(v) \neq \emptyset$ and $|\bigcup_{u,v \in S}(O(u) \cap O(v))| < |S|$, or we can find a set $S' \subseteq V(D)$ so that for all $u \in S'$, there exists $v \in S'$ such that $I(u) \cap I(v) \neq \emptyset$ and $|\bigcup_{u,v \in S'}(I(u) \cap I(v))| < |S'|$. We deal with the first case, and note the second is a similar argument. Let $Q = \bigcup_{u,v \in S}(O(u) \cap O(v))$. Let R denote the set of rows in U which correspond to the vertices of S, and let C denote the set of columns of U which correspond to vertices of Q. Let R' be a set of row vectors obtained from R by restricting these vectors to the entries which occur in C.

Pick distinct row vectors $R'_{i\bullet}$ and $R'_{j\bullet}$ from R', let $U_{i\bullet}$ and $U_{j\bullet}$ be the corresponding rows of U in R, and let x_i and x_j be the vertices in D which correspond to $U_{i\bullet}$ and $U_{j\bullet}$ respectively. Since $O(x_i) \cap O(x_j) \subseteq Q$, the only entries of $U_{i\bullet}$ and $U_{j\bullet}$ which contribute nonzero terms to $\langle U_{i\bullet}, U_{j\bullet}\rangle$ occur in the columns of C. So, $\langle R'_{i\bullet}, R'_{j\bullet}\rangle = \langle U_{i\bullet}, U_{j\bullet}\rangle = 0$. Thus, R' is a set of mutually orthogonal vectors. However, R' is a set of vectors in $\mathbb{F}^{|Q|}$. This is a contradiction, because this is a set of $|S|$ mutually orthogonal vectors in a vector space of dimension $|Q| < |S|$. Thus, D must be strongly quadrangular. ■

In a field of non-zero characteristic, orthogonal vectors need not be linearly independent, so we would not have been able to draw the above contradiction. To emphasize this, consider the matrix

$$A = \begin{pmatrix} 1&1&1&0&0&0&0&0 \\ 1&1&0&1&0&0&0&0 \\ 1&1&0&0&1&0&0&0 \\ 1&1&0&0&0&1&0&0 \\ 1&1&0&0&0&0&1&0 \\ 1&1&0&0&0&0&0&1 \\ 1&0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1&1 \end{pmatrix}.$$

Let D be the digraph whose adjacency matrix is A. Over $GF(2)$, there is only one matrix with pattern A, namely A. One can quickly verify that $AA^\top = I$ over

$GF(2)$. If $S \subseteq V(D)$ corresponding to the first 6 rows of A, then $|\bigcup_{u,v\in S}(O(u)\cap O(v))| = 2 < |S|$, so D is not strongly quadrangular. So, strong quadrangularity is only a necessary condition when dealing with fields of characteristic 0. For this reason we will only deal with fields of characteristic 0 for the rest of the chapter.

To see that strong quadrangularity is in fact a more restrictive condition, consider the following tournament. Let T be a tournament with $V(T) = \{0,1,2,3,4,5,6,x,y\}$ so that $\{0,1,2,3,4,5,6\}$ induces the tournament QR_7, $x \to y$ and $O(y) = I(x) = V(T) - \{x,y\}$. In section 2.5, we saw that T is quadrangular. Now consider the set of vertices $S = \{0,1,5\}$. Since each of $0,1,5$ beat x, we have that for all $u \in S$, there exits $v \in S$ so that $O(u) \cap O(v) \neq \emptyset$. Also,

$$\begin{aligned}\left|\bigcup_{u,v\in S}(O(u)\cap O(v))\right| &= |(O(0)\cap O(1)) \cup (O(0)\cap O(5)) \cup (O(1)\cap O(5))| \\ &= |\{2,x\}\cup\{2,x\}\cup\{2,x\}| \\ &= 2 \\ &< |S|.\end{aligned}$$

So T is not strongly quadrangular. We now construct a class of strongly quadrangular tournaments.

3.2 A class of strongly quadrangular tournaments

In this section we construct a class of strongly quadrangular tournaments. The following lemma is a well known fact of tournaments. It is a corollary to the fact that every n-tournament contains a transitive $(\lfloor \log_2(n) \rfloor + 1)$-subtournament. We prove it here for completeness.

Lemma 3.2 *Let T be a tournament on $n \geq 4$ vertices. Then there must exist distinct $a, b \in V(T)$ such that $O(a) \cap O(b) \neq \emptyset$.*

Proof: Pick a vertex a of maximum out-degree in T. As, $n \geq 4$, $d^+(a) \geq 2$. Pick a vertex b of maximum out-degree in the subtournament W induced on $O(a)$. As $d^+(a) \geq 2$, $d^+_W(b) \geq 1$. Thus, $|O(a) \cap O(b)| = d^+_W(b) \geq 1$. ■

Theorem 3.3 *Pick $l \geq 1$. Let T' be a strong tournament on the vertices $\{1, 2, \ldots, l\}$, and let $T_1, T_2, \ldots, T_l$ be regular or near-regular tournaments of order $k \geq 5$. Construct a tournament T on kl vertices as follows. Let V be a set of kl vertices. Partition the vertices of V into l subsets $V_1, \ldots, V_l$ of size k and place arcs to form copies of $T_1, T_2, \ldots, T_l$ on $V_1, \ldots, V_l$ respectively. Finally, add arcs so that $V_i \Rightarrow V_j$ if and only if $i \rightarrow j$ in T'. Then the resulting tournament, T, is a strongly quadrangular tournament.*

Proof: Pick $S \subseteq V(T)$. Define the set

$$A = \{V_i : \exists u \neq v \in S \ni u, v \in V_i\},$$

and define the set

$$B = \{V_i : \exists! u \in S \ni u \in V_i\}.$$

Let $\alpha = |A|$, and $\beta = |B|$. Then, since each V_i has k vertices, $k\alpha + \beta \geq |S|$. Consider the subtournaments of T' induced on the vertices corresponding to A and B. These are tournaments and so must contain a Hamiltonian path. So, label the elements of A and B so that $A_1 \Rightarrow A_2 \Rightarrow \cdots \Rightarrow A_\alpha$ and $B_1 \Rightarrow B_2 \Rightarrow \cdots \Rightarrow B_\beta$. By definition of A, each A_i contains at least two vertices of S, and so if $x, y \in S$ and $x, y \in A_i$, for $i \leq \alpha - 1$, then $A_{i+1} \subseteq O(x) \cap O(y)$. Thus,

$$\left| \bigcup_{u,v \in S} O(u) \cap O(v) \right| \geq k(\alpha - 1).$$

We now consider three cases depending on β.

First assume that $\beta \geq 2$. Consider the vertices of S in B we see that if $x, y \in S$ and $x \in B_i$ and $y \in B_{i+1}$ then $O(y) \cap B_{i+1} \subseteq O(x) \cap O(y)$. Thus, $|O(x) \cap O(y)| \geq \frac{k-1}{2}$, and so

$$\left| \bigcup_{u,v \in S} O(u) \cap O(v) \right| \geq k(\alpha - 1) + \frac{k-1}{2}(\beta - 1) \geq k(\alpha - 1) + 2\beta - 2 \geq k(\alpha - 1) + \beta.$$

Now, since T' is a tournament, either $A_1 \Rightarrow B_1$ or $B_1 \Rightarrow A_1$. If $A_1 \Rightarrow B_1$, then for vertices $x, y \in A_1$ we know $B_1 \subseteq O(x) \cap O(y)$. Since no vertex of B_1 had been previously counted, we have that

$$\left| \bigcup_{u,v \in S} O(u) \cap O(v) \right| \geq k(\alpha - 1) + \beta + k = k\alpha + \beta \geq |S|.$$

So, assume that $B_1 \Rightarrow A_1$. Then for the single vertex of S in B_1, u, and a vertex v of S in A_1 $O(v) \subseteq O(u) \cap O(v)$. This adds $\frac{k-1}{2}$ vertices which were not previously counted. Also, since T' is strong, some $A_i \Rightarrow V_j$ for some $V_j \notin A$. We counted at most $\frac{k-1}{2}$ vertices in V_j before, and since A_i contains at least two vertices x, y from S these vertices add at least $\frac{k+1}{2}$ vertices which were not previously counted, so

$$\left| \bigcup_{u,v \in S} O(u) \cap O(v) \right| \geq k(\alpha - 1) + \beta + \frac{k-1}{2} + \frac{k+1}{2} = k\alpha + \beta \geq |S|.$$

Now assume that $\beta = 1$. Since T' is strong we know that $A_i \Rightarrow V_j$ for some $V_j \notin A$. So,

$$\left| \bigcup_{u,v \in S} O(u) \cap O(v) \right| \geq k\alpha.$$

Now, if $|S| \leq k\alpha$, then we are done, so assume that $|S| = k\alpha + 1$. So, for every $A_i \in A$, $A_i \subseteq S$. So by Lemma 3.2 we can find two vertices of S in A_1 which compete over a vertex of A_i, adding one more vertex to our count, and

$$\left| \bigcup_{u,v \in S} O(u) \cap O(v) \right| \geq k\alpha + 1 = |S|.$$

For the last case, assume that $\beta = 0$. Then since T' is strong we once again have that some $A_i \Rightarrow V_j$ for some $V_j \notin A$. Thus,

$$\left| \bigcup_{u,v \in S} O(u) \cap O(v) \right| \geq k\alpha \geq |S|.$$

Note that the dual of T' will again be strong, and the dual of each T_i will again be regular. Thus, by appealing to duality in T we have that for all $S \subseteq V(T)$,

$$\left| \bigcup_{u,v \in S} I(u) \cap I(v) \right| \geq |S|,$$

and so T is a strongly quadrangular tournament. ■

Recall that strong quadrangularity is a necessary condition for a digraph to support an orthogonal matrix. To emphasize that strong quadrangularity is not sufficient, consider the strongly quadrangular tournament, T, which the construction in the previous theorem gives on 15 vertices. For this tournament, T_1, T_2 and T_3 are all regular of order 5, and T' is the 3-cycle. Note that up to isomorphism, there is only one regular tournament on 5 vertices, so without loss of generality, assume that T_1, T_2 and T_3 are the rotational tournament with symbol $\{1, 2\}$. We now show that T cannot be the digraph of an orthogonal matrix.

Recall J_5 is the 5×5 matrix of all ones, and O_5 the 5×5 matrix of all 0s. Set

$$RT_5 = \begin{pmatrix} 0\,1\,1\,0\,0 \\ 0\,0\,1\,1\,0 \\ 0\,0\,0\,1\,1 \\ 1\,0\,0\,0\,1 \\ 1\,1\,0\,0\,0 \end{pmatrix}.$$

Then the adjacency matrix M of T is

$$M = \begin{pmatrix} RT_5 & J_5 & O_5 \\ O_5 & RT_5 & J_5 \\ J_5 & O_5 & RT_5 \end{pmatrix}.$$

Now, suppose to the contrary that there exists an orthogonal matrix U whose pattern is M. Observe from the pattern of U that the only entries of U which contribute to $\langle U_i, U_j \rangle$ for $i = 1, \ldots, 5$, $j = 6, \ldots, 10$ are in the first five rows. So, $\langle U_1, U_j \rangle = U_{4,1}U_{4j} + U_{5,1}U_{5,j}$ for $j = 6, \ldots, 10$. Thus, since $0 = \langle U_1, U_j \rangle$ for each $j \neq 1$,

$$U_{4,1} = \frac{-U_{5,1}U_{5,6}}{U_{4,6}} = \frac{-U_{5,1}U_{5,7}}{U_{4,7}} = \frac{-U_{5,1}U_{5,8}}{U_{4,8}} = \frac{-U_{5,1}U_{5,9}}{U_{4,9}} = \frac{-U_{5,1}U_{5,10}}{U_{4,10}}.$$

Since $U_{5,1} \neq 0$ this gives,

$$-\frac{U_{4,1}}{U_{5,1}} = \frac{U_{5,6}}{U_{4,6}} = \frac{U_{5,7}}{U_{4,7}} = \frac{U_{5,8}}{U_{4,8}} = \frac{U_{5,9}}{U_{4,9}} = \frac{U_{5,10}}{U_{4,10}}.$$

So, the vectors $(U_{4,6}, \ldots, U_{4,10})$ and $(U_{5,6}, \ldots, U_{5,10})$ are scalar multiples of each other. Now, note that for $j = 6, \ldots, 10$, we have $0 = \langle U_2, U_j \rangle = U_{1,2}U_{1,j} + U_{5,2}U_{5,j}$. So, by applying the same argument, we see that the vector $(U_{5,6}, U_{5,7}, U_{5,8}, U_{5,9}, U_{5,10})$ is a scalar multiple of $(U_{1,6}, U_{1,7}, U_{1,8}, U_{1,9}, U_{1,10})$. So, $(U_{4,6}, U_{4,7}, U_{4,8}, U_{4,9}, U_{4,10})$ is a scalar multiple of $(U_{1,6}, U_{1,7}, U_{1,8}, U_{1,9}, U_{1,10})$. Now, from the pattern of U we see that only the 6^{th} through 10^{th} columns contribute to $\langle U_{1\bullet}, U_{4\bullet} \rangle$. So, since linearly dependent vectors cannot be orthogonal,

$$\langle U_{1\bullet}, U_{4\bullet} \rangle = \langle (U_{1,6}, U_{1,7}, U_{1,8}, U_{1,9}, U_{1,10}), (U_{4,6}, U_{4,7}, U_{4,8}, U_{4,9}, U_{4,10}) \rangle \neq 0.$$

This contradicts our assumption that U is orthogonal. So, T is not the digraph of an orthogonal matrix.

3.3 Nonexistence and searches

The only known tournament which supports an orthogonal matrix is the 3-cycle (This is a permutation matrix, and hence itself an orthogonal matrix). Otherwise, the problem of determining whether or not there exist tournaments which support orthogonal matrices has proved to be quite difficult. As we have seen, for large values of n we can almost always construct examples of tournaments which meet our necessary conditions. Knowing that almost all tournaments are quadrangular and having a construction for an infinite class of strongly quadrangular tournaments, one may believe that there will exist a tournament which supports an orthogonal matrix. However, attempting to find an orthogonal matrix whose digraph is a given tournament has proved to be a difficult task. In general, aside from the 3-cycle, the existence of a tournament which supports an orthogonal matrix is still an open problem. The following result may lead one to believe non-existence is the answer to this problem.

Theorem 3.4 *Other than the* 3*-cycle, there does not exist a tournament on* 10 *or fewer vertices which is the digraph of an orthogonal matrix.*

Proof: By Theorem 2.34 there exists a quadrangular n-tournament for $n \leq 10$ if and only if n is 1, 2, 3 or 9. Note, in the case $n = 1$ and $n = 2$, the only tournaments are the single vertex and single arc, both of whose adjacency matrices have a column of zeros. Since orthogonal matrices have full rank, these cannot support an orthogonal matrix. When $n = 3$, the 3-cycle is the only quadrangular tournament. The adjacency matrix for this tournament is a permutation matrix and hence orthogonal. Now consider $n = 9$. By Theorem 2.28, if T is quadrangular, $\delta^+(T) \le 1$. If $\delta^+(T) = 0$, then T's adjacency matrix will have a row of zeros, and T cannot be the digraph of an orthogonal matrix. So we must have $\delta^+(T) = 1$. So by Theorem 2.8, T has an arc (x, y) with $O(y) = I(x) = V(T) - \{x, y\}$ and $\gamma(T - \{x, y\}) > 2$. The only 7-tournament with domination number greater than 2 is QR_7, thus $T - \{x, y\} = QR_7$. However, in section 3.2 we observed that this tournament is not strongly quadrangular. Thus, other than the 3-cycle, no tournament on 10 or fewer vertices can be the digraph of an orthogonal matrix. ■

If one believes existence is the answer, they could attempt to simplify the search for an orthogonal matrix whose digraph is a tournament by looking for weighing matrices whose patterns are tournament matrices. An (n, k)-*weighing matrix* is an $n \times n$ matrix M with entries from $\{0, 1, -1\}$ such that $M^\top M = MM^\top = kI$. That is, each row and column of M contains exactly k non-zero entries, and any two rows and any two columns are orthogonal. One first observes that if a tournament is to be the digraph of a weighing matrix, the tournament will be regular. So, we can restrict our search to regular tournaments and hence $(2k + 1, k)$-weighing matrices. The following propositions give us another restriction.

Proposition 3.5 *Let T be a regular n-tournament. If T is the digraph of a weighing matrix, then $|O(u) \cap O(v)|$ and $|I(u) \cap I(v)|$ are even for all distinct $u, v \in V(T)$.*

Proof: Assume T is the digraph of some weighing matrix M. Pick

$u, v \in V(T)$ and let $M_{u\bullet}, M_{v\bullet}$ be the rows of M which correspond to u and v respectively. Note, $0 = \langle M_{u\bullet}, M_{v\bullet}\rangle = M_{u,1}M_{v,1} + M_{u,2}M_{v,2} + \cdots + M_{u,n}M_{v,n}$. Now, $M_{u,i}M_{v,i} \neq 0$ if and only if $i \in O(u) \cap O(v)$. Since $\langle M_{u\bullet}, M_{v\bullet}\rangle = 0$, and every non-zero term of $\sum_{i=1}^{n} M_{u,i}M_{v,i}$ is ± 1, we have that the number of negative terms in the sum must equal the number of positive terms. So, the number of non-zero terms in the sum must be even. Hence, $|O(u) \cap O(v)|$ must be even. Similarly, $|I(u) \cap I(v)|$ must be even. ■

Proposition 3.6 *Let T be an n-tournament. If T is the digraph of a weighing matrix, then $n \equiv 1$ or $3 \pmod 8$.*

Proof: Assume that T is the digraph of some weighing matrix. Note, that this implies T is a regular tournament. Now, choose $u \in V(T)$, and $v \in O(u)$. By Proposition 3.5, $|O(u) \cap O(v)|$ is even. Equivalently, $d^+_{T[O(u)]}(v)$ is even. Since v was chosen from $O(u)$ arbitrarily, $d^+_{T[O(u)]}(x)$ is even for all $x \in O(u)$. Thus, $\binom{d^+_T(u)}{2} = \sum_{x \in O(u)} d^+_{T[O(u)]}(x)$ must be even. So, $2 | \frac{d^+_T(u)(d^+_T(u)-1)}{2}$, and so $4 | d^+_T(u)(d^+_T(u) - 1)$. Since $d^+_T(u)$ and $d^+_T(u) - 1$ must be relatively prime, we have that $4 | d^+_T(u)$ or $4 | d^+_T(u) - 1$. Since T is a regular tournament, $d^+_T(u) = \frac{n-1}{2}$. So, $4 | \frac{n-1}{2}$ or $4 | \frac{n-3}{2}$, and so $8 | n-1$ or $8 | n-3$. Hence $n \equiv 1$ or $3 \pmod 8$. ■

The following condition is well known for (n, k)-weighing matrices in which n is odd. We prove it here for completeness.

Theorem 3.7 [27]*If M is an (n, k) weighing matrix with n odd, then k is a perfect square.*

Proof: Let M be an (n, k)-weighing matrix, and assume n is odd. Then $\det(M^\top M) = \det(kI) = k^n$. Since, $\det(M^\top) = \det(M)$ and $\det(M^\top M) =$

$\det(M^\top)\det(M)$, we have that $\det(M)^2 = k^n$, and so $\det(M) = \pm\sqrt{k^n}$. Since n is odd, $n = 2b+1$ for some b, and so $\det(M) = \pm k^b\sqrt{k}$. Now, the determinant of M is $\sum_{\sigma\in S_n} M_{1,\sigma(1)}M_{2,\sigma(2)}\cdots M_{n,\sigma(n)}$ (where S_n denotes the symmetric group of order n). So, since every entry of M is an integer, $\det(M)$ must be an integer as well. Thus, $k^b\sqrt{k} \in \mathbb{Z}$, and so $\sqrt{k} \in \mathbb{Z}$. That is, k is a perfect square. ■

Since a weighing matrix which has a tournament for its digraph will be a $(2k+1, k)$-weighing matrix, the previous theorem tells us we are looking for a $(2a^2+1, a^2)$-weighing matrix. By examining cases, we get Proposition 3.6 as the following corollary to Theorem 3.7.

Corollary 3.8 *Let T be a regular n-tournament. If T is the digraph of a weighing matrix, then $n \equiv 1$ or $3 \pmod 8$.*

Putting together the fact that we are looking for a $(2a^2+1, a^2)$-weighing matrix with Theorem 3.4, we see that the smallest case we can start searching for is a $(19, 9)$-weighing matrix. One obvious choice for a tournament to search is QR_{19}, since this is a regular 19 vertex tournament with the property that the outsets of every two vertices intersect in exactly 4 vertices. With the aid of Carey Jenkins, we constructed the following algorithm to search for a $(19, 9)$-weighing matrix whose digraph is QR_{19}. A tree diagram of the "branch and bound" technique used in this algorithm is shown in Figure 3.1. In this diagram, circled vertices represent the rows our algorithm has currently appended, and dashed lines show paths which, by design of our algorithm, we need not search.

First construct and store all $2^9 = 512$ possible assignments of -1 to the

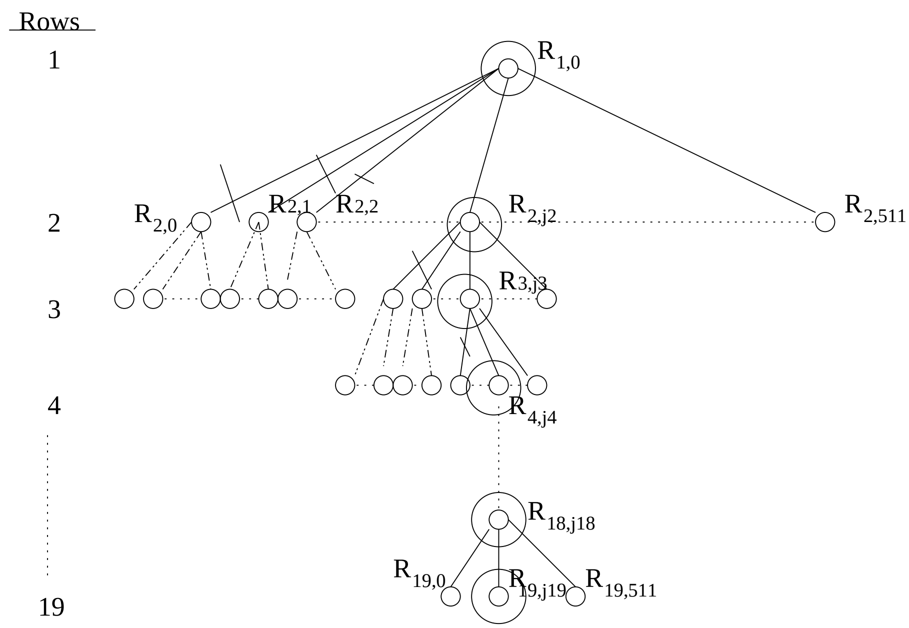

Figure 3.1: The tree diagram for our algorithm

non-zero entries of the first row of the adjacency matrix of QR_{19},

$$R_{1,0} = (0\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 0).$$

Label these as $R_{1,j}$ where j is a number between 0 and 511 which labels each different assignment of -1s to the vector. Next, construct all cyclic shifts of these rows. Label each row created in this way so that $R_{i,j}$ is $R_{1,j}$ shifted $i-1$ positions to the right. Since the adjacency matrix of QR_{19} is circulant, this creates all possible assignments of -1 to the non-zero positions of row i, for $i = 2, \ldots, 19$.

We attempt to build a matrix by appending rows. We begin with the first row $R_{1,0}$ above. Lemma 3.9 shows why we can leave this row fixed. For $j_2 = 0$, compute $\langle R_{1,0}, R_{2,j_2}\rangle$. If $\langle R_{1,0}, R_{2,j_2}\rangle = 0$, we append R_{2,j_2} and move on to find R_{3,j_3} which is orthogonal to $R_{1,0}$ and R_{2,j_2}. If not, increment j_2 and repeat. Continue adding rows in this manner. Assume we have a matrix with i mutually orthogonal rows,

$$\begin{pmatrix} R_{1,0} \\ R_{2,j_2} \\ \vdots \\ R_{i,j_i} \end{pmatrix}.$$

Set $j_{i+1} = 0$, and begin successively checking if $R_{i+1,j_{i+1}}$ is orthogonal to each previously selected row, and incrementing j_{i+1} by 1 if not. If there is some $R_{i+1,j_{i+1}}$ which is orthogonal to each of the previously selected rows, then append it. If j_{i+1} reaches 512, and hence there does not exist a $R_{i+1,j_{i+1}}$ which is orthogonal to each of the previous rows, then return to row i, increment j_i, and search for a new R_{i,j_i} which is orthogonal to each of the previous $i-1$ rows.

At each step the algorithm is maintaining a set of mutually orthogonal rows, so if it finds 19 rows successfully, then it has created a $(19, 9)$-weighing matrix whose digraph is QR_{19}. Otherwise, for some $k < 19$, j_i will reach 512 for all $2 \leq i \leq k$, and we will have completely searched all possible combinations of the first k rows without finding a set of k mutually orthogonal rows, and the algorithm terminates showing no such matrix exists.

The C++ code for our implementation of this algorithm can be found in appendix B. The algorithm was implemented in parallel on the University's Beowulf cluster. The search was split by giving the first 16 possible assignments of row 2 to the first processor, the next 16 to the next processor, and so on. After approximately 5 minutes, the algorithm terminated showing that QR_{19} is not the digraph of a weighing matrix.

Lemma 3.9 *Let M be an (n,k)-weighing matrix, and let M' be the matrix obtained from M by multiplying any set of the rows of columns by -1. Then M' is an (n,k)-weighing matrix.*

Proof: Let $S \subseteq \{1,\dots,n\}$, be the indices of the columns of M multiplied by -1 to obtain M', and S' the indices of the rows multiplied by -1 to obtain M'. Let D_S be the diagonal matrix whose i,i entry is -1 if $i \in S$ and 1 if $i \notin S$. Define $D_{S'}$ to be the diagonal matrix whose i,i entry is -1 if $i \in S'$ and 1 if $i \notin S'$. Note since $(-1)(-1) = 1 * 1 = 1$, D_S and $D_{S'}$ are their own inverses. Also, $M' = D_{S'}MD_S$. So,

$$
\begin{aligned}
M'^{\top}M' &= (D_{S'}MD_S)^{\top}(D_{S'}MD_S) \\
&= D_S^{\top}M^{\top}D_{S'}^{\top}D_{S'}MD_S \\
&= D_SM^{\top}D_{S'}D_{S'}MD_S \\
&= D_SM^{\top}IMD_S \\
&= D_SM^{\top}MD_S \\
&= D_S(kI)D_S \\
&= kD_SID_s \\
&= kD_SD_S \\
&= kI,
\end{aligned}
$$

and M' is an (n,k)-weighing matrix. ■

4. Fully Indecomposable Tournament Matrices

4.1 Definitions and background

Another combinatorial necessary condition for a matrix to be orthogonal is that it can be written in block form

$$\begin{pmatrix} A_1 & O & O & \cdots & O \\ O & A_2 & O & \cdots & O \\ O & O & A_3 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & A_k \end{pmatrix}$$

where each A_i is fully indecomposable. This is a combinatorial property, so it is also required of a $(0,1)$-matrix to be the pattern of an orthogonal matrix. So, in this chapter, we determine exactly which tournaments have adjacency matrices which are fully indecomposable, and characterize the separable tournament matrices.

Let A be an $n \times n$ matrix, $n \geq 2$. Then A is called *partly decomposable* provided there exist permutation matrices P and Q such that

$$PAQ = \begin{pmatrix} B & O \\ D & C \end{pmatrix}$$

where O is an $r \times s$ matrix of zeros with $r + s = n$, and B and C are square matrices other than the zero matrix. If A cannot be permuted into this form, then we say that A is *fully indecomposable.* If A has a row or column of zeros, or we can independently permute the rows and columns of A so that

$$A = \begin{pmatrix} B & O \\ O & C \end{pmatrix}$$

then we say that A is *separable.* This property has also been called *totally decomposable,* by Brualdi, Harary and Miller, [11], and *disconnected* by Greenberg, Lundgren, and Maybee, [29]. If a matrix is not separable, we say it is *inseparable.*

If we require simultaneous row and column permutations, then we say a matrix which can be permuted into the form $\begin{pmatrix} B & O \\ D & C \end{pmatrix}$ is *reducible,* and *irreducible* otherwise. If M is the adjacency matrix of a digraph D, it is a well known result that M is irreducible if and only if D is strongly connected. However, as mentioned in chapter 1, independent row and column permutations do not necessarily result in the same digraph. So, it is difficult to determine just what these properties mean in digraphs.

Since independent row and column permutations do not necessarily result in isomorphic digraphs, when looking at a matrix property that allows for independent row and column permutations, one typically wants to treat the matrix as a reduced adjacency matrix of a bipartite graph. Given a bipartite graph B with bipartition $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$, the *reduced adjacency matrix* of B is the $m \times n$ $(0,1)$-matrix M with $M_{i,j} = 1$ if and only if $[x_i, y_j] \in E(B)$. By treating a $(0,1)$-matrix as the reduced adjacency matrix of a bipartite graph, independent row and column permutations will result in the reduced adjacency matrix of an isomorphic bipartite graph. For instance, consider the following matrix M. By exchanging the 5^{th} and 7^{th} rows, and swapping the 1^{st} and 3^{rd} and the 2^{nd} and 5^{th} columns we obtain the matrix M'.

$$M = \begin{pmatrix} 0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0\,0 \\ 1\,0\,0\,0\,0\,0\,0 \\ 0\,0\,1\,0\,1\,1\,0 \\ 0\,0\,0\,0\,0\,0\,1 \\ 0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0 \end{pmatrix} \quad M' = \begin{pmatrix} 0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,1\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0\,0 \\ 1\,1\,0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,0\,1\,0 \\ 0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,1 \end{pmatrix}$$

Both M and M' represent the same bipartite graph B shown in Figure 4.1. However, M and M' represent the digraphs D and D' respectively from Figure 4.2, and clearly D and D' are not isomorphic.

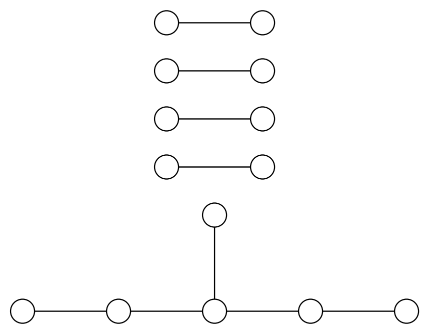

Figure 4.1: A bigraph representation of M and M'

Despite this problem, we are interested in characterizing tournaments with properties which require independent row and column permutations. So, we study how these properties affect digraphs, and in particular, tournaments. For fully indecomposable digraphs, we look at the matrix before the permutations, and for separable matrices we look at competition graphs.

We obtain our characterization of separable tournament matrices by utilizing the characterization of the competition graphs of tournaments via the characterization of the domination graphs of tournaments. We will also use these

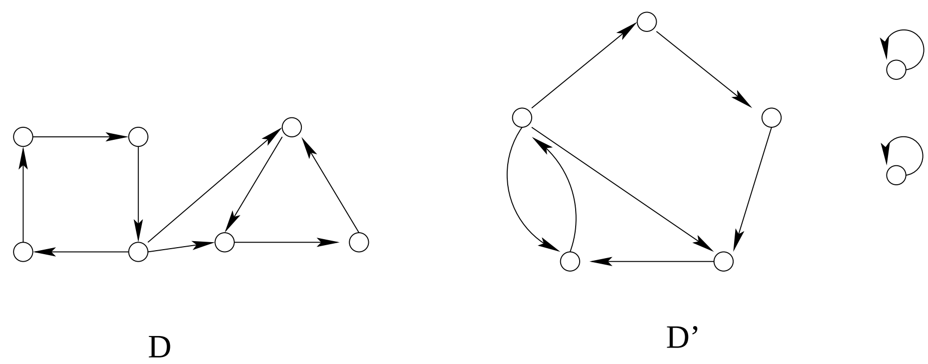

Figure 4.2: Different digraph representations for M and M'.

characterizations to obtain some interesting corollaries about tournaments and tournament matrices. We see exactly when a tournament matrix has a minimum line cover other than the all rows or all columns cover. We characterize the tournaments which are the digraphs of doubly-stochastic matrices, and give a necessary condition for a tournament to be the digraph of an orthogonal matrix. We also characterize the tournaments for which every arc is contained in a cycle factor.

4.2 Fully indecomposable matrices

Let M be the adjacency matrix of a digraph D, and suppose M is partly decomposable. So we can perform independent row and column permutations to create an $r \times s$ block of zeros in M with $r + s = n$. By permuting the rows and columns of M back to their original positions, we see that the block of zeros corresponds to two sets of vertices X and Y, not necessarily disjoint, with $|X| = r$ and $|Y| = s$, and there is no arc (x, y) with $x \in X$ and $y \in Y$. We use this interpretation of the partly decomposable property to get the following theorem.

Theorem 4.1 *Let T be a tournament on n vertices, and let M be its adjacency matrix. Then M is fully indecomposable if and only if T is strongly connected and does not contain a vertex v such that $T - v$ contains a strong component of order* 1*.*

Proof: Suppose M is partly decomposable. So there exist sets $X, Y \subseteq V(T)$ so that there is no arc of T from X to Y, and $|X| + |Y| = n$. Suppose that $|X \cap Y| \geq 2$. Then there exist distinct vertices $u, v \in X \cap Y$. Since T is a tournament either $u \to v$ or $v \to u$. Without loss of generality say $u \to v$. This cannot be since $u \in X$ and $v \in Y$. Therefore $|X \cap Y| \leq 1$. We now consider two cases based on $|X \cap Y|$.

Case 1: Suppose $|X \cap Y| = 0$. Then since $|X| + |Y| = n$, X and Y form a partition of the vertices of T. Thus, since X and Y partition $V(T)$ and no vertex in X beats any vertex in Y, T is not strongly connected.

Case 2: Suppose $|X \cap Y| = 1$. So $X \cap Y = \{u\}$ and $X \cup Y = V(T) - \{v\}$ for some $u \neq v \in V(T)$. Further, since $u \in Y$, $u \Rightarrow X - \{u\}$ and since $u \in X$, $Y - \{u\} \Rightarrow u$. Finally, since no vertex in X beats any vertex in Y, we have $Y \Rightarrow X - \{u\}$. Thus, $T - v$ contains a strong component of order 1, namely u.

For the converse, recall that if a tournament is not strongly connected, it can be partitioned into $m \geq 2$ strong components $T_1, T_2, \ldots, T_m$ such that $T_i \Rightarrow T_j$ if and only if $i < j$. So, if T is not strongly connected, then taking $X = V(T_m)$ and $Y = V(T_1) \cup V(T_2) \cup \cdots \cup V(T_{m-1})$ gives X and Y so that $|X| + |Y| = n$ and there is no arc from X to Y. Now, suppose T contains a vertex v so that $T' = T - v$ is not strongly connected and has a strong component, say T'_i, of order 1. Then take $X = V(T'_i) \cup V(T'_{i+1}) \cup \cdots \cup V(T'_m)$ and $Y = V(T'_1) \cup V(T'_2) \cup \cdots \cup V(T'_i)$. Then $|X| + |Y| = n$ and there is no arc from X to Y. ■

4.3 Separable matrices

Let M be an $n \times n$ $(0,1)$-matrix, and let B be a bipartite graph who's reduced adjacency matrix is M, and D a digraph who's adjacency matrix is M. Then one can quickly verify that M being inseparable is equivalent to B being connected. Unfortunately, there is no direct connection between M being inseparable and a connectedness property of D. However, in [29], Greenberg, Lundgren and Maybee showed that there is a connection between M being inseparable and connectedness in two more graphs other than B. The *row graph* of M is the graph whose vertices correspond to the rows of M, with vertices i and j adjacent if and only if there exists some k so that $M_{i,k} = M_{j,k} = 1$. Similarly, the *column graph* of M is the graph whose vertices correspond to the columns of M, with vertices i and j adjacent if and only if there exists some k so that $M_{k,i} = M_{k,j} = 1$. We now state their result.

Theorem 4.2 [29] *Let M be an $m \times n$ matrix such that each column and row of M has a non-zero element. Then the following are equivalent.*

1) The bipartite graph whose reduced adjacency matrix is M is connected.

2) The row graph of M is connected.

3) The column graph of M is connected.

Note, when M is the adjacency matrix of a digraph D, then the row graph and column graph are equivalent to the competition graph of D and D^r respectively. Thus we can restate Theorem 4.2 as follows.

Theorem 4.2 (restated) *Let D be a digraph with $\delta^+(D) \geq 1$ and $\delta^-(D) \geq 1$. Then the following are equivalent.*

1) The adjacency matrix of D is inseparable.

2) comp(D) is connected.

3) comp(D^r) is connected.

We can use this theorem to determine exactly which tournaments are separable, as the competition graphs of tournaments have been classified. This classification comes from the classification of the domination graphs of tournaments and the correspondence between the competition graphs and domination graphs of tournaments in Theorems 1.7 and 1.8 due to Merz, Fisher, Lundgren, and Reid [44].

Let G be a graph which is not connected, and G_1 and G_2 two connected components of G of order m and n respectively. Then, in the complement of G, the graph induced on $V(G_1) \cup V(G_2)$ contains the complete bipartite graph $K_{m,n}$. In particular $V(G_1)$ and $V(G) - V(G_1)$ forms a bipartition for a spanning biclique of the complement of G. Recall from Theorem 1.8 that the domination graph of a tournament is either a caterpillar or spiked odd cycle. The only biclique which can be contained in a caterpillar or spiked odd cycle is a star, $K_{1,s}$. So, since the compliment of a disconnected graph contains a spanning biclique, Theorems 4.2, 1.7 and 1.8 tell us that a tournament on n vertices with no transmitter or receiver has a separable adjacency matrix, if and only if its dual has a domination graph which contains the star $K_{1,n-1}$.

Lemma 4.3 *Let T be an n-tournament. Then dom(T) contains $K_{1,n-1}$ as a subgraph, if and only if T has a transmitter, or T contains an arc (x, y) so that $O(y) = I(x) = V(T) - \{x, y\}$.*

Proof: Suppose the star $K_{1,n-1}$ is a subgraph of dom(T), and let y be the center of $K_{1,n-1}$. Suppose there exist distinct $u, v \in V(T)$ such that $u \to y$ and $v \to y$, and without loss of generality, suppose $u \to v$. Then neither y nor v dominate u, however $[y, v]$ is an edge in dom(T), a contradiction. Thus, $d^+(y) \geq n - 2$. If $d^+(y) = n - 1$, then y is a transmitter and we are done. So assume $d^+(y) = n - 2$ and hence $d^-(y) = 1$. Let x be the unique vertex in $I(y)$. Now choose $u \in V(T) - \{x, y\}$. Since $I(y) = \{x\}$, we have $y \to u$. Also, since $[y, u]$ is an edge in dom(T), and $x \to y$ we must have $u \to x$. Thus, there exists an arc (x, y) in T with $O(y) = I(x) = V(T) - \{x, y\}$. So these conditions are necessary.

For sufficiency, first suppose T has a transmitter s. Pick a vertex v other than s in $V(T)$. If $u \in V(T) - \{s, v\}$ then $s \to u$, so $[s, v] \in E(\text{dom}(T))$. Since v was chosen arbitrarily, the star $K_{1,n-1}$ is a subgraph of dom(T) with center s. Now, assume T contains and arc (x, y) so that $O(y) = I(x) = V(T) - \{x, y\}$. Choose $w \in V(T) - \{x, y\}$. If $z \in V(T) - \{x, y, w\}$, then $y \to z$, and $w \to x$. So, $[y, w] \in E(\text{dom}(T))$ for each $w \in V(T) - \{x, y\}$. Further, $[x, y] \in E(\text{dom}(T))$ since $y \to z$ for all $z \in V(T) - \{x, y\}$. So dom(T) contains the star $K_{1,n-1}$ as a subgraph with center y. ■

By Lemma 4.3 and Theorems 4.2 and 1.7, if T is a tournament with no transmitter or receiver, T's adjacency matrix is separable if and only if T^r contains an arc (x, y) with $O(y) = I(x) = V(T) - \{x, y\}$. Note however, if T^r contains such an arc so will T. Also, if T has a transmitter or receiver its adjacency matrix has a row or column of zeros, and is separable. This gives us the following theorem.

Theorem 4.4 *Let T be a tournament. Then the adjacency matrix of T is separable if and only if at least one of the following occur.*

1. *T has a transmitter.*

2. *T has a receiver.*

3. *There exists an arc (x, y) in T such that $O(y) = I(x) = V(T) - \{x, y\}$.*

4.4 Corollaries

In this section we give some corollaries due to our characterizations and previous results about fully indecomposable matrices. Our first result involves line covers of tournament matrices. A *line cover* of a matrix, M, is a set of rows and columns of M which contain every non-zero entry of M. The following theorem is taken from [14].

Theorem 4.5 *A matrix M has a minimum line cover other than the all rows or all columns cover if and only if M is partly decomposable.*

The size of a minimum line cover is called the term rank of a matrix. In Chapter 6 we will see that an $n \times n$ tournament matrix can only have real rank, and hence term rank, n or $n-1$. In [22], Doherty, Lundgren and Siewert show that the tournaments whose adjacency matrices have term rank $n-1$ are exactly those with a strong component of order 1. The following corollary shows that not only is the class of tournament matrices with term rank $n-1$ highly restricted, the class of tournaments whose adjacency matrices admit a minimum line cover other than the all rows or all columns cover is fairly restricted.

Corollary 4.6 *Let T be a tournament and M its adjacency matrix. Then M has a minimum line cover other than the all rows or all columns cover if and only if T is not strongly connected, or T contains a vertex v such that $T - v$ has a strong component of order* 1*.*

Our remaining results are concerned with when a matrix has the property that each of its inseparable components is fully indecomposable. Note, when a matrix contains a row or column of zeros, one of the inseparable components will be a submatrix of zeros and, hence, not fully indecomposable. So, in the following theorem, we may assume that our tournaments have no transmitters or receivers.

Theorem 4.7 *Let T be a tournament with $\delta^+(T) \geq 1$, $\delta^-(T) \geq 1$, and adjacency matrix M. Then each inseparable component of M is fully indecomposable if and only if*

1. *T is strongly connected, and*

2. *if T contains a vertex v such that $T-v$ has a strong component of order* 1*, then T contains an arc (x,y) with $O(y) = I(x) = V(T) - \{x,y\}$ and $T - \{x,y\}$ does not contain a strong component of order* 1*.*

Proof: If T is not strongly connected, then by Theorem 4.1, M must be partly decomposable. Further, in this case the only way M could be separable is for T to have a transmitter or receiver, a contradiction to our assumption that $\delta^+(T) \geq 1$ and $\delta^-(T) \geq 1$. Thus 1. is necessary, and we may assume T is strongly connected. If M is fully indecomposable we are done, so assume M is partly decomposable. By Theorem 4.1, M is partly decomposable if and only if T contains a vertex v such that $T - v$ has a strong component of order 1. Suppose M is separable. Since T has no transmitter or receiver, Theorem 4.4 says T must have an arc (x,y) such that $O(y) = I(x) = V(T) - \{x,y\}$. Now, consider the submatrix of M obtained by deleting the row and column of M corresponding to $O(x)$ and $I(y)$ respectively. This matrix has the form

$$A = \begin{bmatrix} 1 & \mathbf{1} \\ \mathbf{1} & M' \end{bmatrix}$$

where M' is an $(n-2) \times (n-2)$ tournament matrix and $\mathbf{1}$ denotes a vector of all ones. Let T' be a tournament with adjacency matrix M'. Note, A is partly decomposable if and only if there exist sets $X, Y \subseteq V(T')$ with no arc from X to Y, and $|X| + |Y| = n - 1$. Now, $|X \cup Y| \leq |V(T')| = n - 2$ and since T' is a tournament, $|X \cap Y| \leq 1$. So,

$$n - 1 = |X| + |Y| = |X \cup Y| + |X \cap Y| \leq n - 2 + 1 = n - 1.$$

Thus, $|X \cup Y| = n - 2 = |V(T')|$ and $|X \cap Y| = 1$, which occurs exactly when T' has a strong component of order 1. Further, since A contains a row and column of all ones, A cannot be separable. Thus, each inseparable component of M is fully indecomposable if and only if T is strongly connected and if T contains a vertex v such that $T - v$ has a strong component of order one, then T contains a arc (x, y) such that $O(y) = I(x) = V(T) - \{x, y\}$ and $T - \{x, y\}$ does not contain a strong component of order 1. ■

In our next two results we look at when tournaments are the digraphs of doubly-stochastic and orthogonal matrices. A real non-negative $n \times n$ matrix is called *doubly-stochastic* if every row sum and column sum is 1. The following theorem is taken from [14].

Theorem 4.8 *A $(0, 1)$-matrix M is the pattern of a doubly-stochastic matrix if and only if every inseparable component of M is a fully indecomposable matrix.*

This theorem together with Theorem 4.7 give us the following corollary.

Corollary 4.9 *A tournament T is the digraph of a doubly-stochastic matrix if and only if,*

1. *T is strongly connected, and*
2. *if T contains a vertex v such that $T - v$ has a strong component of order 1, then T contains an arc (x, y) with $O(y) = I(x) = V(T) - \{x, y\}$ and $T - \{x, y\}$ does not contain a strong component of order 1.*

The following theorem is taken from the linear algebra folklore. We prove it here for completeness.

Theorem 4.10 *If M is an orthogonal matrix, then each inseparable component of M is a fully indecomposable matrix.*

Proof: Let M be an $n \times n$ orthogonal matrix. Note if M is separable, then M can be written in the form

$$\begin{pmatrix} A_1 & O & O & \cdots & O \\ O & A_2 & O & \cdots & O \\ O & O & A_3 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & A_k \end{pmatrix}$$

where each A_i is inseparable. Block multiplication shows that if M is orthogonal each A_i must also be orthogonal. So, we may assume M is inseparable. Now, suppose to the contrary that M is partly decomposable. Say M can be written in the form

$$M = \begin{pmatrix} A & O \\ B & C \end{pmatrix}$$

where O is a $p \times q$ matrix of 0s with $p + q = n$. Then,

$$I_n = \begin{pmatrix} I_p & O \\ O & I_q \end{pmatrix} = \begin{pmatrix} A & O \\ B & C \end{pmatrix} \begin{pmatrix} A^\top & B^\top \\ O^\top & C^\top \end{pmatrix} = \begin{pmatrix} AA^\top & AB^\top \\ BA^\top & (BB^\top + CC^\top) \end{pmatrix}.$$

So, $AA^\top = I_p$, and since A is a square matrix, A is invertible. Then since $AB^\top = O$, $AB^\top_{i\bullet} = \mathbf{0}$ for all $i = 1, \ldots q$, a contradiction to A being invertible. Thus, M cannot be partly decomposable and is hence fully indecomposable.

■

Again, using Theorem 4.7 we get the following corollary.

Corollary 4.11 *If a tournament T is the digraph of an orthogonal matrix, then,*

1. *T is strongly connected, and*

2. if T contains a vertex v such that $T - v$ has a strong component of order 1, then T contains an arc (x, y) with $O(y) = I(x) = V(T) - \{x, y\}$ and $T - \{x, y\}$ does not contain a strong component of order 1*.*

Note, unlike the previous two corollaries, this did not give us a characterization. This result only gives a necessary condition. Recall, as mentioned in Chapter 3, aside from the 3-cycle, it is unknown if a tournament can be the digraph of an orthogonal matrix.

Our final corollary is graph theoretic. A *cycle factor* of a digraph D is a spanning subdigraph of D composed of vertex disjoint cycles. For example, the bold arcs form a cycle factor of the digraph in Figure 4.3.

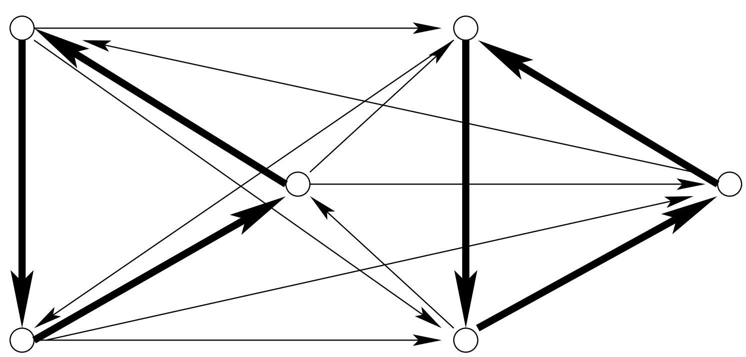

Figure 4.3: An example of a cycle factor in a digraph.

A *non-zero diagonal* of an $n \times n$ matrix M is a set of n non-zero elements of M no two of which occur in the same row or column. If M is the adjacency matrix of a digraph D, then the cycle factors of D are in direct correspondence with non-zero diagonals of M. For example, the following matrix is the adjacency matrix of the digraph in Figure 4.3. The bold ones in this matrix form a non-zero diagonal and correspond to the bold arcs in the cycle factor in Figure 4.3.

$$\begin{pmatrix} 0 & \mathbf{1} & 0 & 1 & 1 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 1 & 1 \\ \mathbf{1} & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 1 & 0 & 0 & \mathbf{1} \\ 1 & 0 & 0 & \mathbf{1} & 0 & 0 \end{pmatrix}$$

The following theorem is taken from [14].

Theorem 4.12 *Let M be an $n \times n$ $(0,1)$-matrix. Then every 1 of M is on a non-zero diagonal of M if and only if each inseparable component of M is fully indecomposable.*

This theorem together with Theorem 4.7, and the correspondence between non-zero diagonals and cycle factors give us the following corollary.

Corollary 4.13 *Let T be a tournament. Then every arc of T is contained in a cycle factor if and only if,*

1. *T is strongly connected, and*

2. *if T contains a vertex v such that $T - v$ has a strong component of order 1, then T contains an arc $x \to y$ with $O(y) = I(x) = V(T) - \{x, y\}$ and $T - \{x, y\}$ does not contain a strong component of order 1.*

5. Domination Graphs of Complete Paired Comparison Digraphs

5.1 Definitions and background

In Chapter 4 we observed how a classification of the domination graphs of tournaments gave us quite a bit of information about tournament matrices. A generalized tournament matrix is a non-negative matrix M so that $M_{i,j} + M_{j,i} = 1$ for all $i \neq j$, and $M_{i,i} = 0$ for all i. In this chapter we characterize the domination graphs of the digraphs associated with generalized tournament matrices, which we call complete paired comparison digraphs. A *complete paired comparison digraph,* D, is a complete symmetric directed graph so that for each arc (x, y) we associate a real number between 0 and 1, denoted w_{xy}, such that $w_{yx} = 1 - w_{xy}$. We also refer to a complete paired comparison digraph as a *pcd,* rather than the awkward cpcd.

Like the round robin competition model for tournaments, we can think of a pcd as a model in which each vertex competes with all others, where if $x, y \in V(D)$, w_{xy} denotes the probability that x will beat y. Based on this we define a concept of domination in pcds. If D is a pcd, and $x, y, z \in V(D)$ we say that x and y *dominate* z if $w_{xz} + w_{yz} \geq 1$. This is analogous to a dominant pair in a tournament, when we ask the question as to which pairs of vertices $\{x, y\}$ dominate D. We say that vertices x and y form a *dominant pair* in a complete paired comparison digraph D if for all $z \in V(D) - \{x, y\}$ we have $w_{xz} + w_{yz} \geq 1$. A tournament can be considered a pcd with arc weights 0 or 1.

So, a dominant pair in a tournament is also a dominant pair in the tournament when it is treated as a pcd.

If D is a pcd, then we define the *domination graph* of D, denoted $\text{dom}(D)$, on the same vertices of D with $[x, y]$ an edge of $\text{dom}(D)$ if and only if $\{x, y\}$ is a dominant pair. By studying the domination graphs of pcds, we can get a feel for who a dominant competitor may be in the model. In [51] Moon and Pullman also attempt to rank competitors in this model using the eigenvalues of generalized tournaments.

Surprisingly, there is a distinction between domination graphs of pcds in which some competitors are equally matched (i.e., the arcs between them have weight .5) and domination graphs of pcds having no equally matched competitors. We characterize domination graphs of pcds in which $w_{xy} \neq .5$ for all $x, y \in D$. We also characterize domination graphs of pcds in which the domination graphs are connected graphs, and in which the domination graphs have no isolated vertices.

5.2 Preliminary results

In this section we give some preliminary results that will be used throughout the chapter. A key to this study is the following lemma.

Lemma 5.1 *Let D be a complete paired comparison digraph. For any 2 vertex disjoint edges in dom(D), say $[r, s]$ and $[u, v]$, we have that in D*

$$w_{ru} = w_{us} = w_{sv} = w_{vr}.$$

Proof: Since $\{r, s\}$ and $\{u, v\}$ are dominant pairs in D, we know that $w_{vr} + w_{ur} \geq 1$, $w_{rv} + w_{sv} \geq 1$, $w_{su} + w_{ru} \geq 1$, and $w_{vs} + w_{us} \geq 1$. Also, by the definition of a pcd, $w_{ru} + w_{ur} = 1$, $w_{us} + w_{su} = 1$, $w_{sv} + w_{vs} = 1$, and $w_{vr} + w_{rv} = 1$. Thus,

$$w_{ru} = 1 - w_{ur} \leq w_{vr} = 1 - w_{rv} \leq w_{sv} = 1 - w_{vs} \leq w_{us} = 1 - w_{su} \leq w_{ru},$$

and so

$$w_{ru} = w_{us} = w_{sv} = w_{vr}.$$

■

Recall U_n is the rotational tournament on the vertices $\{0, 1, 2 \ldots, n-1\}$ with $i \to j$ if and only if $j - i$ is odd modulo n. In [44], Merz et. al. show that U_n is the unique tournament which has an n-cycle as its domination graph. We now prove a similar result for an analogous pcd. Choose $0 \leq p \leq 1$. We define the pcd $U_{n,p}$ on vertex set $\{1, 2, \ldots, n\}$ by $w_{ij} = p$ if and only if $j - i$ is odd modulo n, and $w_{ij} = 1 - p$ otherwise.

Lemma 5.2 *If n is odd, and C_n is an induced subgraph of the domination graph of some complete paired comparison digraph D, then the vertices which induce the cycle in dom(D) induce $U_{n,p}$ in D, for some p, $0 \leq p \leq 1$.*

Proof: We assume $V(C_n) = \{1, 2, \ldots, n\}$ and $E(C_n) = \{[i, i+1] : 1 \leq i \leq n-1\} \cup \{[n, 1]\}$. Suppose that $n \geq 5$. Consider the arc $(1, j)$, $2 \leq j \leq n-2$. Assume that $j - 1$ is odd. Let $w_{1j} = p$. Apply Lemma 5.1 to edges $[n, 1]$ and $[j, j+1]$ to see that $w_{1j} = w_{jn} = w_{n(j+1)} = w_{(j+1)1} = p$. Apply Lemma 5.1 to edges $[1, 2]$ and $[j+1, j+2]$ to see that $w_{(j+2)1} = w_{1(j+1)} = 1 - w_{(j+1)1} = 1 - p$. Thus, $w_{1(j+2)} = p$, and $w_{1(j+1)} = 1 - p$. This implies that all arc weights w_{1j} are p when $j - 1$ is odd and $1 - p$ when $j - 1$ is even, $2 \leq j \leq n$. By an identical argument with i in place of 1, if $w_{ij} = q$ for $j-i$ odd, then all arc weights w_{ij} are q when $j-i$ is odd and $1-q$ when $j-i$ is even. Thus, $p = w_{12} = 1-w_{21} = 1-q$. So, $p+q=1$. Consequently, in D, $w_{ij} = p$ if and only if $j-i$ is odd modulo n, and $w_{ij} = 1-p$ otherwise. Thus, $V(C_n)$ induces $U_{n,p}$ in D.

To complete the proof, suppose that $n = 3$. Then, since $\{1,3\}$, $\{1,2\}$ and $\{2,3\}$ form dominant pairs, $w_{12} + w_{32} \geq 1$, $w_{13} + w_{23} \geq 1$, and $w_{21} + w_{31} \geq 1$. This means that $w_{12}+(1-w_{23}) \geq 1$, $w_{23}+(1-w_{31}) \geq 1$, and $w_{31}+(1-w_{12}) \geq 1$. Thus,

$$w_{12} \geq w_{23} \geq w_{31} \geq w_{12}.$$

So, the result follows. ■

Lemma 5.3 *Let D be a complete paired comparison digraph with at least 4 vertices. Then, dom(D) $= K_n$ if and only if $w_{xy} = .5$ for all distinct $x, y \in V(D)$.*

Proof: Suppose dom(D) $= K_n$, and pick distinct $x, y \in V(D)$. As $n \geq 4$, there are vertex disjoint edges $[x, x']$ and $[y, y']$ in dom(D). By Lemma 5.1, $w_{xy} = w_{yx'} = w_{x'y'} = w_{y'x}$. Now, apply Lemma 5.1 to edges $[x, y']$ and $[x'y]$ to see that $w_{yx} = w_{xx'} = w_{x'y'} = w_{y'y}$. Therefore, $w_{xy} = w_{x'y'} = w_{yx}$. But $w_{yx} = 1 - w_{xy}$, so $w_{xy} = w_{yx} = .5$, as desired. The converse is immediate. ■

Note, it was shown in Lemma 5.2 that if D is a pcd with 3 vertices x, y, z so that $w_{xy} = w_{yz} = w_{zx} > .5$, then dom($D$) $= K_3$. So, sufficiency of Lemma 5.3 requires our pcd have 4 or more vertices.

Theorem 5.4 *Let D be a complete paired comparison digraph, and $S \subseteq V(D)$.*

Let D' be the pcd induced on S, then the subgraph of dom(D) induced on S is a subgraph of dom(D').

Proof: Let $\{x, y\}$ be a dominant pair in D, with $x, y \in S$. Then, for all $v \in V(D) - \{x, y\}$, $w_{xv} + w_{yv} \geq 1$. In particular, this is true of all $v \in S$. Thus, $\{x, y\}$ form a dominant pair in D'. This proves our result. ■

If D is a pcd, $v \in V(D)$, and $S \subseteq V(D)$, then we define the set $O_S^+(v)$ by

$$O_S^+(v) = \{x \in S : w_{vx} > .5\}.$$

If $S = V(D)$, then $O_{V(D)}^+(v)$ will be abbreviated as $O^+(v)$.

Lemma 5.5 *Let D be a complete paired comparison digraph, $v \in V(D)$, and $S \subseteq V(D)$. Then, $O_S^+(v)$ forms an independent set in dom(D).*

Proof: Let $x, y \in O_S^+(v)$. Then, $w_{xv} < .5$, and $w_{yv} < .5$, so $w_{xv} + w_{yv} < 1$. That is, $\{x, y\}$ does not form a dominant pair. ■

Lemma 5.6 *Let D be a pcd. There exists a pcd D' on the same vertices as D such that dom(D') =dom(D), $w_{xy} < 1$ for all $x, y \in V(D')$, and w_{xy} is greater than, less than, or equal to .5 in D' if and only if it is greater than, less than, or equal to .5 in D respectively.*

Proof: If $w_{xy} < 1$ for all $x, y \in V(D)$, we are done. So, assume $w_{xy} = 1$ for some $x, y \in V(D)$. Let $k = \min\{|w_{uv} - .5| : u, v \in V(D), w_{uv} \neq .5\}$, and set $\epsilon = \frac{k}{2}$. For simplicity, for the rest of this proof we will let w'_{xy} denote the weight of arc (x, y) in D' and w_{xy} the weight of arc (x, y) in D. We construct D' by setting $V(D') = V(D)$, and for distinct $u, v \in V(D')$ assigning weights to (u,v) as follows. If $w_{uv} = .5$, set $w'_{uv} = .5$. If $w_{uv} > .5$, set $w'_{uv} = w_{uv} - \epsilon$, and if $w_{uv} < .5$, set $w'_{uv} = w_{uv} + \epsilon$. So, $w'_{xy} < 1$ for all $x, y \in V(D')$, and w'_{xy} is less than, greater than, or equal to .5 if and only if w_{xy} is. We now show dom(D') =dom(D).

Pick $[x,y] \in E(\text{dom}(D))$. So, for all $z \in V(D)$, $w_{xz} + w_{yz} \geq 1$. Choose $z \in V(D)$. If $w_{xz} = w_{yz} = .5$, then $w'_{xz} + w'_{yz} = .5 + .5 = 1$. If exactly one of w_{xy} and w_{yz} is .5, then the other must be strictly greater than .5 or else $w_{xz} + w_{yz} < .5 + .5 = 1$. So, $w'_{xz} + w'_{yz} = w_{xz} + w_{yz} - \epsilon$. By our choice of ϵ, $\epsilon < w_{xz} + w_{yz} - 1$, so $w'_{xz} + w'_{yz} \geq 1$. So, assume w_{xz} and w_{yz} are not .5. By a similar argument, at least one of w_{xz} and w_{yz} must be strictly greater than .5. So,

$$w'_{xz} + w'_{yz} = \begin{cases} w_{xz} + \epsilon + w_{yz} - \epsilon, & \text{if } w_{xz} > .5, w_{yz} < .5, \\ w_{xz} - \epsilon + w_{yz} + \epsilon, & \text{if } w_{xz} < .5, w_{yz} > .5, \\ w_{xz} + \epsilon + w_{yz} + \epsilon, & \text{if } w_{xz} > .5, w_{yz} > .5. \end{cases}$$

In all three cases, $w'_{xz} + w'_{yz} \geq 1$ since $w_{xz} + w_{yz} \geq 1$. Thus, $[x,y] \in E(\text{dom}(D'))$

Now choose distinct $x, y \in V(D')$ so that $[x,y] \notin E(\text{dom}(D))$. So, for some $z \in V(D)$, $w_{xz} + w_{yz} < 1$. So either $w_{xz} < .5$ or $w_{yz} < .5$, or else $w_{xz} + w_{yz} \geq 1$. So,

$$w'_{xz} + w'_{yz} = \begin{cases} w_{xz} + w_{yz} + \epsilon, & \text{if } w_{xz} = .5, w_{yz} < .5, \\ w_{xz} + \epsilon + w_{yz}, & \text{if } w_{xz} < .5, w_{yz} = .5, \\ w_{xz} + \epsilon + w_{yz} - \epsilon, & \text{if } w_{xz} > .5, w_{yz} < .5, \\ w_{xz} - \epsilon + w_{yz} + \epsilon, & \text{if } w_{xz} < .5, w_{yz} > .5, \\ w_{xz} - \epsilon + w_{yz} - \epsilon, & \text{if } w_{xz} < .5, w_{yz} < .5. \end{cases}$$

In the first two cases, our choice of epsilon gives $\epsilon < 1 - (w_{xz} + w_{yz})$, and so $w_{xz} + w_{yz} + \epsilon < 1$. In the last three cases, we have $w'_{xz} + w'_{yz} < 1$, since $w_{xz} + w_{yz} < 1$, and so $[x,y] \notin E(\text{dom}(D))$. Thus dom($D$) =dom($D'$). ■

The next proposition indicates why a characterization of domination graphs of pcds is difficult. Consider the situation for tournaments. There are strict requirements on a graph G so that there exist a tournament T for which dom(T) contains G as an induced subgraph. This is not so in the context of pcds since any graph will do as seen in the next proposition.

Proposition 5.7 *Let G be a graph, then there exists a complete paired comparison digraph D for which dom(D) contains G as an induced subgraph.*

Proof: Let G be a graph on n vertices, and construct a pcd D in the following way. Start with $V(D) = V(G)$. For each $x, y \in V(G)$, let $w_{xy} = .5$. Now, for each pair $\{i, j\}$ of nonadjacent distinct vertices in G, add a vertex v_{ij} (same as v_{ji}) to $V(D)$, and set $w_{v_{ij}i} = w_{v_{ij}j} = 1$. Also, for each $z \in V(G)$ with $[i, z] \in E(G)$ or $[j, z] \in E(G)$, set $w_{zv_{ij}} = 1$. Set all other weights to .5. We now show the construction gives the desired result.

Consider two vertices $x, y \in V(G)$. Suppose that $[x, y] \in E(G)$. If $[x, y] \notin$ dom(D), then there is a $z \in V(D)$ so that $w_{xz} + w_{yz} < 1$. Because arc weights are $0, .5$, or 1, this implies that at least one of w_{xz} or w_{yz} is 0. Suppose $w_{xz} = 0$; then by construction $z = v_{xk}$ for some vertex k. But, since $[x, y] \in E(G)$, the construction yields $w_{yz} = 1$. So, $w_{xz} + w_{yz} = 1$, a contradiction. Hence $[x, y]$ is in dom(D).

On the other hand, if $[x, y] \notin E(G)$, then using $z = v_{xy}$ we see that $w_{xz} + w_{yz} = 0 + 0 = 0$ so that $\{x, y\}$ is not a dominating pair in D. That is, $[x, y] \notin E(\text{dom}(D))$. So, G is an induced subgraph of dom(D). ■

5.3 Complete paired comparison digraphs with no arc weight .5

As Proposition 5.7 shows, characterizing the domination graphs of pcds is not an easy task. However, the problems seem to arise from equally matched competitors. So, in this section, we will assume that $w_{xy} \neq .5$ for every arc (x, y) in our pcd. This is not such a bad assumption for it is rare that two competitors are truly evenly matched. This assumption also gives a strong relation between tournaments and pcds, as we shall see.

Let D be a pcd with no arc weight equal to .5. With D we associate a digraph T defined on the same vertices of D, where $x \to y$ in T if $w_{xy} > .5$. So for all $x, y \in V(T)$, we have exactly one of $x \to y$ or $y \to x$, i.e. T is a tournament. We refer to T as D's *associated tournament.*

Lemma 5.8 *Let D be a complete paired comparison digraph with no weight equal to .5, and T the associated tournament. Then dom(D) is a subgraph of dom(T). Furthermore, if D has only two weights, $a > .5$, and $1 - a$, then dom(D) =dom(T).*

Proof: Let $\{x, y\}$ be a dominant pair in D. Then for each $v \in V(D) - \{x, y\}$, $w_{xv} + w_{yv} \geq 1$. So, since $w_{xv} \neq .5$ and $w_{yv} \neq .5$, either $w_{xv} > .5$ or $w_{yv} > .5$. This implies that either $x \to v$ or $y \to v$ in T. Thus, $\{x, y\}$ is a dominant pair in T. So, dom(D) is a subgraph of dom(T).

Now, suppose that D has only two weights, $a > .5$ and $1 - a$. Let $\{x, y\}$ be a dominant pair in T. That is, for any other vertex v, $x \to v$ or $y \to v$, i.e. $w_{xv} > .5$ or $w_{yv} > .5$. As $a > .5$, this implies that $w_{xv} = a > 1 - a$ or $w_{yv} = a > 1 - a$, so $w_{xv} + w_{yv} \geq a + (1 - a) = 1$. This means that $\{x, y\}$ is a dominant pair in D. Consequently, dom(D) = dom(T). ■

We point out that if the pcd D has more than two weights, we may have proper containment in the previous lemma. For instance, consider the pcd D

in Figure 5.1. Shown next to D is its associated tournament T. As shown in Figure 5.2, dom(T) is a spiked 3-cycle, while dom(D) is only a 2-path.

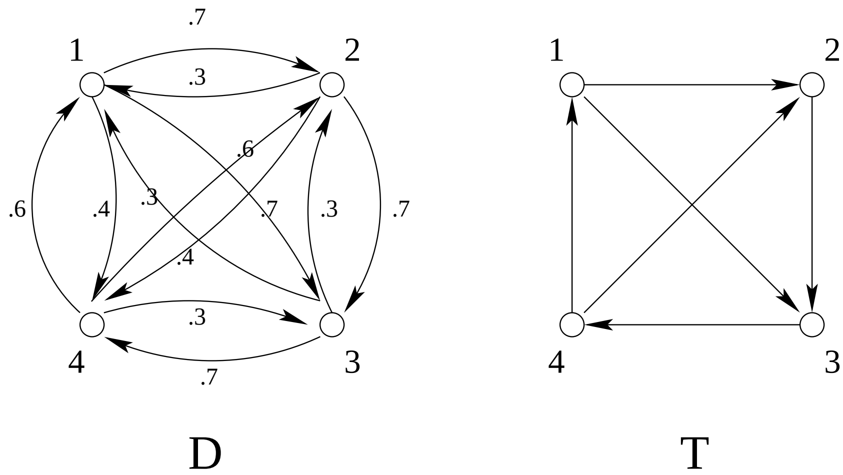

Figure 5.1: A pcd and its associated tournament

Recall from Theorem 1.8, that the domination graph of a tournament is either a spiked odd cycle with or without isolated vertices, or a forest of caterpillars. This result together with Lemma 5.8 give us Theorem 5.9.

Theorem 5.9 *Let D be a complete paired comparison digraph in which no arc has weight equal to .5. Then dom(D) is a spiked odd cycle with or without isolated vertices, or a forest of caterpillars.*

Proof: Note that a subgraph of a forest of caterpillars is a forest of caterpillars. Also, any subgraph of a spiked odd cycle is a either a spiked odd cycle, or a forest of caterpillars. Hence, the result follows from Lemma 5.8 and Theorem 1.8. ■

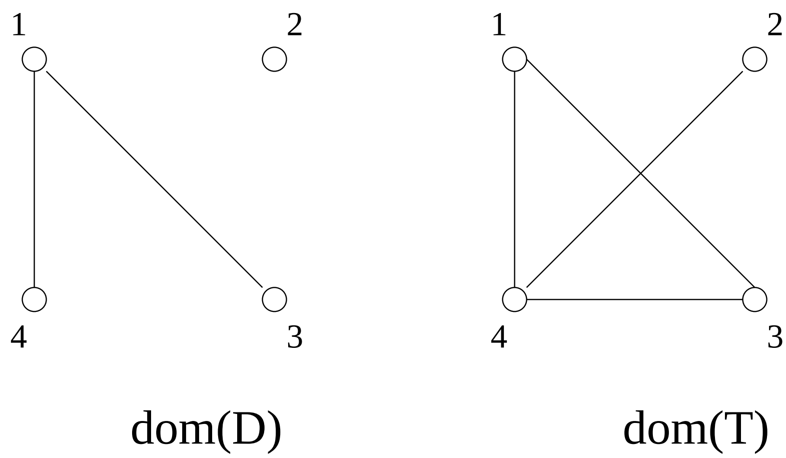

Figure 5.2: The domination graphs of a pcd and its associated tournament

We obtain Lemma 5.11 by essentially deleting an edge from the graphs constructed by Merz et. al. in Theorem 5.10.

Theorem 5.10 [44] *Any graph G consisting of a spiked odd cycle with possibly some isolated vertices is the domination graph of some tournament.*

Recall, given a caterpillar G, we define a *spine* of G as a path of maximum length in G.

Lemma 5.11 *If G is a caterpillar, then there exists a complete paired comparison digraph D in which no arc has weight .5 and dom$(D) = G$.*

Proof: Let C be a caterpillar with a spine $v_1, v_2, \ldots, v_m$. Construct the spiked odd cycle C' by adding to C the edge $[v_m, v_1]$ if m is odd, or the edge $[v_m, v_2]$ if m is even. From Theorem 5.10 we know that there exists a tournament with C' as its domination graph. In particular, the proof of Theorem 5.10 shows that the tournament T formed in the following way has C' as its domination graph. Define T on the same vertices as C', and orient the arcs on T so that U_m or U_{m-1}, whichever of m or $m-1$ is odd, is the subtournament on $v_1, \ldots, v_m$ or $v_2, \ldots, v_m$ respectively. Furthermore, if y is pendent to the cycle in C' and $[y, x] \in E(C')$, then let $x \to y$ in T, and for all z in $V(C') - \{y\}$, let $y \to z$ if $z \to x$ and let $z \to y$ if $x \to z$. It does not matter which direction the arcs between the pendant vertices have.

We now construct our pcd D from T. First, let $V(D) = V(T)$. Choose $.5 < a < 1$, and if $x \to y$ in T, then set $w_{xy} = a$, and $w_{yx} = 1 - a$. Finally, let $a < b < 1$ and change $w_{v_{m-1}v_m} = b$, and $w_{v_m v_{m-1}} = 1 - b$. Then for all $x, y \in V(D) - \{v_m\}$ we know from Lemma 5.8 that $[x, y] \in E(\text{dom}(D))$ if and only if $[x, y] \in E(\text{dom}(T))$ since all the arcs except (v_{m-1}, v_m) have weight a or $1 - a$, and $w_{v_{m-1}v_m} > a$. Furthermore, $\{v_m, v_{m-1}\}$ is a dominating pair, since it is one in T, and all arcs in $A(D) - \{(v_m, v_{m-1}), (v_{m-1}, v_m)\}$ have weight either a or $1 - a$. However, if $x \neq v_{m-1}$ then, $\{x, v_m\}$ does not form a dominant pair since $w_{xv_{m-1}} + w_{v_m v_{m-1}} < 1$. Since v_m is the end of the spine, it has no pendant vertices in C, and so $\text{dom}(D) = C$. ∎

Before we characterize the domination graphs of pcds with no arc weight .5 we will first do it for the case in which the domination graph is a connected graph. Theorem 5.12 below is taken from [45]. Comparing this result to Theorem 5.13 below, we see that in pcds, although the domination graph is a subgraph of the domination graph of some tournament, our characterization is less restrictive.

Theorem 5.12 [45] *A connected graph is the domination graph of a tournament* *if and only if it is a spiked odd cycle, a star, or a caterpillar with a triple end.*

Theorem 5.13 *If G is a connected graph, then $G =$dom(D) for some complete paired comparison digraph D with no arc having weight .5 if and only if G is a spiked odd cycle or a caterpillar.*

Proof: From Theorem 5.12 and Lemma 5.8 we know that for any spiked odd cycle there exists a pcd with no arc weight .5 whose domination graph is that spiked odd cycle. Also, from Lemma 5.11 we know that for any caterpillar, there exists a pcd with no arc weight .5 whose domination graph is that caterpillar. Also, Theorem 5.9 insures that these are the only possible connected domination graphs. Thus, the result follows. ■

For a positive integer n and graph H, we let nH denote the graph which consists of n disjoint copies of H. In particular, nK_1 is a graph consisting of n isolated vertices.

Theorem 5.14 *If G is a collection of isolated vertices, then there exists a pcd D such that dom$(D) = G$ if and only if G is not $2K_1$ or $3K_1$.*

Proof: First note that if a pcd has only two vertices, then they vacuously dominate all other vertices in the pcd. Thus $2K_1$ cannot be the domination graph of a pcd. Now suppose there exists a pcd D with dom$(D) = 3K_1$. Let $V(D) = \{1, 2, 3\}$. Then, $w_{12} + w_{32} < 1$, $w_{21} + w_{31} < 1$, $w_{13} + w_{23} < 1$. So,

$$w_{12} + w_{21} + w_{23} + w_{32} + w_{13} + w_{31} < 3.$$

But this contradicts the fact that

$$w_{12} + w_{21} + w_{23} + w_{32} + w_{13} + w_{31} = 3.$$

Thus, no such pcd exists.

Now, assume that G consists of 7 or more isolated vertices. The quadratic residue tournament on 7 vertices, QR_7, has $\gamma(QR_7) = 3$, and so has no dominant pairs. Thus the domination graph of QR_7 is empty. Consequently, any tournament of order $n \geq 7$ that contains such a subtournament of order 7, all of whose vertices dominate the remaining $n-7$ vertices, has a domination graph consisting of n isolated vertices. So, pick a tournament T with $\mathrm{dom}(T) = G$, and let D be a pcd with no arc weight .5 whose associated tournament is T. Since $\mathrm{dom}(D)$ is a subgraph of $\mathrm{dom}(T)$ and $\mathrm{dom}(T)$ has no edges, $\mathrm{dom}(D) = \mathrm{dom}(T) = G$.

We now consider the case where G is $4K_1$. Let $V(G) = \{1, 2, 3, v\}$. Construct a pcd D on the vertices of G in the following way. Let $\{1, 2, 3\}$ induce $U_{3,.7}$ in D and let $w_{vi} = .6$ for each $i \in \{1, 2, 3\}$. Then, no pair of vertices $\{i, j\}$ from $\{1, 2, 3\}$ can be dominant, since for each such pair $w_{iv} + w_{jv} = .8$. Also, $w_{v(i-1)} + w_{i(i-1)} = .6 + .3 = .9$ for each $i = 1, 2, 3$ so v is not part of any dominant pair. These are all possible combinations, so $\mathrm{dom}(D) = 4K_1$.

Now, assume that $G = 5K_1$. Let D be the pcd with $V(D) = \{1, 2, 3, 4, 5\}$ and arc weights assigned as follows using addition modulo 5. For each $i \in V(D)$ set $w_{i(i+1)} = .6$ and $w_{i(i+3)} = .7$. We now examine all possible pairs of vertices. Pick $i \in V(D)$. Since the pair $\{i, i+2\}$ is equivalent to the pair $\{i, i+3\}$ and the pair $\{i, i+1\}$ is equivalent to the pair $\{i, i+4\}$ we only need to look at the pairs $\{i, i+1\}$ and $\{i, i+2\}$. Then,

$$w_{i(i+2)} + w_{(i+1)(i+2)} = .3 + .6 = .9$$

and

$$w_{i(i+4)} + w_{(i+2)(i+4)} = .4 + .3 = .7.$$

Therefore no two vertices form a dominant pair, and so $\text{dom}(D) = 5K_1$.

Now suppose $G = 6K_1$, and let $V(G) = \{1, 2, 3, 4, 5, v\}$. Construct a pcd D on the vertices of G in the following way using addition modulo 5. Let $\{1, 2, 3, 4, 5\}$ induce $U_{5,.7}$ in D and set $w_{vi} = .6$ for each $i \in \{1, 2, 3, 4, 5\}$. So, if $i, j \in \{1, 2, 3, 4, 5\}$ then $w_{iv} + w_{jv} = .4 + .4 = .8$ so no pair of vertices not containing v is dominant. Now, if $i = 1, 2, 3, 4,$ or 5 then $w_{v(i-1)} + w_{i(i-1)} = .6 + .3$ and $\{v, i\}$ is not a dominant pair, as desired. Thus, $\text{dom}(D) = 6K_1$. This completes the proof. ■

In the proof of Theorem 5.14, none of the pcds we construct have arc weights of .5. So, this result is also true in the context of pcds having no arc weight .5.

Lemma 5.15 *If G is a forest of caterpillars with at least one nontrivial component, then there exists a complete paired comparison digraph D with no arc weight of .5 for which dom(D) = G.*

Proof: Let $A_1, A_2, \ldots, A_k$ be the nontrivial caterpillars of G, with spines $v_{1,1}, v_{1,2}, \ldots, v_{1,m_1};\ v_{2,1}, \ldots, v_{2,m_2}; \cdots ; v_{k,1}, \ldots, v_{k,m_k}$ respectively. Form the spiked odd cycle G' by adding to G the edges $[v_{i,m_i}, v_{i+1,1}]$ for each $i = 1, \ldots, k-1$, and adding the edge $[v_{k,m_k}, v_{1,1}]$ if $\sum_{i=1}^{k} m_i$ is odd, and the edge $[v_{k,m_k}, v_{1,2}]$ if $\sum_{i=1}^{k} m_i$ is even. Now, let G'' be the graph which consists of G' and the isolated vertices of G. Note from Theorem 5.12, we know that there exists a tournament T such that $\text{dom}(T) = G''$. In particular, we can use the tournament T defined in the following way. Let $V(T) = V(G)$, and orient the arcs on the subtournament defined by the vertices on the cycle of G' to be the rotational tournament U_l, where l is the number of vertices in the cycle of G'. Now, if y is pendant to the cycle in G' at some vertex x, then in T let $x \to y$, and for all z in the cycle in G' if $x \to z$ in T then let $z \to y$ in T, and if $z \to x$ in T then let $y \to z$ in T. If v is isolated in G, then let $u \to v$ in T for all $u \in V(G')$. Orient arcs between isolated vertices arbitrarily. One can verify that, $\text{dom}(T) = G''$.

We now construct our pcd, D, in the following way. Let $V(D) = V(T)$. Let $.5 < a < 1$, and if $x \to y$ in T, then set $w_{xy} = a$, and $w_{yx} = 1 - a$. Now, choose $a < b < 1$ and for all $i = 1, \ldots, k$ set $w_{v_{i,m_i-1}v_{i,m_i}} = b$ and $w_{v_{i,m_i}v_{i,m_i-1}} = 1 - b$. Then, for all $u, v \in V(D) - \{v_{i,m_i}\}$, $\{u, v\}$ is a dominant pair in T if and only if $\{u, v\}$ is a dominant pair in D, from Lemma 5.8, since for all arcs incident with u or v, they either have weight a, $1-a$ or $b > a$. Also, $\{v_{i,m_i-1}, v_{i,m_i}\}$ is a dominant pair in D if and only if it is a dominant pair in T, for all $i = 1, \ldots, k$. This follows from Lemma 5.8 and the fact that for all arcs incident to v_{i,m_i} and v_{i,m_i-1}, other than (v_{i,m_i-1}, v_{i,m_i}) and (v_{i,m_i}, v_{i,m_i-1}) the weights are either a or $1 - a$, for all $i = 1, \ldots, k$. Furthermore, for all $i = 1, \ldots, k$, if $x_i \neq v_{i,m_i-1}$ then, $\{x_i, v_{i,m_i-1}\}$ does not form a dominant pair in D since $w_{x_i v_{i,m_i-1}} + w_{v_{i,m_i}v_{i,m_i-1}} < 1$. Since v_{i,m_i} is the end of a spine for all $i = 1, \ldots, k$, it is incident only to v_{i,m_i-1} in G. Thus, $\text{dom}(D) = G$. ■

Theorem 5.16 *A graph G is the domination graph of a complete paired comparison digraph with no arc having weight .5 if and only if G is a spiked odd cycle with or without isolated vertices, or a forest of caterpillars other than $2K_1$ or $3K_1$.*

Proof: From Theorem 5.9 we know that G must be a spiked odd cycle with or without isolated vertices or a forest of caterpillars. From Lemma 5.8 and Theorem 5.10 there is a pcd with no arc weight of .5 which has any spiked odd cycle with or without isolated vertices as its domination graph. From Lemma 5.15 and Theorem 5.14, for any forest of caterpillars other than $2K_1$ and $3K_1$, we can find a pcd with no arc weight of .5 which has that forest of caterpillars as its domination graph. Thus, the result follows. ∎

5.4 Connected domination graphs

Recall from Proposition 5.7 any graph we wish can be an induced subgraph of a domination graph of a pcd. In the proof of Proposition 5.7 the graphs we construct have several isolated vertices. So, in our attempt to characterize the domination graphs of pcds, a logical first step is to see which connected graphs can be the domination graphs of pcds. This characterization together with Theorem 5.4 is a large step in characterizing the domination graphs of pcds.

In the next theorem, we refer to the graph $NC7$. This is the smallest tree which is not a caterpillar and is shown in Figure 5.3. One should note that a tree is a caterpillar if and only if $NC7$ is not a subgraph.

Theorem 5.17 *If G is a connected graph, and G is the domination graph of a complete paired comparison digraph D, then $NC7$ is not an induced subgraph of G.*

Proof: Assume to the contrary that $NC7$ is an induced subgraph of G, and G is the domination graph of some pcd D. Let $\{1, 2, \ldots, 7\}$ denote the set

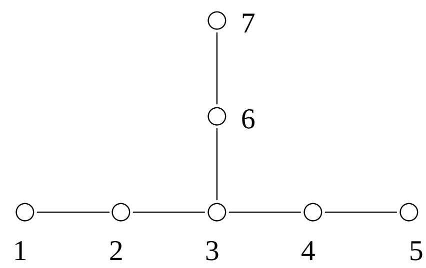

Figure 5.3: NC7

of vertices which induce $NC7$. By applying Lemma 5.1 to the pairs of dominant pairs given in the following table in the order they appear, we force the situation in Figure 5.4, where each arc has weight $0 \leq x \leq 1$ in D.

step	pairs
1	$\{1,2\},\{6,7\}$
2	$\{2,3\},\{6,7\}$
3	$\{1,2\},\{3,6\}$
4	$\{3,4\},\{6,7\}$
5	$\{4,5\},\{6,7\}$
6	$\{4,5\},\{3,6\}$
7	$\{1,2\},\{3,4\}$
8	$\{1,2\},\{4,5\}$

Now, applying Lemma 5.1 to the pairs $[2,3]$ and $[4,5]$ we get that $x = w_{34} = w_{43} = 1 - w_{34}$ so $x = .5$. This implies that $w_{ij} = .5$ for each i, j except perhaps for $\{i,j\} = \{1,2\},\{4,5\},\{6,7\}$. Since $w_{13} = .5$, $w_{31} = .5$. As $\{2,3\}$ is a dominant pair, $w_{21} \geq .5$. Similarly, $w_{45} \geq .5$ (as $\{3,4\}$ is a dominant pair), and $w_{67} \geq .5$ (as $\{3,6\}$ is a dominant pair). This implies that $\{4,2\}$, $\{4,6\}$, and $\{2,6\}$ are dominant pairs, in $D[\{1,2,\ldots,7\}]$. If G has only 7 vertices, then $G \neq$dom(D), a contradiction.

Suppose that G has more than 7 vertices. Let $v \in V(G) - \{1,2,\ldots,7\}$. Since

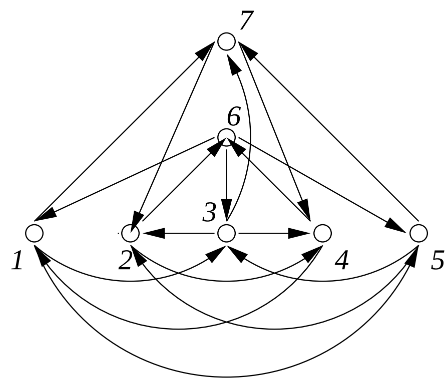

Figure 5.4: The arcs with weight x

G is connected there exists a shortest path P from $\{1, 2, \ldots, 7\}$ to v, where P is given by $u_1, \ldots, u_m = v$ in G, and $u_1 \in \{1, 2, \ldots, 7\}$. We next show that $w_{2v} \geq .5$ and $w_{6v} \geq .5$.

Consider the case in which $u_1 \neq 3$. We show that $w_{2v} \geq .5$, $w_{6v} \geq .5$ and $w_{3v} \geq .5$ by induction on $m \geq 2$. Suppose that $m = 2$ (so $u_2 = v$). If $u_1 = 6$, use Lemma 5.1 on edges $[2, 3]$ and $[u_1, u_2] = [6, v]$ to deduce that $w_{2v} = .5$ and $w_{3v} = .5$. Since $\{3, 6\}$ is a dominating pair and $w_{3v} = .5$, $w_{6v} \geq .5$, as desired. If $u_1 = 2$ proceed similarly using edges $[3, 6]$ and $[2, v]$. If $u_1 \neq 2, 3, 6$, use Lemma 5.1 on edges $[u_1, v]$ and $[2, 3]$ to deduce $w_{2v} = w_{3v} = .5$, then use Lemma 5.1 on edges $[u_1, v]$ and $[3, 6]$ to deduce that $w_{6v} = .5$ as desired.

Now, suppose that $m > 2$ and that $w_{2u_i} \geq .5$, $w_{6u_i} \geq .5$ and $w_{3u_i} \geq .5$ for $i = 2, \ldots, k-1$, where $k \leq m$. Use Lemma 5.1 on $[2, 3]$ and $[u_{k-1}, u_k]$ to deduce $w_{2u_k} = w_{3u_{k-1}} \geq .5$. Again, use Lemma 5.1 on $[3, 6]$ and $[u_{k-1}, u_k]$ to deduce $w_{6u_k} = w_{3u_{k-1}} \geq .5$ and $w_{3u_k} = w_{6u_{k-1}} \geq .5$, as desired. This completes the case in which $u_1 \neq 3$.

Now, consider the case in which $u_1 = 3$. We show that $w_{2v} = w_{6v} = w_{1v} = w_{7v} = .5$ by induction on $m \geq 2$. Suppose $m = 2$ (so $u_2 = v$). Use Lemma 5.1 on edges $[1, 2]$ and $[3, v]$ and then on the edges $[6, 7]$ and $[3, v]$ to deduce that $w_{2v} = w_{1v} = w_{6v} = w_{7v} = .5$, as required. Now suppose that $w_{2u_i} = w_{1u_i} = w_{6u_i} = w_{7u_i} = .5$ for $i = 2, \ldots, k-1$, where $k \leq m$. Use Lemma 5.1 on edges $[1, 2]$ and $[u_{k-1}, u_k]$ and then on edges $[6, 7]$ and $[u_{k-1}, u_k]$ to deduce that $w_{2u_k} = w_{1u_k} = .5$ (since $w_{1u_{k-1}} = .5$) and that $w_{6u_k} = w_{7u_k} = .5$ (since $w_{6u_{k-1}} = .5$). By induction, $w_{2v} = w_{6v} = w_{1v} = w_{7v} = .5$, as desired.

In particular, $w_{2v} + w_{6v} \geq 1$, for all $v \in V(G) - \{2, 6\}$. Recall that $\{2, 6\}$ is a dominant pair in $D[\{1, 2, \ldots, 7\}]$. Thus, $\{2, 6\}$ is a dominant pair, but this means that $\text{dom}(D) \neq G$, a contradiction. This completes the proof. ■

This result, together with the fact that a tree is a caterpillar if and only if it has no $NC7$ as a subgraph gives us the following corollary.

Corollary 5.18 *If a tree is the domination graph of a complete paired comparison digraph, then it is a caterpillar.*

Our next lemma helps to simplify many of the proofs of the following results.

Lemma 5.19 *Let D be a complete paired comparison digraph. Suppose that G is a subgraph of dom(D) such that for every $x \in V(G)$, $d_G(x) \geq 2$. Also, suppose that $w_{xy} = .5$ in D for all distinct x, y in $V(G)$. Let P denote a shortest path between $V(G)$ and a vertex v in $V(\text{dom}(D)) - V(G)$ given by $u_1, u_2, \ldots, u_m = v$, where $u_1 \in V(G)$ and $m \geq 2$. Then, for $m \geq 3$ and for all $x \in V(G)$, $w_{u_i x} = .5$*

for all $i = 2, \ldots, m$, *and for* $m = 2$, $w_{u_2 x} = .5$ *for all* $x \in V(G) - \{u_1\}$ *and* $w_{u_1 u_2} \geq .5$.

Proof: Suppose $m = 2$ (so $v = u_2$). Pick $x \in V(G) - \{u_1\}$. By assumption there is a vertex $y \in V(G) - \{u_1\}$ so that $[x, y] \in E(G)$. Use Lemma 5.1 on edges $[x, y]$ and $[u_1, u_2]$ to deduce $w_{vx} = w_{xu_1} = w_{u_1 y} = w_{yv}$. Since $w_{xu_1} = .5$, $w_{vx} = .5$. So, $w_{vx} = .5$ for all $x \in V(G) - \{u_1\}$. As $d_G(u_1) \geq 2$, u_1 is adjacent to some $z \in V(G)$. So, $\{u_1, z\}$ is a dominant pair and $w_{zv} = .5$, so $w_{u_1 v} \geq .5$.

Assume $m \geq 3$. Note, $[u_1, u_2]$ is a shortest path from u_2 to $V(G)$. So the case $m = 2$ above yields $w_{u_2 x} = .5$ for all $x \in V(G) - \{u_1\}$. Since $d_G(u_1) \geq 2$, we can choose $y \in V(G)$ such that $[u_1, y] \in E(G)$. Applying Lemma 5.1 to the edges $[u_1, y]$ and $[u_2, u_3]$ we deduce $w_{u_2 u_1} = w_{u_1 u_3} = w_{u_3 y} = w_{y u_2} = .5$, since $y \in V(G) - \{u_1\}$. Thus, $w_{u_2 z} = .5$ for all $z \in V(G)$. Inductively assume that $w_{u_i z} = .5$ for all $z \in V(G)$ and for $i = 2, \ldots, k-1$, where $3 \leq k \leq m$. Apply Lemma 5.1 to edges $[x, y]$ and $[u_{k-1}, u_k]$ to deduce $w_{u_k x} = w_{x u_{k-1}} = w_{u_{k-1} y} = w_{y u_k}$. Since $w_{u_{k-1} y} = .5$ by the induction hypothesis, $w_{u_k x} = .5$. That is, as x was arbitrary, $w_{u_k z} = .5$ for all $z \in V(G)$. So, by induction, for all $z \in V(G)$ and for all i, $2 \leq i \leq m$, $w_{u_i x} = .5$. ■

Theorem 5.20 *If G is a connected graph and G contains an even cycle, C_{2k}, as an induced subgraph, then G is not the domination graph of any complete paired comparison digraph.*

Proof: Suppose, to the contrary, that some pcd D has G as its domination graph. Let $S = \{v_1, v_2, \ldots, v_{2k}\}$ be the set of vertices which induce the even cycle, and let the cycle be given by $[v_1, v_2], [v_2, v_3], \ldots, [v_{2k-1}, v_{2k}], [v_{2k}, v_1]$. We first do the case for $V(G) = S$. If $k = 2$ then two applications of Lemma 5.1 yields $w_{v_i v_j} = .5$ for all $i \neq j$ in $\{1, 2, 3, 4\}$. As $V(G) = S$, $\text{dom}(D) = K_4 \neq C_4 = G$, a contradiction. So, assume that $k \geq 3$. Fix a vertex v_j. We claim that

$$w_{v_j v_i} = w_{v_i v_{j-1}} = w_{v_{j-1} v_{i-1}} = w_{v_{i-1} v_j} \text{ when } i - j \pmod{2k} \text{ is even.} \tag{5.1}$$

Use induction on i. For $i - j = 2 \pmod{2k}$, apply Lemma 5.1 to edges $[v_{j-1}, v_j]$ and $[v_{j+1}, v_{j+2}]$ to deduce $w_{v_j v_{j+2}} = w_{v_{j+2} v_{j-1}} = w_{v_{j-1} v_{j+1}} = w_{v_{j+1} v_j}$ (arithmetic mod $2k$), as desired. Now, suppose that (5.1) holds for $i - j$ even where $2 \leq i - j < j - 2 \pmod{2k}$. Apply Lemma 5.1 to edges $[v_{j-1}, v_j]$ and $[v_i, v_{i+1}]$ to obtain

$$w_{v_j v_i} = w_{v_i v_{j-1}} = w_{v_{j-1} v_{i+1}} = w_{v_{i+1} v_j} (\text{arithmetic mod } 2k), \tag{5.2}$$

then apply Lemma 5.1 to edges $[v_{j-1}, v_j]$ and $[v_{i+1}, v_{i+2}]$ to obtain

$$w_{v_{j-1} v_{i+1}} = w_{v_{i+1} v_j} = w_{v_j v_{i+2}} = w_{v_{i+2} v_{j-1}} (\text{arithmetic mod } 2k). \tag{5.3}$$

Thus, as $w_{v_{i+1} v_j}$ appears in both (5.2) and (5.3),

$$w_{v_j v_{i+2}} = w_{v_{i+2} v_{j-1}} = w_{v_{j-1} v_{i+1}} = w_{v_{i+1} v_j}.$$

So, (5.1) holds by induction.

Statement (5.1) implies that

$$w_{v_j v_{j+2}} = w_{v_j v_{j+4}} = \cdots = w_{v_j v_{j-2}} (\text{arithmetic mod } 2k). \tag{5.4}$$

But, a similar statement is true for v_{j+2}, i.e.

$$w_{v_{j+2} v_{j+4}} = w_{v_{j+2} v_{j+6}} = \cdots = w_{v_{j+2} v_j} (\text{arithmetic mod } 2k). \tag{5.5}$$

Apply Lemma 5.1 to edges $[v_j, v_{j+1}]$ and $[v_{j+2}, v_{j+3}]$ to deduce

$$w_{v_j v_{j+2}} = w_{v_{j+2} v_{j+1}} = w_{v_{j+1} v_{j+3}} = w_{v_{j+3} v_j}. \tag{5.6}$$

Apply Lemma 5.1 to edges $[v_{j+1}, v_{j+2}]$ and $[v_{j+3}, v_{j+4}]$ to deduce

$$w_{v_{j+1} v_{j+3}} = w_{v_{j+3} v_{j+2}} = w_{v_{j+2} v_{j+4}} = w_{v_{j+4} v_{j+1}}. \tag{5.7}$$

Since $w_{v_{j+1}v_{j+3}}$ appears in both (5.6) and (5.7), $w_{v_j v_{j+2}} = w_{v_{j+2} v_{j+4}}$. This implies that all values in (5.4) and (5.5) are equal. In particular, $w_{v_j v_{j+2}} = w_{v_{j+2} v_j} = .5$. Thus, $w_{v_r v_s} = .5$ if $r - s$ (or $s - r$) is even. Further, if $r - s$ is odd, then by (5.1), $w_{v_r v_s} = w_{v_{r-1} v_s} = .5$ since $r - 1 - s$ is even. Similarly, if $s - r$ is odd, then $w_{v_r v_s} = .5$. In conclusion, $w_{xy} = .5$ for all $x, y \in S$. So, if $V(G) = S$, then all pairs of vertices are dominant in D, and $\text{dom}(D) = K_{2k} \neq C_{2k} = G$, a contradiction.

So, assume that S is properly contained in $V(G)$. Let $z \in V(G) - S$. Since G is connected, there exists a shortest path P from S to z, given by $u_1, u_2, \ldots, u_m = z$ in G, where $u_1 \in S$. By Lemma 5.19, $w_{tz} = .5$ for all $t \in S - \{u_1\}$, and $w_{u_1 z} \geq .5$. Now, choose $s, t \in S$ such that $[s, t] \notin E(G)$. Since $w_{sz} + w_{tz} \geq .5 + .5 = 1$ for all $z \in V(D)$, $\{s, t\}$ must be a dominant pair in D. Thus, $G \neq \text{dom}(D)$, a contradiction. ■

Lemma 5.21 *Let D be a complete paired comparison digraph. If dom(D) is connected, contains a 3-cycle C, and contains a vertex v of distance at least 2 from C, then C is contained in a larger clique in dom(D). Further, if P is a path from u to C, then some vertex of $V(P) - V(C)$ is contained in this clique.*

Proof: Let C be a 3-cycle in dom(D) given by $[1,2]$, $[2,3]$ and $[3,1]$, and suppose there is a vertex v not on C of distance at least 2 from C. Say, P is a path of shortest distance from C to v given by $u_1, u_2, \ldots, u_m = v$, where $m \geq 3$ and $u_1 \in V(C)$. Use Lemma 5.1 with each edge of C and $[u_2, u_3]$ to deduce that $w_{xy} = .5$ for each $x \in \{1,2,3\}$ and $y \in \{u_2, u_3\}$. Then, use Lemma 5.1 with $[u_1, u_2]$ and $[a, b]$ where $a, b \in \{1,2,3\} - \{u_1\}$ to deduce that $w_{au_1} = w_{bu_1} = .5$. Also, note that $w_{ab} = w_{ba} = .5$ for if $w_{ab} > .5$ then $w_{ba} + w_{u_1a} < 1$, but $[b, u_1] \in E(\text{dom}(D))$. Thus, $w_{xy} = .5$ for all $x, y \in \{1,2,3,u_2\}$. We show $w_{u_2z} \geq .5$ and $w_{iz} \geq .5$ for all $z \in V(\text{dom}(D)) - \{u_2, i\}$ for $i = 1,2,3$. This will imply that $\{u_2, i\}$ is a dominant pair, so that u_2 is adjacent to each vertex of C in dom(D).

Choose $x \in V(D) - \{1,2,3,u_2\}$. From Lemma 5.19, $w_{ix} \geq .5$ for each $i = 1,2,3$. If $[x, u_2] \in E(\text{dom}(D))$, then from Lemma 5.19, $w_{ix} = .5$ for each $i = 1,2,3$. Also, since $\{u_1, u_2\}$ is a dominant pair and $w_{u_1x} = .5$, $w_{u_2x} \geq .5$. If $[u_1, x] \in E(\text{dom}(D))$, then apply Lemma 5.1 to $[x, u_1]$ and $[u_2, u_3]$ to see that $w_{u_2x} = w_{u_1u_2}$. Also from Lemma 5.19, we know that $w_{u_1u_2} = .5$ and so $w_{u_2x} = .5$. If $[x, u_1], [x, u_2] \notin E(\text{dom}(D))$, then since dom$(D)$ is connected, there exists $a \in V(D)$, $a \neq u_1, u_2$ so that $[x, a] \in E(\text{dom}(D))$. Applying Lemma 5.1 to $[u_1, u_2]$ and $[x, a]$ we see that $w_{u_2x} = w_{xu_1}$. By Lemma 5.19, $w_{xu_1} = .5$, so $w_{u_2x} = .5$. Therefore, for any $x \in V(D)$, $w_{u_2x} + w_{ix} \geq .5 + .5 = 1$ as desired. So, $\{1,2,3,u_2\}$ induces K_4 in dom(D). ∎

The *chorded 4-cycle* is the only simple graph on 4 vertices with 5 edges (shown in Figure 5.5), and is sometimes referred to as a "kite."

The *bowtie* is the graph on 5 vertices shown in Figure 5.6.

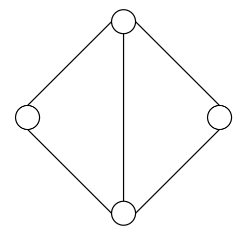

Figure 5.5: The chorded 4-cycle.

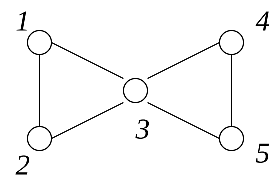

Figure 5.6: The bowtie.

Theorem 5.22 *If D is a complete paired comparison digraph and dom(D) is connected and contains a 3-cycle C, then either dom(D) is a spiked 3-cycle or there is a clique of order at least 4 in dom(D) that contains C.*

Proof: Let C be given by $[1,2]$, $[2,3]$ and $[3,1]$. If dom(D) is not a spiked odd cycle, then in dom(D), either C is contained in a chorded 4-cycle, a bowtie, or there is a vertex v not on C but of distance at least 2 from C. Lemma 5.21 treats the latter case.

Suppose that C is contained in a bowtie and no vertex v is of distance at least 2 from C. Let $B = \{1,2,3,4,5\}$ be the set of vertices which induce the bowtie as shown in Figure 5.6. By applying Lemma 5.2 to $\{1,2,3\}$ we know that $w_{12} = w_{23} = w_{31}$. By applying Lemma 5.1 to $[1,2]$ and $[3,4]$, then to $[1,2]$

and $[3,5]$, and then to $[1,2]$ and $[4,5]$ we see that $w_{31} = w_{14} = w_{42} = w_{23}$, $w_{31} = w_{15} = w_{52} = w_{23}$ and $w_{42} = w_{25} = w_{51} = w_{14}$. But then, $w_{25} = w_{42} = w_{23} = w_{52}$. Thus, $w_{25} = w_{52} = .5$. So, by applying Lemma 5.2 to $\{3,4,5\}$ and Lemma 5.1 to the pairs, $[2,3]$ and $[4,5]$ give us that $w_{xy} = .5$ for all $x, y \in B$. If $V(D) = B$, then dom$(D) = K_5$. So, assume there exists another vertex $v \in V(D)$. Then, v is adjacent to some vertex in B, and so by Lemma 5.19, $w_{xv} \geq .5$ for all $x \in B$. Thus, for each $x, y \in B$, $w_{xv} + w_{yv} \geq .5 + .5 = 1$. Since v was arbitrary we deduce that $\{x, y\}$ is a dominant pair for all $x, y \in B$. Thus, B induces K_5 in dom(D), and C is contained in this K_5.

Now, assume that C is contained in a chorded 4-cycle C' and no vertex is of distance at least 2 from C. By applying Lemma 5.1 to the two vertex disjoint pairs of edges in the 4-cycle we deduce that $w_{xy} = .5$ for all $x, y \in C'$. So, if $V(D) = C'$, dom$(D) = K_4$. So, assume there exists another vertex $v \in V(D)$. Then by Lemma 5.19, $w_{xv} \geq .5$ for all $x \in C'$. Thus, $w_{xv} + w_{yv} \geq .5 + .5 = 1$ for all $x, y \in C'$. Since v was arbitrarily chosen, we deduce that $\{x, y\}$ is a dominant pair for all pairs of vertices $x, y \in C'$, and, so, the vertices of C' induce K_4 in dom(D). Thus, C is contained in a larger clique. ■

Theorem 5.23 *If G is a connected graph with an induced cycle, C, of order $n = 2k+1 \geq 5$, and $G =$dom(D) for some complete paired comparison digraph D, then G is a spiked odd cycle.*

Proof: Suppose, to the contrary, that there exists a pcd D with dom$(D) = G$, and G is not a spiked odd cycle. Then either G contains a vertex v adjacent to no vertex of C or it does not. If it does not, then either $G = C$ (and we are done) or G contains a vertex x not on C which is on a cycle which shares at least one vertex with C. Let C' be the smallest such cycle. If C' is a 3-cycle, then we use Lemma 5.21 to obtain a contradiction as follows.

Since C has at least 5 vertices, and shares at most two of them with C', there must be a vertex u on C of distance at least 2 from C'. Take C' to be the 3-cycle, u the vertex, and the path from C' to u contained in C to be the path in Lemma 5.21. This implies that C' must be contained in a larger clique, and this clique contains a vertex of $V(C) - V(C')$. Thus, there must be an edge between every vertex in C' and some vertex of C which is not in C'. Since $V(C) \cap V(C') \neq \emptyset$, some of these edges form chords in C contradicting it being an induced cycle.

Theorem 5.20 implies that C' is not an even cycle. So, suppose C' is an odd cycle of size 5 or greater. If x is the only vertex in C' not in C, then there must be at least three vertices in C which are not on C'. If there were only one, then this vertex together with x and the two vertices adjacent to x on C would form a 4-cycle contradicting the minimality of C'. If there were only two, then C would be an even cycle, contradicting the fact that C is an odd cycle. Thus, C' is an odd cycle of order at least 5 such that there is a vertex on C not adjacent to C'. Taking C' as C and this vertex as v we continue the proof.

We now treat the case where G contains a vertex v adjacent to no vertex of C. Let S be the set of vertices which induces C in G. From Lemma 5.2 we know that in D, S induces a $U_{n,p}$, for some $0 \leq p \leq 1$. We will prove that $p = .5$. We assume $S = \{v_1, \ldots, v_n\}$, where $w_{v_iv_j} = p$ if and only if $j - i$ is odd modulo n and $w_{v_iv_j} = 1 - p$ otherwise. Without loss of generality, assume that $p \geq .5$.

Suppose $p > .5$. Since G is connected, there is a shortest path P from C to v, given by $u_1, u_2, \ldots, u_m = v$ in G, $m \geq 3$ with $u_1 \in S$. Without loss of generality we may assume that $u_1 = v_1$.

Apply Lemma 5.1 to edges $[u_1, u_2]$ and $[v_r, v_{r+1}]$ $(2 \leq r \leq n-1)$ to deduce that $w_{u_1 v_r} = w_{u_2 v_{r+1}}$. Since $\{v_i : i \text{ is even}, 2 \leq i \leq 2k\} \subseteq O_S^+(u_1)$, we see that $\{v_i : i \text{ is odd}, 3 \leq i \leq 2k+1\} \subseteq O_S^+(u_2)$. That is, for all i, $2 \leq i \leq 2k+1$,

$$w_{u_2 v_i} = \begin{cases} p & \text{if } i \text{ is odd,} \\ 1-p & \text{if } i \text{ is even.} \end{cases} \tag{5.8}$$

Since $\{v_{2k+1}, v_1\}$ is a dominant pair, and since $w_{v_{2k+1} u_2} = 1 - p$, we deduce that $w_{v_1 u_2} \geq p$. Apply Lemma 5.1 to edges $[u_2, u_3]$ and $[v_r, v_{r+1}]$ $(2 \leq r \leq n$, arithmetic modulo $n)$ to deduce that $w_{u_2 v_r} = w_{u_3 v_{r+1}}$. By (5.8) this implies that for all i, $2 \leq i \leq 2k+1$,

$$w_{u_3 v_i} = \begin{cases} p & \text{if } i \text{ is even,} \\ 1-p & \text{if } i \text{ is odd,} \end{cases} \tag{5.9}$$

and (when $i = n$) $w_{u_3 v_1} = p$. Thus, $\{v_1, v_2, v_4, \ldots, v_{2k}\} \subseteq O_S^+(u_3)$. By Lemma 5.5, $O_S^+(u_3)$ is an independent set in G that contains edge $[v_1, v_2]$, a contradiction. Thus, $p = .5$.

As $n \geq 5$, there exist vertices $x, y \in S$ so that x and y are not adjacent in C. By Lemma 5.19, $w_{xv} + w_{yv} = .5 + .5 = 1$, for all such vertices v. Also, by the above argument, $w_{xw} + w_{yw} = .5 + .5 = 1$ for all $w \in S - \{x, y\}$. To show that $\{x, y\}$ is a dominant pair in D it remains to show that $w_{xz} + w_{yz} \geq 1$ for all vertices z adjacent to a vertex of C. Let z be a vertex not on C, but adjacent to, say, v_i on C. If $v_i \neq x, y$, then by Lemma 5.19 (where $m = 2, u_2 = z, x \neq u_1 \neq y$), $w_{xz} + w_{yz} = .5 + .5 = 1$. If $v_i = x$, then by Lemma 5.19 (where

$m = 2, u_2 = z, u_1 = x, y \neq u_1$) $w_{xz} + w_{yz} \geq .5 + .5 = 1$. So, $\{x, y\}$ is a dominant pair in D. But, then $[x, y] \in E(\text{dom}(D))$, contradicting the choice of x and y. Consequently, there is no such vertex v, i.e. G is a spiked odd cycle. ■

A connected graph G of order $m+n$ is called a *spiked clique* if $V(G)$ can be written as $V_1 \cup V_2$, where $|V_1| = n$, $|V_2| = m$, $G[V_1]$ is a clique of order at least 4, and each vertex in V_2 has degree 1. A spiked clique with $n = 4$ and $m = 5$ is shown in Figure 5.7. Equivalently, we can think of a spiked clique as a graph in which the removal of all pendant vertices results in a clique of order 4 or more.

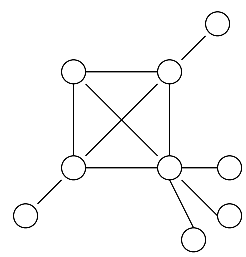

Figure 5.7: An example of a spiked clique.

Theorem 5.24 *If G is a connected graph which contains a maximal clique K of order $m \geq 4$, and $G =$dom(D) for some complete paired comparison digraph D, then G is a spiked clique.*

Proof: Suppose, to the contrary, that there exists a pcd D with $\text{dom}(D) = G$, and G is not a spiked clique. By Lemma 5.3, $w_{xy} = .5$ for each $x, y \in V(K)$. Since G is not a spiked clique, either G contains a vertex v which is not adjacent to any vertex in K or it does not. If it does not, then there must be some cycle C containing at most 2 vertices of K in G.

First, suppose this cycle has order 3. If C and K share exactly 1 vertex then C together with any pair of vertices of K form a bowtie. If C and K share exactly 2 vertices, then C together with any vertex of K forms a chorded 4-cycle. From the proof of Theorem 5.22 we see that any bowtie or chorded 4-cycle must be contained in a clique. However, since the pair or single vertex which forms the bowtie or chorded 4-cycle were chosen arbitrarily, each vertex of C is adjacent to every vertex in K, a contradiction to K being maximal.

Now, the cycle cannot be of even length since this contradicts Theorem 5.20. If the cycle is odd of order at least 5, then Theorem 5.23 assures us that G is a spiked odd cycle, contradicting the existence of K in G. So, assume that there exists $v \in V(G)$ with v not adjacent to any vertex of K. Then, since G is connected, there exists a shortest path P from K to v given by $u_1, u_2, \ldots, u_m = v$ where $m \geq 3$ and $u_1 \in V(K)$. We now show that $\{u_2, x\}$ is a dominant pair for every $x \in K$ to draw a contradiction.

Pick $x \in V(K)$. First note, by Lemma 5.19, that $w_{xy} + w_{u_2y} = .5 + .5 = 1$ for all $y \in V(K) - \{x\}$. Now select $z \in V(G) - V(K)$. By Lemma 5.19, $w_{xz} \geq .5$. We show $w_{u_2z} \geq .5$. If $[z, u_2] \in E(G)$, then since $w_{u_1z} = .5$, Lemma 5.19 and the fact that $\{u_1, u_2\}$ is a dominant pair implies that $w_{u_2z} \geq .5$. If $[u_1, z] \in E(G)$, then apply Lemma 5.1 to $[z, u_1]$ and $[u_2, u_3]$ to see that $w_{u_2z} = w_{u_1u_2}$. Lemma 5.19 yields $w_{u_1u_2} = .5$, so $w_{u_2z} = .5$ as desired. Now, if z is not adjacent to u_1 or u_2 in G, then since G is connected, there exists $a \in V(G)$ such that $[z, a]$ is in G. By applying Lemma 5.1 to the edges $[u_1, u_2]$ and $[z, a]$ we see that $w_{u_2z} = w_{zu_1}$. By Lemma 5.19, $w_{zu_1} = .5$ and so $w_{u_2z} = .5$ as desired. Thus, for each $z \in V(G) - V(K)$ $w_{xz} + w_{u_2z} \geq .5 + .5 = 1$. Thus, $\{x, u_2\}$ is a dominant pair, a contradiction. ■

The previous two theorems show us that if D is a pcd such that dom(D) is connected and contains an induced cycle or clique, then dom(D) is a spiked odd cycle or a spiked clique. This together with the following lemma and some results from the Section 5.3 will yield a classification of connected domination graphs of pcds in Theorem 5.26.

Lemma 5.25 *If G is a spiked clique, then there exists a complete paired comparison digraph D for which dom(D) $= G$.*

Proof: Take the vertex set of D to be $V(G)$. Let S denote the set of vertices in $V(G)$ that induces the clique in G. If $[x, y] \in E(G)$ with $x \in S$ and $y \in V(G) - S$, define $w_{xy} = b$, where $.5 < b \leq 1$ (so, $w_{yx} = 1 - b$). For all other pairs of distinct vertices u and v in $V(G)$ define $w_{uv} = .5$. Pick distinct x, y in $V(G)$. We check that $w_{xz} + w_{yz} \geq 1$ for all $z \in V(G) - \{x, y\}$ if and only if $[x, y] \in E(G)$. Suppose that $[x, y] \in E(G)$. If $x, y \in S$, then for all $z \in V(G) - \{x, y\}$,

$$w_{xz} + w_{yz} = \begin{cases} b + .5 & \text{, if } z \in S \text{ and } z \text{ is adjacent to } x \text{ or } y, \\ .5 + .5 & \text{, otherwise.} \end{cases}$$

If exactly one of x, y is in S, say $x \in S$ and $y \notin S$, then for all $z \in V(G) - \{x, y\}$,

$$w_{xz} + w_{yz} = \begin{cases} b + .5 & \text{, if } [z, x] \in E(G), z \notin S, \\ .5 + .5 & \text{, otherwise.} \end{cases}$$

Since $b > .5$, $w_{xz} + w_{yz} \geq 1$ for all $z \in V(G) - \{x, y\}$, as desired.

On the other hand, suppose that $[x, y] \notin E(G)$. Since G is a spiked clique, at least one of x, y is not in S, say $y \notin S$. Now, choose $z \in S$ such that

$[y, z] \in E(G)$. Then $w_{yz} = 1 - b$, and since $z \in S$, $w_{xz} \leq .5$. So, $w_{xz} + w_{yz} \leq .5 + (1 - b) < .5 + .5 = 1$. Thus, $\{x, y\}$ is not a dominant pair, as desired. ■

Together, Lemma 5.25, and Theorems 5.13, 5.17, 5.20, 5.22, 5.23 and 5.24, give us the following theorem classifying which connected graphs are the domination graphs of complete paired comparison digraphs. In comparison to the analogous result for tournaments, Theorem 5.12, we see that there are several new connected graphs which are the domination graphs of complete paired comparison digraphs, namely caterpillars without triple ends and spiked cliques.

Theorem 5.26 *Let G be a connected graph. Then there exists a complete paired comparison digraph D such that $dom(D) = G$ if and only if G is a spiked odd cycle, a caterpillar, or a spiked clique.*

5.5 Domination graphs with no isolated vertices

Proposition 5.7 tells us that any graph can be an induced subgraph of the domination graph of a pcd. However, in the proof, we essentially are carving the desired graph from a clique, and leaving an isolated vertex in the graph for each edge we delete. In this section we determine the graphs which comprise the connected components of the domination graphs which contain no isolated vertices, adding to the classes of the previous section the class of spiked bicliques. We also show exactly which combinations of these classes are possible.

In light of Proposition 5.7, the results of this section essentially characterize the basic graphs which can be the domination graphs of pcds. In the following section we expand slightly on these results with some counting arguments for obtaining graphs outside of this classification. We now open the section with two results which limit the number of spiked cliques our graphs can contain.

Lemma 5.27 *Let D be a complete paired comparison digraph. If K is a clique of order 3 or more in dom(D) and $[u, v] \in E(dom(D))$ which is vertex disjoint from K, then $w_{zu} = w_{zv} = .5$ for all $z \in V(K)$.*

Proof: Let K be a clique of order 3 or more in dom(D), and $[u, v] \in E(\text{dom}(D))$ which is vertex disjoint from K. Choose distinct vertices $x, y, z \in V(K)$. Since K is a clique, $[x, y], [y, z], [x, z] \in E(\text{dom}(D))$. So, by Lemma 5.1,

$$w_{xu} = w_{uy} = w_{yv} = w_{vx},$$

$$w_{xu} = w_{uz} = w_{zv} = w_{vx}, \text{ and}$$

$$w_{uy} = w_{yv} = w_{vz} = w_{zu}.$$

So, $w_{zv} = w_{xu} = w_{yv} = w_{vz} = 1 - w_{zv}$, and so $w_{uz} = w_{zv} = .5$. Since x, y, z were chosen arbitrarily from $V(K)$, the result holds. ∎

Theorem 5.28 *Let D be a complete paired comparison digraph such that dom(D) has no isolated vertices. Then, dom(D) does not contain more than one maximal clique of size at least four.*

Proof: Suppose to the contrary that there exist two maximal cliques of size 4 or more. Let K_1 and K_2 denote the sets of vertices which induce these cliques. Let D_1 and D_2 be the pcds induced on K_1 and K_2 respectively. Since K_1 and K_2 induce cliques in dom(D), by Theorem 5.4, dom(D_1) and dom(D_2) must also be cliques. So, by Lemma 5.3, $w_{xy} = .5$ for all $x, y \in K_1$ and $x, y \in K_2$. We show $w_{xy} = .5$ for all $x, y \in K_1 \cup K_2$.

If $K_1 \cap K_2 = \emptyset$, then by Lemma 5.27, $w_{xy} = .5$ for each $x \in K_1$ and $y \in K_2$, and so $w_{xy} = .5$ for each $x, y \in K_1 \cup K_2$. So, assume $K_1 \cap K_2 \neq \emptyset$. First consider when $|K_1 \cap K_2| = 1$. Let a be the unique vertex in $K_1 \cap K_2$. Then $K_1 - \{a\}$ and $K_2 - \{a\}$ induce disjoint cliques of order 3 or more. So, by Lemma 5.27, $w_{xy} = .5$ for all $x \in K_1 - \{a\}$ and $y \in K_2 - \{a\}$. So, since $w_{xa} = w_{ya} = .5$, for all $x \in K_1$ and $y \in K_2$, $w_{xy} = .5$ for all $x \in K_1$, and $y \in K_2$. Now, assume $|K_1 \cap K_2| \geq 2$. Choose distinct vertices $b, c \in K_1 \cap K_2$, $x \in K_1 - K_2$ and $y \in K_2 - K_1$. Then $[x, b]$ and $[y, c]$ are disjoint edges in dom(D), and so by Lemma 5.1 $w_{xy} = w_{yb} = w_{bc} = w_{cx}$. Since $b \in K_2$, $w_{yb} = .5$, and so $w_{xy} = .5$. Since x and y were chosen arbitrarily, $w_{xy} = .5$ for all $x \in K_1 - K_2$ and $y \in K_2 - K_1$. Thus, $w_{xy} = .5$ for all $x, y \in K_1 \cup K_2$.

If $V(D) = K_1 \cup K_2$ then $w_{xz} + w_{yz} = 1$ for all $x, y, z \in V(D)$, so $\{x, y\}$ is a dominant pair for all $x, y \in V(D)$, and dom(D) is a complete graph, contradicting dom(D) having two maximal cliques. So, assume $V(D) \neq K_1 \cup K_2$, and let $v \in V(D) - (K_1 \cup K_2)$. Since dom$(D)$ has no isolated vertices, there exists $u \in V(D)$ such that $[u, v] \in E(\text{dom}(D))$. If $u \in K_1 \cup K_2$, then by Lemma 5.19 (with $m = 2$), $w_{va} = .5$ for all $a \in K_1 \cup K_2 - \{u\}$ and $w_{uv} \geq .5$. If $[u, v]$ is disjoint from $K_1 \cup K_2$, then by Lemma 5.27, $w_{xv} = .5$ for all $x \in K_1$ and $w_{yv} = .5$ for all $y \in K_2$. Thus, $w_{xv} + w_{yv} \geq .5 + .5 = 1$ for all $x, y \in K_1 \cup K_2$. Since v was chosen arbitrarily, this is true for all $v \in K_1 \cup K_2$. Further, since $w_{xy} = .5$ for all $x, y \in K_1 \cup K_2$, $w_{xz} + w_{yz} = 1$ for all $x, y, z \in K_1 \cup K_2$. Thus, $K_1 \cup K_2$ induces a clique in dom(D), a contradiction to K_1 and K_2 inducing maximal cliques in dom(D). ■

Theorem 5.29 *Let D be a complete paired comparison digraph, and let G be a component of dom(D). Let D' be the complete paired comparison digraph induced on the vertices of G. If G is not a spiked odd cycle or a caterpillar, then dom(D') is a spiked clique.*

Proof: From Theorem 5.4, Theorem 5.26 and the fact that G is connected, we know that dom(D) must be a subgraph of a spiked odd cycle, a caterpillar, or a spiked clique. Subgraphs of spiked odd cycles and caterpillars are spiked odd cycles and caterpillars. So, if G is neither of these, we must have dom(D') is a spiked clique. ■

Lemma 5.30 *Let D be a complete paired comparison digraph such that some component S of dom(D) contains an odd cycle C, and let $[x, y]$ be an edge of dom(D) which is vertex disjoint from C. Then in D, $w_{vx} = w_{vy} = .5$ for each $v \in V(S)$.*

Proof: Pick a vertex in C and call it c_1, then trace all other vertices in C in a clockwise order, labeling them $c_2, c_3, \ldots, c_{2k+1}$. From Lemma 5.1 applied to the edges $[x, y]$ and $[c_1, c_2]$, $w_{xc_1} = w_{c_1y} = w_{yc_2} = w_{c_2x}$. Also, Lemma 5.1 applied to $[c_2, c_3]$ and $[x, y]$ gives $w_{c_2x} = w_{xc_3} = w_{c_3y} = w_{yc_2}$. Now choose i and assume that for each odd $j < i$,

$$w_{xc_1} = w_{xc_j} = w_{c_jy} = w_{yc_{j+1}} = w_{c_{j+1}x}.$$

Then, by Lemma 5.1 applied to $[x, y]$ and $[c_{i-1}, c_i]$,

$$w_{xc_1} = w_{c_{i-1}x} = w_{xc_i} = w_{c_iy} = w_{yc_{i-1}}.$$

So, by induction,

$$w_{xc_1} = w_{xc_i} = w_{c_iy} = w_{yc_{i+1}} = w_{c_{i+1}x}$$

for all odd i. In particular, $w_{xc_{2k+1}} = w_{xc_1}$, and from Lemma 5.1 applied to $[x, y]$ and $[c_{2k+1}, c_1]$ we have that

$$w_{xc_{2k+1}} = w_{xc_1} = w_{c_1y} = w_{yc_{2k+1}} = w_{c_{2k+1}x} = 1 - w_{xc_{2k+1}}.$$

Thus, $w_{xc_1} = w_{c_1x} = .5$, and so $w_{xv} = w_{yv} = .5$ for all $v \in C$.

If $V(C) = V(S)$ we are done, so assume that there exists a vertex $v \in V(S) - V(C)$. Then, since S is a connected component of dom(D), there is a shortest path P from C to v given by $u_1, u_2, \ldots, u_m$ where $u_m = v$. From Lemma 5.1 applied to the pairs $[x, y]$ and $[u_1, u_2]$,

$$.5 = w_{xc_1} = w_{xu_1} = w_{u_1y} = w_{yu_2} = w_{u_2x}.$$

Now, assume that $w_{xu_j} = w_{yu_j} = .5$ for all $j \leq i$. Then Lemma 5.1 applied to the pairs $[u_i, u_{i+1}]$ and $[x, y]$ yields

$$w_{xu_{i+1}} = w_{u_{i+1}y} = w_{yu_i} = .5$$

and so $w_{yu_{i+1}} = 1 - w_{u_{i+1}y} = 1 - .5 = .5$. Thus, for all u_i, $w_{xu_i} = w_{yu_i} = .5$. In particular, $w_{xv} = w_{yv} = .5$. Since v was arbitrary, this holds for all $v \in V(S) - V(C)$. Thus, $w_{vx} = w_{vy} = .5$ for all $v \in V(S)$. ■

Theorem 5.31 *Let G be a graph with no isolated vertices, which contains a component S which contains an odd cycle, and S is not a spiked odd cycle or spiked clique. There does not exist a complete paired comparison digraph D such that dom$(D) = G$.*

Proof: Suppose to the contrary, that there exists a pcd D such that $\text{dom}(D) = G$. If $G = S$, then from Theorem 5.26 the result follows, so suppose G has more than one component. Let D' be the pcd induced on $V(S)$. Then, from Theorem 5.29, $\text{dom}(D')$ is a spiked clique. Let H be the subset of vertices of S which are not pendant in $\text{dom}(D')$. Then, for all $x, y \in H$, $w_{xy} = .5$, by Lemma 5.3, and since $[x, y] \in E(\text{dom}(D'))$, $w_{xz} + w_{yz} \geq 1$ for all $x, y \in H$, and $z \in V(S)$.

Now, choose $v \in V(G) - V(S)$. Since G has no isolated vertices there exists $u \in V(G) - V(S)$ such that $[u, v] \in E(G)$. From Lemma 5.30 we know that $w_{us} = w_{vs} = .5$ for all $s \in V(S)$. So, for each $x, y \in H$, we have that $w_{xv} + w_{yv} \geq 1$ for all $v \in V(G) - \{x, y\}$. Thus, H induces a clique in $\text{dom}(D)$. By Theorem 5.4, S is a subgraph of $\text{dom}(D')$. So, since $\text{dom}(D')$ is a spiked clique, with its clique induced on H, and H induces a clique in $\text{dom}(D)$, S must be a spiked clique when $|H| \geq 4$, and a spiked 3-cycle when $|H| = 3$. This contradicts the choice of S, so no such pcd exists. ■

Recall a graph is bipartite if and only if it contains no odd cycles. This gives the following corollary.

Corollary 5.32 *Let D be a complete paired comparison digraph. If dom(D) has no isolated vertices, then any component which is not a spiked odd cycle or spiked clique is bipartite.*

Theorem 5.33 *Let D be a complete paired comparison digraph such that dom(D) has no isolated vertices and dom(D) contains a component which is not a caterpillar, spiked odd cycle, or spiked clique. Then dom(D) has at least* *two bipartite components.*

Proof: Let S be the component which is neither a caterpillar, spiked odd cycle, or spiked clique. From Theorem 5.26 we know that dom(D) must have another component, and from Corollary 5.32 we know that S is bipartite. So, suppose to the contrary, that no other component is bipartite. Then every other component of dom(D) contains an odd cycle. Since S is connected, for each $x \in V(S)$ there exists a y in S such that $[x, y] \in E(S)$. So, since all components of dom(D) other than S contain an odd cycle, $w_{xv} = .5$ for all $x \in V(S)$ and $v \notin V(S)$ by Lemma 5.30. Now, from Theorem 5.29 we know that the pcd D' induced on S has a spiked clique as its domination graph. Let H denote the vertices which form the clique in dom(D'). Note $|H| \geq 4$ by definition of spiked clique. Then $w_{xv} + w_{yv} \geq 1$ for all $x, y \in H$ and $v \in V(S)$. Thus, for all $x, y \in H$ and $v \in V(D)$, $w_{xv} + w_{yv} \geq 1$, and so H is a clique in dom(D). A contradiction to S being bipartite. Thus, there must exist some other component of dom(D) which does not contain an odd cycle, and so dom(D) contains at least 2 bipartite components. ■

Lemma 5.34 *Let D be a complete paired comparison digraph. Suppose that dom(D) contains a bipartite component B. Let $X \cup Y$ be the bipartition of B, and $u, v \in X$. Then, $w_{uz} = w_{vz}$ for each vertex $z \in V(D) - V(B)$ such that z is not isolated in dom(D). Also, $w_{ac} = w_{bc}$ for each $a, b \in Y$, and $c \in V(D) - V(B)$ such that c is not isolated in dom(D).*

Proof: Choose $u, v \in X$ and $z \in V(D) - V(B)$ such that z is not isolated. Since z is not an isolated vertex in dom(D), B is a component and $z \notin V(B)$, there must exist a $y \in V(D) - V(B)$ such that $[y, z]$ is an edge of dom(D). Since B is connected, there exists a path P from u to v in dom(D) given by $u_1, u_2, \ldots, u_m$ where $u = u_1$ and $u_m = v$. Since B is bipartite, for all odd i, $u_i \in X$ and $u_{i+1} \in Y$. Note, since $v \in X$, m is odd. Now, from Lemma 5.1, $w_{uz} = w_{zu_2} = w_{u_2 y} = w_{yu_1}$. So, assume that for each odd i, $w_{uz} = w_{u_i z} = w_{zu_{i+1}} = w_{u_{i+1} y} = w_{yu_i}$. Lemma 5.1 applied to $[z, y]$ and $[u_{i+1}, u_{i+2}]$ gives $w_{uz} = w_{u_{i+1} y} = w_{yu_{i+2}} = w_{u_{i+2} z}$. Thus, for all odd i, $w_{u_i z} = w_{uz}$. So, $w_{vz} = w_{uz}$. Since z was chosen arbitrarily $w_{uz} = w_{vz}$ for all $u, v \in X$, and $v \notin V(B)$ with v not isolated. By symmetry, $w_{ac} = w_{bc}$ for each $a, b \in Y$, and $c \in V(D) - V(B)$ such that c is not isolated in dom(D). This completes the proof. ■

Theorem 5.35 *Let D be a complete paired comparison digraph, and let G and H be non-trivial bipartite components of dom(D) with bipartitions $V(G) = G_1 \cup G_2$ and $V(H) = H_1 \cup H_2$. Then for $i, j \in \{1, 2\}$, for all distinct $u, u' \in G_i$, and for all distinct $v, v' \in H_j$, $w_{uv} = w_{u'v'}$.*

Proof: Pick $i, j \in \{1, 2\}$. From Lemma 5.34 we know that $w_{uz} = w_{u'z}$ for all $u, u' \in G_i$ and $z \in V(H)$. Furthermore, $w_{vz} = w_{v'z}$ for each $v, v' \in H_j$ and $z \in V(G)$. Thus,

$$w_{uv} = 1 - w_{vu} = 1 - w_{v'u} = w_{uv'} = w_{u'v'}$$

for all $u, u' \in G_i$ and $v, v' \in H_j$. ■

We define a *spiked biclique* as a graph G such that the graph obtained by removing all pendant vertices of G is a biclique. Some examples of spiked bicliques are shown in Figure 5.8. Note bicliques themselves are spiked bicliques. So, the smallest spiked biclique is the single edge.

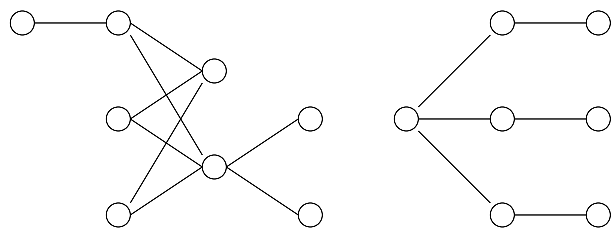

Figure 5.8: Some spiked bicliques

Theorem 5.36 *Let D be a complete paired comparison digraph such that dom(D) contains no isolated vertices. Any component of dom(D) which is not a spiked odd cycle, spiked clique or caterpillar, is a spiked biclique.*

Proof: Let G be a component of dom(D) which is not a spiked odd cycle, spiked clique or caterpillar. From Corollary 5.32 we know that G is bipartite with bipartition $X \cup Y$. Now, dom(D) must contain another component for otherwise dom(D) $= G$, a contradiction to Theorem 5.26. Let H be the subgraph of G formed by removing the pendant vertices of G. Note H is connected, and if $|V(H)| > 1$, H is a bipartite graph, with bipartition $(X \cap V(H)) \cup (Y \cap V(H))$. Suppose, to the contrary, $[x, y] \notin E(H)$ for some $x \in X \cap V(H)$ and $y \in Y \cap V(H)$. So there must be some vertex of D, call it v, so that $w_{xv} + w_{yv} < 1$. Let D' be the pcd induced on $V(G)$. Since G is not a spiked odd cycle or caterpillar, dom(D') is a spiked clique by Theorem 5.29. Since H is obtained by removing the pendant vertices of G, $V(H)$ is contained within the vertices of dom(D') which induce the clique. Hence, $w_{xu} + w_{yu} \geq 1$ for all $x, y \in V(H)$ and $v \in V(G)$. So, $v \in V(D) - V(G)$. By Lemma 5.30, if u is in a component of dom(D) containing an odd cycle $w_{xu} + w_{yu} = .5 + .5 = 1$. So, v must be in some bipartite component, K, of dom(D).

By Lemma 5.34, $w_{xv} = w_{x'v'}$ for all $x' \in X \cap V(H)$ and $v' \in V(K)$. Similarly, $w_{yv} = w_{y'v'}$ for all $y' \in Y \cap V(H)$ and $v' \in V(K)$. Thus, since $w_{xv} + w_{yv} < 1$, $w_{x'v'} + w_{y'v'} < 1$ for all $x' \in X \cap V(H)$, $y' \in Y \cap V(H)$, and $v' \in X_0$. So, $E(H) = \emptyset$. Since H is a connected graph, H must be a single vertex. Since H is the graph obtained from G by removing the pendant vertices of G, G must be a star. However, this implies that G is a caterpillar, a contradiction to our choice of G. Thus, $[x, y] \in E(H)$ for all $x \in X \cap V(H)$ and $y \in Y \cap V(H)$, and so G is a spiked biclique. ■

The spiked biclique is an interesting new class which not only increases the kinds of graphs which can can be domination graphs of pcds from the last section, but puts quite a bit of distance between the domination graphs of pcds and the domination graphs of tournaments. Recall that a tournament cannot have an even cycle or $NC7$ in its domination graph. However, since the 4-cycle is a biclique, it can be the domination graph of a pcd. Also, $NC7$ is a spiked biclique (in fact, it is the graph on the right in Figure 5.8), so $NC7$ can be the domination graph of a pcd. So, the trees which are the domination graphs of pcds do not necessarily need to be caterpillars. However, trees which are not caterpillars must be spiked bicliques, and hence are fairly restricted.

Lemma 5.37 *Let G be a star on $n \geq 3$ vertices with center u. Then there exists a complete paired comparison digraph, D, such that $O^+(u) \neq V(D) - \{u\}$ and dom$(D) = G$.*

Proof: We construct D as follows. Let $V(D) = V(G)$. Let p and p' be real numbers so that $.5 < p' < p < 1$. Choose $v \in V(D) - \{u\}$, and set $w_{vu} = p'$, and $w_{uv} = 1 - p'$. For $x \in V(D) - \{u, v\}$ set $w_{ux} = p$ and $w_{xu} = 1 - p$, and set $w_{xv} = p'$ and $w_{vx} = 1 - p'$. For each $x, y \in V(D) - \{u, v\}$ set $w_{xy} = p$ or $1 - p$ arbitrarily, to create a pcd. We claim $\text{dom}(D) = G$.

Let $x, y \in V(D) - \{u, v\}$. If $x \neq y$ then $w_{uy} + w_{xy}$ is $p + (1 - p)$ or $p + p$. In either case $w_{uy} + w_{xy} \geq 1$, since $p > .5$. Also, $w_{uv} + w_{xv} = 1 - p' + p' = 1$. So, $[u, x] \in E(\text{dom}(D))$ for each $x \in V(D) - \{u, v\}$. Now, $w_{ux} + w_{vx} = p + p' > 1$ for each $x \in V(D) - \{u, v\}$, so $[u, v] \in E(\text{dom}(D))$. So, $E(G) \subseteq E(\text{dom}(D))$. By Lemma 5.5, $O^+(u) = V(D) - \{u, v\}$ is an independent set in $\text{dom}(D)$. So, we need only show that $[v, x]$ is not an edge for $x \in V(D) - \{u, v\}$. This follows since $w_{vu} + w_{xu} = p' + 1 - p = 1 - (p - p') < 1$. So, $\text{dom}(D) = G$. ∎

Lemma 5.38 *Let G be a spiked odd cycle or caterpillar on 3 or more vertices. There exists a complete paired comparison digraph D so that dom$(D) = G$, and if $u \in V(D)$ there is some $v \in V(D)$ so that $w_{uv} < .5$.*

Proof: Suppose that no such pcd exists. By Theorem 5.26, there exists a pcd D so that $\text{dom}(D) = G$. So for every such D there must be some vertex $u \in V(D)$ so that $O^+(u) = V(D) - \{u\}$. By Lemma 5.5, $V(D) - \{u\}$ forms an independent set in $\text{dom}(D)$. So, since G is a caterpillar or spiked odd cycle, G must be a star centered at u. So, for every pcd such that $\text{dom}(D)$ is a star on $n \geq 3$ vertices, $O^+(u) = V(D) - \{u\}$, where u is the center of the star. This is a contradiction to Lemma 5.37. So such a pcd must exist. ∎

Lemma 5.39 *Let G be a graph with no isolated vertices whose connected components form some collection of the following:*

1. At most one spiked clique.

2. Spiked odd cycles.

3. Caterpillars on 3 *or more vertices.*

Then, there exists a complete paired comparison digraph D such that dom$(D) = G$. Further, for each vertex u of D not contained in the spiked clique, there exists a vertex v so that $w_{uv} < .5$.

Proof: We construct D as follows. Let $V(D) = V(G)$. Let $C_1, \ldots, C_k$ denote the caterpillars of G. Let K denote the spiked clique in G. Let $S_1, \ldots, S_m$ denote the spiked odd cycles of G. Note we only require at least one of $C_1, \ldots, C_k, K, S_1, \ldots, S_m$ exist. By Theorem 5.26, there exists a pcd D_K such that $\text{dom}(D_K) = K$. Also, by Lemma 5.38 there exist pcds $D_{C_1}, \ldots, D_{C_k}, D_{S_1}, \ldots, D_{S_m}$ so that for $i = 1, \ldots, k$ and $j = 1, \ldots, m$ $\text{dom}(D_{C_i}) = C_i$ and for every vertex u of D_{C_i} there exists a vertex v of D_{C_i} so that $w_{uv} < .5$, and $\text{dom}(D_{S_j}) = S_j$ and for each $x \in V(D_{S_j})$ there exists a $y \in V(D_{S_j})$ so that $w_{xy} < .5$. Weight the arcs in D so that $V(K)$ induces D_K, $V(C_i)$ induces D_{C_i} for $i = 1, \ldots, k$ and $V(S_j)$ induces D_{S_j} for $j = 1, \ldots, m$. Give the remaining arcs of D weight .5. We claim $\text{dom}(D) = G$.

Let $D' \in \{D_{C_1}, \ldots, D_{C_k}, D_K, D_{S_1}, \ldots, D_{S_m}\}$. If $[x, y] \in E(\text{dom}(D'))$, then $w_{xv} + w_{yv} \geq 1$ for all $v \in V(D')$. Also, $w_{xz} + w_{yz} = .5 + .5 = 1$ for each $z \notin V(D')$. If $[x, y] \notin E(\text{dom}(D'))$, then by Theorem 5.4, $[x, y] \notin E(\text{dom}(D))$. So each of $C_1, \ldots, C_k, K, S_1, \ldots, S_m$ are induced subgraphs of $\text{dom}(D)$. Note, if G has only one component we are done. So, assume G has 2 or more components, and pick $u, v \in V(D)$ so that u and v are in separate components of G. So, at most one of u, v can be in K, and without loss of generality, say u is not in K. So u is in one of $C_1, \ldots, C_k, S_1, \ldots, S_m$. Let $D'' \in \{D_{C_1}, \ldots, D_{C_k}, D_K, D_{S_1}, \ldots, D_{S_m}\}$ so that $u \in V(D'')$. By Lemma 5.38, there is some $t \in V(D'')$ so that $w_{ut} < .5$. Thus, $w_{vt} + w_{ut} < .5 + .5 = 1$, and $[u, v] \notin E(\text{dom}(D))$. Thus, $\text{dom}(D) = G$. ∎

Note that in proving the construction above we did not require the existence of any particular component.

Lemma 5.40 *Let G be a graph with exactly two components, one of which is a spiked clique and the other is a single edge. There does not exist a complete paired comparison digraph whose domination graph is G.*

Proof: Suppose there exists a pcd, D, with $\text{dom}(D) = G$. Let K be the set of vertices which induce the spiked clique in $\text{dom}(D)$, and $\{u, v\}$ the set of vertices which induce the single edge. Without loss of generality, assume $w_{uv} \geq w_{vu}$. Let K' be the set of vertices in K which induce the clique. By Lemma 5.27, $w_{xu} = w_{xv} = .5$ for all $x \in K'$. If $y \in K - K'$, and z is the vertex adjacent to y in $\text{dom}(D)$, then $z \in K$ and by Lemma 5.1, $w_{uy} = w_{yv} = w_{vz} = .5$. So, $w_{ut} = w_{vt} = .5$ for all $t \in K$. So, $w_{us} + w_{ts} \geq 1$ for all $t \in K$ and $s \in V(D) - \{u, t\}$. Thus, $[u, t] \in E(\text{dom}(D))$ for each $t \in K$, and $\text{dom}(D)$ is connected, a contradiction. Thus no such pcd exists. ∎

Lemma 5.41 *Let G be a graph with exactly two components, one of which is a spiked odd cycle, and the other is a single edge. Then there exists a complete paired comparison digraph D so that dom$(D) = G$.*

Proof: Construct D as follows. Let $V(D) = V(G)$. Let S denote the spiked odd cycle of G, and let $[u, v]$ be the single edge component of G. By

Lemma 5.38, there exists a pcd D_1 so that $\text{dom}(D_1) = S$, and for all $u \in V(D_1)$ there exists a $v \in V(D_1)$ so that $w_{uv} < .5$. Let $V(S)$ induce D_1 and set all other arc weights to .5. We claim $\text{dom}(D) = G$.

Pick $[x, y] \in E(S)$. Then $w_{xz} + w_{yz} \geq 1$ for all $z \in V(D_1)$, and $w_{xa} + w_{ya} = .5 + .5 = 1$ for each $a \notin V(D_1)$. Thus $[x, y] \in E(\text{dom}(D))$. If $[s, t] \notin E(S)$, then $[s, t] \notin E(\text{dom}(D_1))$, and so $[s, t] \notin E(\text{dom}(D))$ by Theorem 5.4. Now, for each $z \in V(D_1)$, $w_{uz} + w_{vz} = .5 + .5 = 1$, and since $V(D_1) = V(D) - \{u, v\}$, $[u, v] \in E(\text{dom}(D))$. Also, if $b \in V(D_1)$, there exists $c \in V(D_1)$ so that $w_{bc} < .5$. So, $w_{uc}+w_{bc} = w_{vc}+w_{bc} < 1$, and $[u, b], [v, b] \notin E(\text{dom}(D))$. Thus, $\text{dom}(D) = G$.

■

Lemma 5.42 *Let G be a graph with at least two components each of which is a spiked biclique. Then there exists a complete paired comparison digraph D so that for all $u \in V(D)$ there exists some $v \in V(D)$ such that $w_{uv} < .5$, and dom$(D) = G$.*

Proof: Construct D as follows. Let $V(D) = V(G)$. Let $S_1, \ldots, S_k$ be the components of G, and let $X_i \cup Y_i$ be the bipartition of S_i for $i = 1, \ldots, k$. Choose $.5 < p < 1$. For $i < j$ weight the arcs of D as follows. For $x \in X_i$, $x' \in X_j$, $y \in Y_i$ and $y' \in Y_j$ set

$$w_{xx'} = p,\ w_{x'x} = 1 - p,$$

$$w_{yy'} = p,\ w_{y'y} = 1 - p,$$

$$w_{y'x} = p,\ w_{xy'} = 1 - p,$$

$$w_{x'y} = p, \text{ and } w_{yx'} = 1 - p.$$

For each $i = 1, \ldots, k$ let S_i' denote the biclique of S_i. For each $i = 1, \ldots, k$, and each distinct $u, v \in V(S_i')$, set $w_{uv} = .5$. For each $a \in V(S_i) - V(S_i')$ let b be its unique neighbor in S_i, and set $w_{ba} = p$, $w_{ab} = 1 - p$, and $w_{au} = .5$ for each $u \neq b$ in S_i. Note from our construction, for each $u \in V(S_i)$, there is some vertex $v \in V(S_j)$ with $i \neq j$ so that $w_{uv} = 1 - p < .5$. So, for each $u \in V(D)$ there is some $v \in V(D)$ such that $w_{uv} < .5$. We finish by showing $\text{dom}(D) = G$.

Let $[u, v] \in E(G)$, and say $[u, v] \in E(S_j)$. Pick $z \in V(D) - \{u, v\}$. Suppose $z \in V(S_j)$. If z is not a pendant vertex adjacent to either of u or v in S_j, then $w_{uz} + w_{vz} = .5 + .5 = 1$. If z is a pendant vertex adjacent to one of u or v in S_j, then $w_{uz} + w_{vz} = .5 + p > 1$. Now suppose $z \notin V(S_j)$. Since S_j is bipartite, one of u, v is in X_j and the other is in Y_j. Without loss of generality, assume $u \in X_j$ and $v \in Y_j$. Since $z \notin V(S_j)$, $z \in V(S_i)$ for some $i \neq j$. Since S_i is bipartite, $z \in X_i$ or $z \in Y_i$. If $z \in X_i$, then $w_{uz} + w_{vz} = (1 - p) + p = 1$ for $i < j$ and $w_{uz} + w_{vz} = p + (1 - p) = 1$ for $i > j$. If $z \in Y_i$, then $w_{uz} + w_{vz} = p + (1 - p) = 1$ for $i < j$ and $w_{uz} + w_{vz} = (1 - p) + p = 1$ for $i > j$. Thus, $[u, v] \in E(\text{dom}(D))$, and $E(G) \subseteq E(\text{dom}(D))$.

Now pick $[u', v'] \notin E(G)$. First suppose $u', v' \in V(S_i)$ for some i. Suppose u' and v' are in the same bipartition of S_i. If $u', v' \in X_i$, then for every $a \in X_q$ and $b \in Y_r$, with $q < i < r$, $u', v' \in O^+(a)$ and $u', v' \in O^+(b)$. Note, since G has at least two components, at least one of X_q and Y_r must exist. So, $[u', v'] \notin E(\text{dom}(D))$ by Lemma 5.5. So, assume u' and v' are in different bipartitions of S_i. Then either u' or v' is a pendant vertex of S_i. Without loss of generality, assume v' is the pendant, and c is its unique neighbor in S_i. Since $[u, v] \notin E(S_i)$, $c \neq u$, so $w_{uc} + w_{vc} = .5 + (1 - p) < 1$. So, $[u', v'] \notin E(\text{dom}(D))$.

Now suppose $u' \in V(S_i)$ and $v' \in V(S_j)$ for some $i < j$. If $u' \in X_i$ and $v' \in X_j$, then since Y_j must be non-empty there is some $y \in Y_j$ so that $w_{u'y} + w_{v'y} = (1-p) + .5 < 1$. If $u' \in Y_i$ and $v' \in Y_j$, then there is some $x \in X_j$, so that $w_{u'x} + w_{v'x} = (1-p) + .5 < 1$. If $u' \in X_i$ and $v' \in Y_j$, then there is some $y \in Y_i$ so that $w_{u'y} + w_{v'y} = .5 + (1-p) < 1$. Finally, if $u' \in Y_i$, and $v' \in X_j$, then there is some $x \in X_i$ so that $w_{u'x} + w_{v'x} = .5 + (1-p) < 1$. Thus, $[u', v'] \notin E(\text{dom}(D))$, and $\text{dom}(D) = G$. ■

Lemma 5.43 *Let G be a graph with at least two components, each of which is a caterpillar or a spiked biclique. Then there exists a complete paired comparison digraph D so that dom$(D) = G$, and for each $u \in V(D)$, there exists $v \in V(D)$ so that $w_{uv} < .5$.*

Proof: Note, if every component of G is a caterpillar, then the result follows from Lemma 5.15, so assume G has at least one component which is a spiked biclique. Construct D as follows. Let $V(D) = V(G)$. Let $S_1, \ldots, S_m$ be the spiked bicliques of G including single edges, but not stars. Let $C_1, \ldots, C_k$ be the caterpillars of G including stars, but not including single edges. We consider two cases. By Lemma 5.39, there exists a pcd, D_C, so that for each $u \in V(D_C)$ there exists a $v \in V(D_C)$ so that $w_{uv} < .5$ and $\text{dom}(D_C) = C_1 \cup C_2 \cup \cdots \cup C_k$. Let $q = \max\{w_{ab} : a, b \in V(D_C)\}$. Note, by Lemma 5.6 we may assume $q < 1$. Weight the arcs of D so that $V(C_1), \ldots, V(C_k)$ induce D_C. We consider two cases. In the first $m \geq 2$, and in the second $m = 1$ and $k \geq 1$. If $m \geq 2$, then by Lemma 5.42, there exists a pcd D_S so that $\text{dom}(D_S) = S_1 \cup S_2 \cup \cdots \cup S_m$. Weight the arcs of D so that $V(S_1) \cup V(S_2) \cup \cdots \cup V(S_m)$ induce D_S. In the case where $m = 1$, let S_1' denote the biclique of S_1. For each $x, y \in V(S_1')$ set $w_{xy} = .5$. Let l be a real number such that $.5 < l < 1$. If $a \in V(S_1) - V(S_1')$, let b be its unique neighbor in S_1 and set $w_{ba} = l$, $w_{ab} = 1 - l$ and set $w_{ac} = .5$ for each $c \in V(S_1) - \{a, b\}$.

For both the case $m = 1$ and $m \geq 2$, let $X_i \cup Y_i$ denote the bipartition of S_i for $i = 1, \ldots, m$, and let $X_j' \cup Y_j'$ denote the bipartition of C_j for $j = 1, \ldots, k$. Choose a real number p with $q < p < 1$. For $i = 1, \ldots, m$, $j = 1, \ldots, k$, $x \in X_i$, $x' \in X_j'$, $y \in Y_i$ and $y' \in Y_j'$, set

$$w_{x'x} = p,\ w_{xx'} = 1 - p,$$

$$w_{y'y} = p,\ w_{yy'} = 1 - p$$

$$w_{xy'} = p,\ w_{y'x} = 1 - p$$

$$w_{yx'} = p, \text{ and } w_{x'y} = 1 - p$$

So, if $u \in V(S_i)$ for some i, then there is some $v \in V(D_C)$ so that $w_{uv} < .5$. So, for all $u \in V(D)$ there exists some $v \in V(D)$ so that $w_{uv} < .5$, and if $u \in V(D_C)$ there exists $v \in V(D_C)$ so that $w_{uv} < .5$ by Lemma 5.39. We finish the proof by showing $\text{dom}(D) = G$.

Pick $[u, v] \in E(G)$. First suppose $[u, v] \in E(C_j)$ for some j. Then, since $\text{dom}(D_C) = C_1 \cup C_2 \cup \cdots \cup C_k$, $w_{uz} + w_{vz} \geq 1$ for each $z \in V(D_C)$. Also, if $z \notin V(D_C)$, then $z \in V(S_i)$ for some i, and so $w_{uz} + w_{vz} = p + (1 - p) = 1$, and $[u, v] \in E(\text{dom}(D))$. Now suppose $[u, v] \in E(S_i)$ for some i. If $m \geq 2$, then since $\text{dom}(D_S) = S_1 \cup \cdots \cup S_m$, $w_{uz} + w_{vz} \geq 1$ for each $z \in V(D_S)$, and $w_{uz'} + w_{vz'} = p + (1 - p)$ for each $z' \in V(D_C)$, so $[u, v] \in E(\text{dom}(D))$. Now, consider when $m = 1$. Pick $z \in V(S_1)$. If z is not a pendant vertex adjacent to either of u or v in S_1, then $w_{uz} + w_{vz} = .5 + .5 = 1$. If z is a pendant vertex adjacent to one of u or v in S_j, then $w_{uz} + w_{vz} = .5 + l > 1$. If $z' \notin V(S_1)$, then $w_{uz'} + w_{vz'} = p + (1-p) = 1$. Thus, $[u, v] \in E(\text{dom}(D))$. So $E(G) \subseteq E(\text{dom}(D))$.

Now pick distinct $u', v' \in V(D)$ so that $[u', v'] \notin E(G)$. If $u', v' \in V(D_C)$, then $[u', v'] \notin E(\text{dom}(D))$ by Theorem 5.4. Similarly, in the case where $m \geq 2$, if $u', v' \in V(D_S)$, then $[u', v'] \notin E(\text{dom}(D))$ by Theorem 5.4. Consider the case where $m = 1$. If $u', v' \in V(S_1)$, then since $[u', v'] \notin E(S_1)$, either u' and v' are in the same bipartition of S_1 or one of u' or v' is pendant in S_1. If u' and v' are in the same bipartition of S_1 then there is some $x \in V(D_C)$ so that $w_{u'x} + w_{v'x} = 1 - p + 1 - p < 1$. So, without loss of generality, assume v' is a pendant in S_1, and y is its unique neighbor in S_1. Then since $[u, v] \notin E(S_1)$, $y \neq u$ and $w_{u'y} + w_{v'y} = .5 + (1 - l) < 1$. So, $[u', v'] \notin E(\text{dom}(D))$. Now, for both cases, assume $u' \in V(S_i)$ for some i and $v' \in V(C_j)$ for some j. If $u' \in X_i$ and $v' \in X'_j$, then since Y_j must be non-empty there is some $y \in Y_j$ so that $w_{u'y} + w_{v'y} = .5 + (1 - p) < 1$. If $u' \in Y_i$ and $v' \in Y'_j$, then there is some $x \in X_i$, so that $w_{u'x} + w_{v'x} = .5 + (1 - p) < 1$. If $u' \in X_i$ and $v' \in Y'_j$, then there is some $x \in X'_j$ so that $w_{u'x} + w_{v'x} \leq q + (1 - p) < 1$. Finally, if $u' \in Y_i$, and $v' \in X'_j$, then there is some $y \in Y'_j$ so that $w_{u'y} + w_{v'y} \leq (1 - p) + q < 1$. Thus, $[u', v'] \notin E(\text{dom}(D))$, and $\text{dom}(D) = G$. ■

Lemma 5.44 *Let G be a graph with no isolated vertices and at least 3 components which form some collection of the following.*

1. *At most one spiked clique.*

2. *Spiked odd cycles.*

3. At least two bipartite graphs consisting of spiked bicliques and caterpillars.

There exists a complete paired comparison digraph D so that $dom(D) = G$.

Proof: Construct D on the vertices of G by assigning weights in the following ways. Let G_1 be the induced subgraph of G which consists of the spiked clique and spiked odd cycles of G. By Lemma 5.39, there exists a pcd D_1 so that $\text{dom}(D_1) = G_1$. Let G_2 be the induced subgraph of G which consists of spiked bicliques and caterpillars. If $G_2 = \emptyset$, then the result follows from Lemma 5.39. So, assume $G_2 \neq \emptyset$. By Lemmas 5.38, 5.42 and 5.43, (depending on if G contains a just caterpillars, just spiked bicliques, or both) there exists a pcd D_2 so that for each $u \in V(D_2)$ there exists a vertex $v \in V(D_2)$ such that $w_{uv} < .5$, and $\text{dom}(D_2) = G_2$. Note, if $G_1 = \emptyset$, the result follows from Lemma 5.38, 5.42 or 5.43. So, assume $G_1 \neq \emptyset$. Weight the arcs of D so that $V(G_1)$ induces D_1, $V(G_2)$ induces D_2 and $w_{xy} = .5$ for all $x \in V(G_1)$ and $y \in V(G_2)$. We claim $\text{dom}(D) = G$.

Pick $[u, v] \in E(G)$. If $u, v \in V(G_1)$, then since $\text{dom}(D_1) = G_1$, $w_{uz}+w_{vz} \geq 1$ for all $z \in V(G_1) - \{u, v\}$. Also, if $z \notin V(G_1)$, $w_{uz} + w_{vz} = .5 + .5 = 1$. So, $[u, v] \in E(\text{dom}(D))$. By a symmetric argument, if $u, v \in V(G_2)$, then $[u, v] \in E(\text{dom}(D))$. So, $E(G) \subseteq E(\text{dom}(D))$. Now, pick $u', v' \in V(D)$ so that $[u', v'] \notin E(G)$. If $u', v' \in V(G_1)$, then $[u', v'] \notin E(\text{dom}(D_1))$, and so $[u', v'] \notin E(\text{dom}(D))$ by Theorem 5.4. If $u', v' \in V(G_2)$, then by a similar argument, $[u', v'] \notin E(\text{dom}(D))$. So, assume $u' \in V(G_1)$ and $v' \in V(G_2)$. By our construction, there exists some $z \in V(G_2)$ so that $w_{v'z} < .5$. So, $w_{u'z}+w_{v'z} <$

$.5 + .5 = 1$. So, $[u', v'] \notin E(\text{dom}(D))$, and $\text{dom}(D) = G$. ■

We now state our main results.

Theorem 5.45 *Let G be a graph with exactly two components and no isolated vertices. Then there exists a complete paired comparison digraph D such that dom$(D) = G$ if and only if the components of G are one of the following.*

1. *two spiked odd cycles,*
2. *a spiked odd cycle and a caterpillar,*
3. *two caterpillars,*
4. *two spiked bicliques,*
5. *a spiked clique and a spiked odd cycle,*
6. *a spiked clique and a caterpillar other than a single edge,*
7. *a caterpillar and a spiked biclique.*

Proof: The necessary conditions follow from Theorems 5.28, 5.33, and 5.36, and Lemma 5.40. Sufficiency follows from Lemmas 5.39, 5.41 5.42, and 5.43. ■

Theorem 5.46 *Let G be a graph with $n \geq 3$ non-trivial components and no isolated vertices. There exists a complete paired comparison digraph D with dom$(D) = G$ if and only if G is one of the following:*

1. *A spiked clique together with any collection of $n - 1$ spiked odd cycles and caterpillars.*

2. Any collection of n spiked odd cycles and caterpillars.

3. A spiked clique, any collection of spiked odd cycles, and a set of two or more bipartite graphs containing only spiked bicliques and caterpillars.

4. Any collection of spiked odd cycles, and a set of two or more bipartite graphs containing only spiked bicliques and caterpillars.

Proof: The necessary conditions come from Theorems 5.28, 5.33, and 5.36. The sufficient conditions come from Lemmas 5.39 and 5.44. ■

5.6 Isolated vertices in domination graphs

We have a characterization of the graphs which can be the domination graphs of complete paired comparison digraphs when we do not allow isolated vertices in the domination graph. We have also seen that any graph can be an induced subgraph of the domination graph of a complete paired comparison digraph. We conclude this chapter with a lower bound on the number of isolated vertices necessary in the domination graph of a pcd in order to have a graph which did not appear in the previous section's characterization in the domination graph.

Recall from Lemma 5.5, that for a vertex v in a pcd, $O^+(v)$ forms an independent set in the domination graph. An independent set in a graph G corresponds to a clique in the complement of G. So if we want to carve G out of a spiked clique, we are essentially looking to remove cliques from G using $O^+(v)$ for various vertices v. With this strategy we obtain our lower bound using the edge clique cover number of the complement of G. A *clique covering* of a graph H is a collection of cliques $Q_1, Q_2, \ldots, Q_m$ contained in H such that each edge of H is in some Q_i. The *edge clique cover number* of a graph H, denoted $\theta(H)$, is the size of a minimum clique covering of H.

Theorem 5.47 *Let G be a connected graph which is not a spiked clique, spiked odd cycle, caterpillar or spiked biclique. Let H be the graph obtained from G by removing the pendant vertices from G. If G is a component of dom(D) for some complete paired comparison digraph D, then dom(D) contains at least $\theta(\overline{H})$ isolated vertices.*

Proof: Let D' be the pcd induced on $V(G)$. By Theorem 5.29, $\text{dom}(D')$ is a spiked clique. Pick $u, v \in V(H)$ so that $[u, v] \notin E(H)$. By Theorem 5.4, $[u, v] \in E(\text{dom}(D'))$, so there must be some $x \notin V(G)$ so that $w_{ux} + w_{vx} < 1$. We show that x must be an isolated vertex in two cases. First, suppose G is not bipartite. So G, and hence H contains an odd cycle. By Lemma 5.30, if x is in a connected component of $\text{dom}(D)$, then $w_{ux} + w_{vx} = .5 + .5 = 1$. So, $w_{ux} + w_{vx} \not< 1$, and hence x must be isolated in $\text{dom}(D)$.

Now suppose G, and hence H, is bipartite, and x is in a connected component of $\text{dom}(D)$. If x is in a component containing an odd cycle, then since G is a connected component of $\text{dom}(D)$, $w_{yx} = .5$ for all $y \in V(G)$. So, $w_{ux} + w_{vx} = 1$. So, assume that x is in a bipartite component of $\text{dom}(D)$. Let $X \cup Y$ be the bipartition of H, and assume, without loss of generality, that $u \in X$ and $v \in Y$. By Theorem 5.35, $w_{ux} = w_{u'x}$ and $w_{vx} = w_{v'x}$ for all $u' \in X$ and $v' \in Y$. So, if $w_{ux} + w_{vx} < 1$, then $w_{u'x} + w_{v'x} < 1$ for all $u' \in X$ and $v' \in Y$, and so $E(H) = \emptyset$. This contradicts H being a connected graph. Thus, x must be an isolated vertex.

By Lemma 5.5, given a vertex v, $O^+(v)$ is an independent set. To obtain G in $\text{dom}(D)$, we must remove edges from the clique of $\text{dom}(D')$ to obtain H. The above argument shows that we must do so using isolated vertices. Since an independent set in H corresponds to a clique in $\overline{H}$, we need at least $\theta(\overline{H})$ vertices to remove edges from the clique of $\text{dom}(D')$ and obtain H. Thus, $\text{dom}(D)$ must contain at least $\theta(\overline{H})$ isolated vertices in order to contain G as a connected component. ■

6. Ranks of Tournament Matrices

6.1 Definitions and background

Let $\mathbb{R}$ denote the real numbers, $\mathbb{Z}^+$ the non-negative integers, and $\mathcal{B}$ the set $\{0, 1\}$ under Boolean arithmetic (addition and multiplication as usual with the exception that $1+1 = 1$). Let M be an $m \times n$ $(0,1)$-matrix and $\mathcal{S} \in \{\mathbb{R}, \mathbb{Z}^+, \mathcal{B}\}$. The *Schein rank* of M over $\mathcal{S}$, written $r_{\mathcal{S}}(M)$, is the least integer k such that M is the product of an $m \times k$ and $k \times n$ matrix with entries from $\mathcal{S}$. Equivalently, it is the least k such that M can be written as the sum of k rank one matrices in $\mathcal{S}$. We call $r_{\mathbb{R}}(M)$ the *real rank* of M, $r_{\mathbb{Z}^+}(M)$ the *non-negative integer rank* of M, and $r_{\mathcal{B}}(M)$ the *Boolean rank* of M.

A *directed biclique* is an orientation of a biclique with bipartition $X \cup Y$ so that $X \Rightarrow Y$. A *rank one matrix, M,* is a matrix with $r_{\mathcal{S}}(M) = 1$. That is, M can be written as $\mathbf{x}\mathbf{y}^\top$ for some vectors $\mathbf{x}$ and $\mathbf{y}$ over $\mathcal{S}$. Pick $X \subseteq \{1, \ldots, n\}$ and $Y \subseteq \{1, \ldots, n\}$, and let $\mathbf{x}$ and $\mathbf{y}$ be $(0,1)$-vectors with a 1 in the i^{th} position of $\mathbf{x}$ if and only if $i \in X$, and a 1 in the i^{th} position of $\mathbf{y}$ if and only if $i \in Y$. Then, $\mathbf{x}\mathbf{y}^\top$ is the adjacency matrix of the directed biclique $X \Rightarrow Y$. So the square, rank one $(0,1)$-matrices are in direct correspondence with the adjacency matrices of directed bicliques. Thus, for a given directed graph D with adjacency matrix M, the problem of finding $r_{\mathbb{Z}^+}(M)$ is equivalent to finding a minimum set of directed bicliques which partition the arcs of D, and the problem of finding $r_{\mathcal{B}}(M)$ is equivalent to finding a minimum set of directed bicliques which cover the arcs of D. For example, the sum of rank one matrices below corresponds to the directed biclique partition shown in Figure 6.1.

$$\begin{pmatrix} 0&1&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&1&0&0&1\\ 0&0&0&0&0 \end{pmatrix} + \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&1&0&1&0\\ 0&0&0&0&0\\ 0&1&0&1&0 \end{pmatrix} + \begin{pmatrix} 0&0&0&0&0\\ 0&0&1&1&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&1&0&0&1\\ 0&0&1&1&1\\ 0&1&0&1&0\\ 0&1&0&0&1\\ 0&1&0&1&0 \end{pmatrix}$$

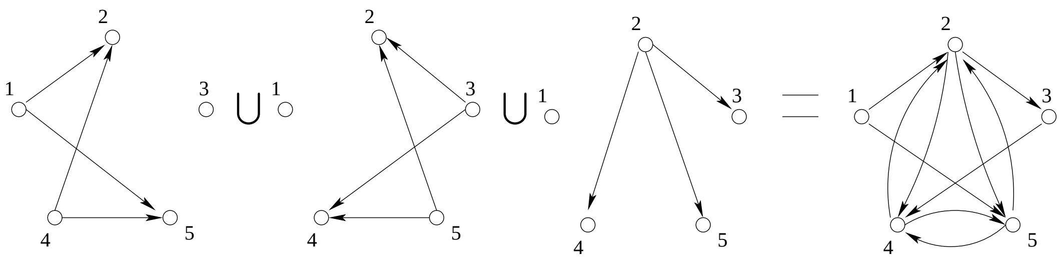

Figure 6.1: A directed biclique partition

Recall, a *line cover* of M is a set of rows and columns of M which contain all the non-zero entries of M. The *term rank* of M, $t(M)$, is the minimum size of a line cover of M. Note, by König's theorem, the problem of finding the term rank of M is equivalent to finding a maximum set of ones, no two of which occur in the same row or column. Such a set of ones is called a set of *independent ones.*

A set of *isolated ones* of a $(0,1)$-matrix M is a set of independent ones, such that no two occur in a submatrix of the form

$$\begin{pmatrix} 1&1\\ 1&1 \end{pmatrix}.$$

We let $is(M)$ denote the size of a largest set of isolated ones. It was observed by Gregory, Jones, Lundgren and Pullman, [30], that $is(M) \leq r_{\mathcal{B}}(M)$. The proof of this follows by noting that a rank one $(0,1)$-matrix can have at most one isolated one.

The graph theoretic interpretation of the ranks discussed in this chapter gives the following inequalities for a $(0,1)$-matrix M,

$$r_{\mathcal{B}}(M) \leq r_{\mathbb{Z}^+}(M) \leq t(M).$$

We get the first inequality since a partition is a cover. A line of a matrix corresponds to the outset or inset of a vertex. So, the arcs involved form the directed biclique $v \Rightarrow O(v)$ or $I(v) \Rightarrow v$ respectively. By assigning an overlapped arc arbitrarily to one of the lines which contain it, a line cover corresponds to a specific directed biclique partition. This gives us the last inequality. Also, by taking advantage of the fact that $\mathbb{Z}^+ \subseteq \mathbb{R}$, we get the following inequality,

$$r_{\mathbb{R}}(M) \leq r_{\mathbb{Z}^+}(M) \leq t(M).$$

In this chapter we examine these inequalities in tournament matrices. In section 6.5 we examine the relationship between the real and Boolean ranks of tournament matrices. For general $(0,1)$-matrices, one can find several examples where the Boolean rank is less than the real rank, and several examples where the real rank is less than the Boolean rank (see [48]). It is a well known result that the real rank of an $n \times n$ tournament matrix is either n or $n-1$ (see [19] or [57]). In appendix A we give two new proofs of this fact. In general, singular tournament matrices tend to be rare. At the same time it is somewhat easier to construct a tournament matrix with less than full Boolean rank. This has lead some to believe that for tournament matrices, the Boolean rank is less than the real rank. This would allow us to merge the two inequalities above, and make rank problems for tournament matrices slightly easier. However, in section 6.5 we show that there exist singular tournament matrices with full Boolean rank, and hence no relationship exists between the real and Boolean ranks of tournament matrices.

In section 6.2 we show how to extend these inequalities to involve domination parameters. In particular, we show how to use irredundance in a tournament as a lower bound for the ranks of the corresponding tournament matrix. We do so by looking at the minimum rank of a matrix. In section 6.3, we show how to formulate the dual of the problem of finding the Boolean row rank of a tournament matrix as an extension of Schütte's problem, and in section 6.4 we examine a dual of the problem of finding the minimum Boolean rank of a tournament matrix.

6.2 Rank and domination

For a tournament T, recall that a set of vertices S is called *dominating* if $O[S] = V(T)$, and S is called *irredundant* if for all $u \in S$ there exists $v \in V(T)$ so that $v \in O[u]$, and $v \notin O[S-u]$. We call this vertex, v, a *private neighbor* of u, and note that we may have $v = u$. We denote by $\gamma(T)$ the size of a smallest dominating set in T, and by $\Gamma(T)$, the size of a largest minimal dominating set in T. Also, we denote the size of a smallest maximal irredundant set and the size of a largest irredundant set by $ir(T)$ and $IR(T)$ respectively. In this section, we show how an irredundant set in a tournament corresponds to a sub-identity matrix in its adjacency matrix. We show that this submatrix gives us a lower bound on the minimum rank of a matrix, and so this gives us a lower bound on both the real and Boolean ranks.

Let M be a $(0,1)$-matrix. Let S be the set of all matrices whose pattern is M. Then the *minimum rank* of M is defined to be

$$mr(M) = \min\{r_{\mathbb{R}}(A) : A \in S\}.$$

Obviously, $mr(M) \leq r_{\mathbb{R}}(M)$. We can equivalently define the Boolean rank as

$$r_{\mathcal{B}}(M) = \min\{r_{\mathbb{Z}^+}(A) : A \in S^+\},$$

where S^+ is the set of all matrices with non-negative integer entries and pattern M. This alternate definition of Boolean rank gives us that $mr(M) \leq r_{\mathcal{B}}(M)$. The equivalence of this definition is discussed below. The result following this discussion gives a lower bound for the minimum rank of a matrix. For more on the minimum rank of a matrix the reader is referred to [32].

Choose an $m \times n$ $(0,1)$-matrix M, with $r_{\mathcal{B}}(M) = k$, and $\min\{r_{\mathbb{Z}^+}(A) : A \in S^+\} = l$, where S^+ is the set of all matrices with non-negative integer entries and pattern M. To see, as claimed above, that $k = l$, let F and G be $m \times k$ and $k \times n$ matrices so that $M = FG$ under Boolean arithmetic. Let $M' = FG$ using integer arithmetic. Then $M' \in S^+$, and $r_{\mathbb{Z}^+}(M') \leq k$. So, $l \leq k$. Now, choose $A \in S^+$ with $r_{\mathbb{Z}^+}(A) = l$, and let B and C be $m \times l$ and $l \times n$ matrices with non-negative integer entries such that $A = BC$. Let B' and C' be the patterns of B and C respectively. Let $A' = B'C'$ under Boolean arithmetic. Since every entry of B and C is non-negative, $A_{i,j} > 0$ if and only if $A' = 1$. So $A' = M$, and so $k \leq l$. Thus $k = l$, as desired.

Lemma 6.1 *Let M be a $(0,1)$-matrix. Let S be a set of independent ones in M such that every two occur together in a submatrix of the form*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $mr(M) \geq |S|$.

Proof: Let A be a matrix whose pattern is M. Permute the rows and columns of A so that the entries of S form a $|S| \times |S|$ diagonal submatrix of A. By performing elementary row operations on A, one can easily verify that the entries of S are pivots for A. So $r_{\mathbb{R}}(A) \geq |S|$. Since A was chosen arbitrarily, $mr(M) \geq |S|$. ■

We now show how to build a set of ones of the form in Lemma 6.1 from an irredundant set.

Lemma 6.2 *Let D be a digraph with adjacency matrix M. Let S be an irredundant set in D, and let $W \subseteq S$ be the set of vertices which are not their own private neighbors. Then M contains a set of $|W|$ ones such that each two occur in a submatrix of the form*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof: For each $u \in W$, let u' be a private neighbor of u. By our choice of W, we may assume that $u' \neq u$. We claim that the ones in M which correspond to the set of the arcs of the form (u, u') in D is our desired set of ones. Suppose they do not. Then there exist distinct $u, v \in W$ with private neighbors u' and v' respectively, so that the ones corresponding to (u, u') and (v, v') are not the only two ones in the submatrix whose rows are indexed by u and v and columns are indexed by u' and v'. So, there must exist another arc in D of either the form (u, v') or (v, u').

$$\begin{array}{c} \\ u \\ v \end{array} \begin{array}{c} u' \; v' \\ \begin{pmatrix} 1 & * \\ * & 1 \end{pmatrix} \end{array}$$

Without loss of generality, suppose (u, v') is an arc of D. Then $v' \in O(u) \subseteq O[S - v]$, a contradiction to v' being a private neighbor of v. Thus, these arcs correspond to the desired set of ones in M. ∎

Let T be a tournament. Suppose S is an irredundant set in T, and $u \in S$ is its own private neighbor. Then $u \notin O[S - u]$. Since T is a tournament, this implies that $u \Rightarrow S - \{u\}$, and u is a transmitter in $T[S]$. Since a tournament can have at most one transmitter, u is the only vertex of S which is its own private neighbor. So, any irredundant set in a tournament has at most one vertex which is its own private neighbor. That is, for a tournament, the set W in Lemma 6.2 differs from S by at most one vertex. So, Lemmas 6.1 and 6.2 give us the following theorem.

Theorem 6.3 *Let T be a tournament with adjacency matrix M. Then,*

$$IR(T) - 1 \leq mr(M) \leq r_{\mathbb{R}}(M) \leq r_{\mathbb{Z}^+}(M) \leq t(M)$$

and

$$IR(T) - 1 \leq mr(M) \leq r_{\mathcal{B}}(M) \leq r_{\mathbb{Z}^+}(M) \leq t(M).$$

We can also add in the inequality in Theorem 1.6 to get the inequalities $ir(T) - 1 \leq \gamma(T) - 1 \leq \Gamma(T) - 1 \leq IR(T) - 1 \leq mr(M) \leq r_{\mathbb{R}}(M) \leq r_{\mathbb{Z}^+}(M) \leq t(M)$ and $ir(T) - 1 \leq \gamma(T) - 1 \leq \Gamma(T) - 1 \leq IR(T) - 1 \leq mr(M) \leq r_{\mathcal{B}}(M) \leq r_{\mathbb{Z}^+}(M) \leq t(M)$. One should note, that while this does give an interesting tie between domination, irredundance, minimum rank, and the four ranks, it does not give the best lower bound. In [34], Hedetniemi, Hedetniemi, McRae and Reid show that $IR(T) \leq n/2$ for an n-tournament T.

In the following section we look at Boolean row rank. We see that almost all tournaments have full Boolean row rank, and show how to express the dual of the problem of finding an n-tournament with minimum Boolean row rank as an extension of Schütte's problem.

6.3 Boolean row rank

In this section we explore the Boolean row ranks, and Boolean column ranks of tournament matrices. The *Boolean row rank* of a $(0,1)$-matrix M, $br(M)$ is the size of a smallest set S of rows of M such that every every row of M can be written as a Boolean sum of rows in S. The Boolean column rank is defined analogously. Boolean row rank has been studied in many situations (see [6], [30], [32], and [35]). Boolean row and column rank are not as well behaved as the real rank or the Boolean rank of a matrix as defined in section 6.1. For instance, unlike with the real rank of a matrix, the Boolean row and column rank of a matrix are not necessarily equal. For a thorough treatment of the algebra behind Boolean row and column rank, the reader is referred to [35].

Let D be a digraph with $V(D) = \{1, \ldots, n\}$ and M its adjacency matrix. If there exists a row of M, say $M_{i\bullet}$, and set of row indices S, such that $M_{i\bullet} = \sum_{j \in S} M_{j\bullet}$, then in D, $O(i) = \bigcup_{j \in S} O(j) = O(S)$. Thus, M has full Boolean rank if and only if there does not exist a vertex $v \in V(D)$ and set $S \subseteq V(D)$ such that $O(v) = O(S)$. In particular, $O(v) = O(S)$ implies that for all $u \in S$, $O(u) \subseteq O(v)$.

A *king* in a tournament T is a vertex u such that for each $v \in V(T) - \{u\}$, either $u \to v$ or there exists some $w \in V(T)$, such that $u \to w$ and $w \to v$. So, a vertex u is not a king if and only if there exists some v such that $O(u) \subseteq O(v)$. An *all kings tournament* is one in which every vertex is a king. So, for no two vertices u, v in an all kings tournament is $O(u) \subseteq O(v)$. Hence an all kings tournament has an adjacency matrix with full Boolean row rank.

Proposition 6.4 *The adjacency matrix of an all kings tournament has full Boolean row rank.*

Kings and all kings tournaments have been studied by several people. For a collection of results on kings in tournaments the reader is referred to [52]. The following probabilistic result is due to Maurer, which gives us the following corollary on Boolean row rank.

Theorem 6.5 [39] *Almost all tournaments are all kings tournaments.*

Corollary 6.6 *Almost all tournament matrices have full Boolean row rank.*

We now look at the minimum possible Boolean row rank of an $n \times n$ tournament matrix, and construct our dual to this problem in Theorem 6.9. To do this we consider the following, specific, Boolean factorization. Let M be an $n \times n$ (0,1)-matrix with $r_{\mathcal{B}}(M) = k$, and let F and G be $n \times k$ and $k \times n$ matrices so that $M = FG$ under Boolean arithmetic. If we can permute the rows and columns of F, so that F contains a $k \times k$ sub-identity matrix, then the rows of G must be rows of M. This tells us that every row of M can be written as a Boolean sum of the rows of G (which are rows of M), and so $br(M) \leq k$.

Further, if we know that $br(M) = k$, then we can write M as FG where G are rows of M, and F contains a $k \times k$ identity matrix as a submatrix. Since this is a special case of a Boolean factorization, $r_{\mathcal{B}}(m) \leq br(M)$ for any $(0, 1)$-matrix M.

Let M be an $n \times n$ tournament matrix, and suppose that $br(M) = k$. So we can write $M = FG$ where F and G are $n \times k$ and $k \times n$ respectively. Further, we can permute the rows and columns of F so that the first k rows form a $k \times k$ identity matrix. So, we can write F and G in block form as

$$F = \begin{bmatrix} I_k \\ A \end{bmatrix}, G = [T_k \; B]$$

where T_k is a $k \times k$ tournament matrix, and A and B are arbitrary. We use this interpretation of Boolean row rank to attack the following problem.

For a given n, let

$$br(n) = \min\{br(M) : M \text{ is an } n \times n \text{ tournament matrix}\}.$$

What are the values of $br(n)$?

In the following section we will look at a similar parameter, $b(n)$, the minimum Boolean rank of any $n \times n$ tournament matrix. An upper bound on $b(n)$ was found by Bain, Lundgren, and Maybee in [4]. They show that, asymptotically, $b(n) \leq n^{\frac{2}{3}}$. They obtain this bound by a construction. In particular their construction gives the $(m^3 + m^2 + m) \times (m^3 + m^2 + m)$ tournament matrix $\tilde{T}$ with factorization

$$\tilde{T} = \begin{bmatrix} T_k \\ Q_{n-1} \\ Q_{n-2} \\ \vdots \\ Q_1 \end{bmatrix} \left[I_k \; Q_1^\top Q_2 \; Q_1^\top Q_3 \cdots Q_1^\top T_k \right]$$

where I_k is the $(m^2+m+1) \times (m^2+m+1)$ identity matrix, each Q_i is a sum of permutation matrices, and T_k is an $(m^2+m+1) \times (m^2+m+1)$ tournament matrix. For a proof that $\tilde{T}$ is in fact a tournament matrix, the reader is referred to [4]. So, $\tilde{T}^\top$ can be written in the form FG where the first (m^2+m+1) rows of F form an identity matrix. Thus, Bain et. al.'s construction gives us the following.

Theorem 6.7 *For a given n,*

$$br(n) \leq n^{\frac{2}{3}}.$$

The only lower bound on $br(n)$ is $b(n)$ which has as its current best lower bound $n^{\frac{1}{\sqrt{\log_2(n)}}}$. We now show how the dual of the problem of finding $br(n)$ is an extension of Schütte's problem. While this interpretation of the dual has yet to improve the lower bound for $br(n)$, the nature of the problem does lead us to believe that $br(n)$ should be much larger than $n^{\frac{1}{\sqrt{\log_2(n)}}}$.

Let T be a k-tournament, and let $\mathcal{F}$ be a set of subsets of $V(T)$. We say that $\mathcal{F}$ is a *Schütte family* if for all $S \in \mathcal{F}$, $|S| \geq 2$, $T[S]$ contains no transmitter, and for all distinct S and S' in $\mathcal{F}$, there exists $u \in S$ such that $u \Rightarrow S'$ or there exists $v \in S'$ such that $v \Rightarrow S$. We denote by $Sc(T)$ the size of a largest Schütte family in a tournament T, and let $Sc(k) = \max\{Sc(T) : T \text{ is a } k\text{-tournament}\}$. We show that finding $b(n)$ is equivalent to finding $Sc(k)$, but first need the following lemma.

Lemma 6.8 *There exists an $n \times n$ tournament matrix M, with a row and column of zeros, such that $br(M) = br(n)$.*

Proof: Note, for $n \geq 2$, the adjacency matrix of the transitive tournament gives an example of a tournament matrix with Boolean row rank less than n, and so for $n \geq 2$, $br(n) < n$. Now choose n, and let M be a tournament matrix with $br(M) = br(n)$. Let S be a minimum set of rows so that every row of M is a Boolean sum of rows in S. We now show that if M does not have a row or column of zeros, then we can replace one of its rows or columns with a row or column of zeros.

A row of zeros would simply be an empty sum of the rows of S, so we can append a row of zeros to M without changing $br(M)$. If we add a column of n ones with a zero in the new zero row, then each non-zero row will have a one in its new entry, and the rows of S can still be used to generate the remaining rows of M. So, we can create an $(n+1) \times (n+1)$ tournament matrix with a row of zeros and Boolean row rank $br(M)$. Now, select a row of M not in S. Say the i^{th} row. Create M' by removing the i^{th} row and column of M. Then $br(M') \leq br(M)$, since S can still be used to generate the remaining rows of M'. Now, create M'' by appending a row of zeros and a column of $n-1$ ones. By the above argument, $br(M'') = br(M') \leq br(M) = br(n)$. Since M'' is $n \times n$ we have the $br(M'') = br(n)$, and M'' is an $n \times n$ tournament matrix with a row of zeros.

Now, assume M does not contain a column of zeros. If the rows of S cannot be used to create a row of ones, then there must be some entry in which every row of S has a zero. Since all rows of M are sums of rows in S, this implies that M has a column of zeros, a contradiction. So, we can append a row of ones to M without raising the Boolean row rank. Appending a column of zeros to M

will not change the fact that all rows of M are sums of rows of S, since this will leave a zero in the same entry for all rows. So, we can create an $(n+1)\times(n+1)$ tournament matrix with a column of zeros and Boolean row rank $br(n)$. This together with an argument similar to the one above, allows us to replace a row and column of M with a row of $n-1$ ones and a column of zeros without changing the Boolean row rank. Thus, we can find an $n \times n$ tournament matrix with a row and column of zeros, and Boolean row rank $br(n)$. ∎

Theorem 6.9 *Choose k and let $n = k + Sc(k) + 2$. Then $br(n) = k$.*

Proof: Let M be an $n \times n$ tournament matrix with $br(M) = br(n) = k$. By Lemma 6.8, we may assume M contains a row and column of zeros. So, $M = FG$ for $n \times k$ and $k \times n$ matrices F and G, with

$$F = \begin{bmatrix} I_k \\ A \\ \mathbf{0}^\top \\ \mathbf{1}^\top \end{bmatrix}, \text{ and } G = [T_k \; B \mathbf{1} \, \mathbf{0}]. \tag{6.1}$$

So,

$$M = FG = \left[\begin{array}{c|c|c} T_k & B & \mathbf{0}\,\mathbf{1} \\ \hline AT_k & AB & \mathbf{0}\,\mathbf{1} \\ \hline \mathbf{1}^\top & \mathbf{1}^\top & 0\;1 \\ \mathbf{0}^\top & \mathbf{0}^\top & 1\;0 \end{array}\right].$$

Let T be the k-tournament with $V(T) = \{1, 2, \ldots, k\}$ whose adjacency matrix is T_k. For $i = 1, \ldots, n-k-2$, let S_i be the set with $j \in S_i$ if and only if $A_{i,j} = 1$. We claim that $\{S_1, \ldots, S_{n-k-2}\}$ is a Schütte family, and that any Schütte family can be used to produce matrices F and G of the form in (6.1).

Note that $A_{i\bullet}T_k$ is the Boolean sum of the rows of T_k, $\sum_{\{j:A_{i,j}=1\}}(T_k)_{j\bullet}$. This results in a row vector with a 1 in the j position if and only if $j \in O(S_i)$. Since M is a tournament matrix, we have that $(AT_k)^\top = B^c$. So, $B_{j,i} = 0$ if and only if $(AT_k)_{i,j} = 1$. Thus, $B_{j,i} = 0$ if and only if $j \in O(S_i)$. So, B_i is a column vector with a 1 in the j^{th} position if and only if $j \in O(S_i)^c$.

Since M is a tournament matrix, $(AB)_{i,j} = 1$ if and only if $(AB)_{j,i} = 0$. So, $A_{i\bullet}B_j = 1$ if and only if $A_{j\bullet}B_i = 0$. The inner product of two Boolean vectors is 0 if and only if the sets corresponding to the vectors have an empty intersection, so we can rewrite the above equivalence as $S_i \cap O(S_j)^c \neq \emptyset$ if and only if $S_j \cap O(S_i)^c = \emptyset$. Since T is a tournament, $S_i \cap O(S_j)^c \neq \emptyset$ if and only if there is some $v \in S_i$ such that $v \Rightarrow S_j$. If there exists $v \in S_i$ so that $v \Rightarrow S_j$, then there cannot exist $u \in S_j$ so that $u \Rightarrow S_i$ or else we would have $u \rightarrow v$ and $v \rightarrow u$. Thus, the statement "$S_i \cap O(S_j)^c \neq \emptyset$ if and only if $S_j \cap O(S_j)^c = \emptyset$" is equivalent to the statement "there exists $v \in S_i$ such that $v \Rightarrow S_j$ or there exists $u \in S_j$ such that $u \Rightarrow S_i$." Thus "$(AB)_{i,j} = 1$ if and only if $(AB)_{j,i} = 0$" is equivalent to the statement "there exists $v \in S_i$ such that $v \Rightarrow S_j$ or there exists $u \in S_j$ such that $u \Rightarrow S_i$."

Now, since M is a tournament matrix, $(AB)_{i,i} = 0$. Further $(AB)_{i,i} = 0$, if and only if $A_{i\bullet}B_i = 0$, if and only if $S_i \cap O(S_i)^c = \emptyset$. This is true if and only if for all $u \in S_i$ there exists $v \in S_i$ such that $v \rightarrow u$, which is equivalent to saying $\delta^-(T[S_i]) \geq 1$, or $T[S_i]$ has no transmitter. Thus $\{S_1, \ldots, S_{n-k-2}\}$ is a Schütte family in T.

Now, suppose $\mathcal{F}$ is a Schütte family of size $n - k - 2$ in a k-tournament T. Construct the $n \times k$ matrix F' so that the first k rows of F' form I_k, the last two rows of F' are $\mathbf{0}^\top$ and $\mathbf{1}^\top$, and for $k + 1 \leq i \leq n - 2$, $F'_{i,j} = 1$ if and only if $j \in S_{i-k}$. Construct G' by letting the first k columns induce the adjacency matrix of T, letting the last two columns of G' be $\mathbf{1}$ and $\mathbf{0}$, and for

$k + 1 \leq j \leq n - 2$ setting $G'_{i,j} = 1$ if and only if $i \in O(S_j)^c$. By the above equivalencies, one can easily verify that FG is an $n \times n$ tournament matrix. Thus each Boolean factorization of a tournament matrix of the form in equation (6.1), is equivalent to a Schütte family in a subtournament. So, $br(n) = k$ if and only if $n = k + Sc(k) + 2$. ■

In looking for Schütte families we have found that they tend to be somewhat small. There are several extra, natural restrictions that occur. For instance, if $\mathcal{F}$ is a Schütte family in a tournament T, and $S_i \in \mathcal{F}$, then $|S_i| \geq 3$. This follows, for otherwise $T[S_i]$ would be the 2-tournament and have a transmitter. We must also have that $S_i \subseteq O(v)$ for some $v \in V(T)$. This tells us that $|S_i| \leq \Delta^+(T)$. While this is not a great upper bound for $|S_i|$, in practice we have found that tournaments with large maximum out-degree tend to have a smaller values of $Sc(T)$. We find that $Sc(T)$ seems to be larger in regular tournaments, and conjecture that $Sc(k)$ is maximized, when $k \equiv 3 \pmod 4$, by a doubly-regular tournament.

We finish this section with the following open problems. The third is motivated by the fact that the construction of Bain et. al. in [4] is our current best upper bound for both $b(n)$ and $br(n)$.

Open Problem 6.10 *Study how the structure of a given tournament T affects $Sc(T)$.*

Open Problem 6.11 *Find values or bounds for $Sc(k)$.*

Open Problem 6.12 *Is $br(n) = b(n)$?*

6.4 A conjecture about tournament codes

In this section we expand on some of the ideas presented in the previous section. In particular we show how finding the Boolean rank of a tournament matrix is related to covering a smaller digraph with a large number of directed bicliques. Recall that we define $b(n)$ to be

$$b(n) = \min\{r_{\mathcal{B}}(M) : M \text{ is an } n \times n \text{ tournament matrix}\}.$$

We prove our problem is in fact a dual, and give some results on the structure of the directed biclique cover of the smaller digraph. Finally we use these to make a conjecture about $b(n)$, but first we discuss a different dual to this problem.

In [3], Bain, Lundgren, and Maybee show that a dual to the problem of finding $b(n)$ is the problem of finding a tournament code of length k with maximum size. A *tournament code* length k is a set of codewords of length k from the set $\{0, 1, *\}$ such that if a and b are distinct codewords then exactly one of the following is true:

1. there exists j such that $(a_j, b_j) = (0, 1)$,
2. there exists j such that $(a_j, b_j) = (1, 0)$.

Tournament codes get their name from the fact that if one defines a digraph whose vertices are the codewords, and letting $a \to b$ if there is some j such that $(a_j, b_j) = (0, 1)$ then the digraph is a tournament. For a given k we define

$$tc(k) = \max\{|C| : C \text{ is a tournament code of length } k\}.$$

Tournament codes have been studied by van Lint in [60] and by Tang, Golumb and Graham in [59]. Each have independently shown that that $tc(k) \geq k^{\frac{3}{2}}$, and that $tc(k) < k^{\log_2(k)}$. In [4], Bain et. al., also proved the lower bound using Boolean rank and the following equivalences.

Theorem 6.13 [3]*Let C be a tournament code of length k and $|C| = n$. Let T be the tournament associated with the code C, and let M be the adjacency matrix of T. Then $M = FG$ where F is $n \times k$ and G is $k \times n$, where the arithmetic is Boolean.*

Theorem 6.14 [3] *Let M be an $n \times n$ tournament matrix with Boolean factorization $M = FG$ where F is $n \times k$ and G is $k \times n$. Then there exists a tournament code C of length k with $|C| = n$.*

These two theorems show that $b(n) = k$ if and only if $tc(k) = n$. In this section we give yet another dual to the problem of determining $b(n)$. We do so by examining the factorization of a tournament matrix M into the product FG. In particular, we show that the product GF is the adjacency matrix of an oriented graph (recall that we assume all graphs are simple for this thesis). So, if M is an $n \times n$ tournament matrix with $r_{\mathcal{B}}(M) = k$, and $M = FG$ with F an $n \times k$ matrix and G a $k \times n$ matrix, then $M' = GF$ is a $k \times k$ matrix whose digraph can be covered with n directed bicliques.

In the following results and proofs, the equivalence between directed biclique covers of a digraph and the Boolean rank of its adjacency matrix is a useful tool. Due to this, we want to be able to move back and forth easily between graph theoretic techniques and matrix theoretic techniques. To do so, we introduce the following notation. Let M be an $n \times n$ $(0, 1)$-matrix. We define $D(M)$ to be the digraph whose adjacency matrix is M.

Theorem 6.15 *Let M be an $n \times n$ $(0,1)$-matrix such that $D(M)$ is an oriented graph, and suppose $M = FG$ where F is $n \times k$ with no zero columns and G is $k \times n$ with no zero rows, and the arithmetic is Boolean. Then $A = GF$ is the adjacency matrix of an oriented graph. That is,*

1. *$A_{i,i} = 0$ for all $i = 1, \ldots, k$, and*
2. *if $A_{i,j} = 1$, then $A_{j,i} = 0$.*

Further, $A_{i,j} = A_{j,i} = 0$ for some $i \neq j$, if and only if the corresponding directed biclique cover of $D(M)$ contains two directed bicliques $X_i \Rightarrow Y_i$ and $X_j \Rightarrow Y_j$ such that $X_i \cap Y_j = \emptyset$ and $X_j \cap Y_i = \emptyset$.

Proof: Let $A = GF$, and suppose $A_{i,i} = 1$ for some i. Then $G_{i\bullet}F_i = 1$, and so $G_{i,m} = F_{m,i} = 1$ for some m. However, $M = \sum_{j=1}^{k} F_j G_{j\bullet}$. So, if $F_{m,i} = G_{i,m} = 1$, then $M_{i,i} = \sum_{j=1}^{k} F_{j,i}G_{i,j} = F_{m,i}G_{i,m} = 1$, a contradiction to the fact that $M_{i,i} = 0$ for all i. Thus, $A_{i,i} = 0$ for all $i = 1, \ldots, k$.

Now, assume to the contrary that $A_{i,j} = A_{j,i} = 1$ for some $i \neq j$. So,

$$G_{i\bullet}F_j = G_{j\bullet}F_i = 1.$$

Consider the product of the rank one matrices $F_jG_{j\bullet}$ and $F_iG_{i\bullet}$. This gives

$$\begin{aligned}(F_iG_{i\bullet})(F_jG_{j\bullet}) &= F_i(G_{i\bullet}F_j)G_{j\bullet} \\ &= F_i(1)G_{j\bullet} \\ &= F_iG_{j\bullet}.\end{aligned}$$

Since F has no zero columns and G has no zero rows, $F_iG_{j\bullet}$ is not the zero matrix. Let $X_i \Rightarrow Y_i$ and $X_j \Rightarrow Y_j$ be the directed bicliques defined by $D(F_iG_{i\bullet})$ and

$D(F_jG_{j\bullet})$ respectively, and let D be the digraph with vertices $X_i \cup Y_i \cup X_j \cup Y_j$ and whose arcs are only those appearing in these two directed bicliques. Then, since $F_iG_{j\bullet}$ is not the zero matrix, the above equation implies that there is a directed walk of length 2 in D from some $x \in X_i$ to some $y \in Y_j$. Let v be the interior vertex of this walk. Then $v \in Y_i$ and $v \in X_j$. A symmetric argument shows that there is a directed walk of length 2 from some $x' \in X_j$ to some $y' \in Y_i$. Let u be the interior vertex of this walk. So, $u \in Y_j$ and $u \in X_i$. Since $v \in X_j$ and $u \in Y_j$, $v \to u$. Also, since $u \in X_i$ and $v \in Y_i$, $u \to v$. However, u and v are also vertices of $D(M)$, a contradiction to $D(M)$ being an oriented graph. Thus, if $A_{i,j} = 1$, then $A_{j,i} = 0$.

Now, suppose $A_{i,j} = A_{j,i} = 0$. Again, we consider the product of the rank one matrices $F_iG_{i\bullet}$ and $F_jG_{j\bullet}$. This gives,

$$\begin{aligned}(F_iG_{i\bullet})(F_jG_{j\bullet}) &= F_i(G_{i\bullet}F_j)G_{j\bullet}\\ &= F_i(0)G_{j\bullet}\\ &= O_{n\times n},\end{aligned}$$

and similarly,

$$(F_jG_{j\bullet})(F_iG_{i\bullet}) = O_{n\times n}.$$

As before, let $X_i \Rightarrow Y_i$ and $X_j \Rightarrow Y_j$ be the directed bicliques defined by $D(F_iG_{i\bullet})$ and $D(F_jG_{j\bullet})$ respectively, and let D be the digraph with vertices $X_i \cup Y_i \cup X_j \cup Y_j$ and whose arcs are only those appearing in these directed bicliques. Then, these equations state that there are no walks of length 2 from X_i to Y_j or from X_j to Y_i in D. Thus, $X_j \cap Y_i = \emptyset$, and $Y_j \cap X_i = \emptyset$.

Conversely, suppose the directed biclique cover of $D(M)$ which is defined by F and G contains two directed bicliques $X_i \Rightarrow Y_i$ and $X_j \Rightarrow Y_j$ such that $X_i \cap Y_j = \emptyset$ and $X_j \cap Y_i = \emptyset$. Let these bicliques correspond to $D(F_i G_{i\bullet})$ and $D(F_j G_{j\bullet})$ respectively. Since, $X_i \cap Y_j = \emptyset$ and $X_j \cap Y_i = \emptyset$, there is no walk of length 2 in the digraph D defined on the vertices and arcs of these directed bicliques. So,

$$(F_i G_{i\bullet})(F_j G_{j\bullet}) = O_{n\times n}.$$

Letting $a = G_{i\bullet} F_j$ we have that

$$aF_i G_{j\bullet} = F_i a G_{j\bullet} = F_i(G_{i\bullet} F_j) G_{j\bullet} = O_{n\times n}.$$

since F_i and $G_{j\bullet}$ are non-empty, $F_i G_{j\bullet} \neq O_{n\times n}$, and so $G_{i\bullet} F_j = a = 0$. Similarly, we must have that $G_{j\bullet} F_i = 0$. ■

In particular, if M in Theorem 6.15 is a tournament matrix, then A is an oriented graph. Theorem 6.15 also gives us a dual to the problem of finding $b(n)$. Given k, let D be a k vertex oriented graph. Let $f(D)$ be the size of a largest directed biclique cover of D such that for any two directed bicliques $X_i \Rightarrow Y_i$ and $X_j \Rightarrow Y_j$ of C, either $X_i \cap Y_j \neq \emptyset$ or $X_j \cap Y_i \neq \emptyset$. Let $f(k) = \max\{f(D) : D \text{ is a } k \text{ vertex oriented graph}\}$. Then by Theorem 6.15, $f(k) = n$ if and only if $b(n) = k$.

Let M be a tournament matrix. Suppose M can be factored as $M = FG$, and $M' = GF$. Let C be the directed biclique cover of M' defined by GF. That is,

$$C = \{D(G_i F_{i\bullet}) : i = 1, \ldots, n\}.$$

The following two results tell us more about the structure of C.

Theorem 6.16 *Let M be an $n \times n$ tournament matrix and suppose $M = FG$ where F is $n \times k$ and G is $k \times n$. Let $M' = GF$. Let C be the directed biclique cover of M' defined by GF. Then, for any $i \neq j$, $D(G_i F_{i\bullet}) \nsubseteq D(G_j F_{j\bullet})$.*

Proof: Suppose, to the contrary, that for some $i \neq j$, $D(G_i F_{i\bullet}) \subseteq D(G_j F_{j\bullet})$. Then, since $D(G_i F_{i\bullet})$ and $D(G_j F_{j\bullet})$ are directed bicliques, we must have that $G_i \leq G_j$ and $F_{i\bullet} \leq F_{j\bullet}$. Now, since $M = FG$ has a zero diagonal, we have that $F_{j\bullet} G_j = 0$. Thus, $M_{j,i} = F_{j\bullet} G_i \leq F_{j\bullet} G_j = 0$, and $M_{i,j} = F_{i\bullet} G_j \leq F_{j\bullet} G_j = 0$. This is a contradiction to M being a tournament matrix. Thus, for all $i \neq j$, $D(G_i F_{i\bullet}) \nsubseteq D(G_j F_{j\bullet})$.

■

Theorem 6.17 *Let M be an $n \times n$ tournament matrix and suppose $M = FG$ where F is $n \times k$ and G is $k \times n$. Let $M' = GF$. Let C be the directed biclique cover of M' defined by GF. If $X_1 \Rightarrow Y_1, X_2 \Rightarrow Y_2, \ldots, X_m \Rightarrow Y_m$ are in C, then $\bigcup_{i=1}^{m} X_i \cup Y_i$ is not a directed biclique.*

Proof: Choose $m \geq 2$ elements of C. Without loss of generality, assume these are $D(G_1 F_{1\bullet}), D(G_2 F_{2\bullet}), \ldots, D(G_m F_{m\bullet})$. Assume to the contrary that the union of these form a directed biclique. Then it will have bipartition $(\bigcup_{i=1}^{m} X_i) \cup (\bigcup_{i=1}^{m} Y_i)$. So, the directed biclique will be $D((G_1 + G_2 + \cdots + G_m)(F_{1\bullet} + F_{2\bullet} + \cdots + F_{m\bullet}))$. Since C is a cover of $D(M')$, this implies that $(G_1 + G_2 + \cdots + G_m)(F_{1\bullet} + F_{2\bullet} + \cdots + F_{m\bullet}) \leq M'$. So, $\sum_{1 \leq i,j \leq m} G_i F_{j\bullet} \leq M'$. Choose $i \neq j$. Since $M = FG$ is a tournament matrix, $F_{i\bullet} G_j = 0$ if and only if $F_{j\bullet} G_i = 1$. Without loss of generality, assume that $F_{j\bullet} G_i = 1$. So, $G_i F_{j\bullet}$ must have a 1 on the diagonal. So, $\sum_{1 \leq i,j \leq m} G_i F_{j\bullet}$ has a 1 on the diagonal. However, $\sum_{1 \leq i,j \leq m} G_i F_{j\bullet} \leq M'$ and

M' has a zero diagonal by Lemma 6.15, a contradiction. Thus, $\bigcup_{i=1}^{m} X_i \cup Y_i$ is not a directed biclique. ■

It is our current belief that if D is a directed graph, and C a directed biclique cover of the arcs of D subject to the conditions in Theorems 6.16 and 6.17, then one can find a system of distinct representatives of arcs for the directed bicliques in C. This would imply that $|C| \leq |A(D)|$. If D is an oriented graph on k vertices, then the maximum number of arcs D could have is $\binom{k}{2}$, and we would have $|C| \leq \binom{k}{2}$. So, we would have $k^2 - k - 2|C| \geq 0$. Since, as a function of k, this is an increasing function for $k \geq \frac{1}{2}$, and is equal to 0 for $k = \frac{1+\sqrt{1+8|C|}}{2}$, we would have that $k \geq \frac{1+\sqrt{1+8|C|}}{2}$. By the dual provided by Theorem 6.15, this would imply that $b(n) \geq \frac{1+\sqrt{1+8n}}{2}$. This gives us the following conjecture about the Boolean rank of tournament matrices, and tournament codes.

Conjecture 6.18

$$\sqrt{n} \leq b(n) \leq n^{\frac{2}{3}},$$

and

$$k^{\frac{3}{2}} \leq tc(k) \leq k^2.$$

If Conjecture 6.18 is true, then it will be a serious improvement over the previous lower bound on $b(n)$ of $n^{\frac{1}{\sqrt{\log_2(n)}}}$, and previous upper bound on $tc(k)$ of $k^{\log_2(k)}$.

6.5 A construction for a singular tournament matrix with full Boolean rank

In this section we answer the question of Siewert [58], "Does there exist a singular tournament matrix with full Boolean rank?" The following 9×9 matrix answers the question in the positive.

$$\left(\begin{array}{ccc|ccc|ccc} 0 & \mathbf{1} & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \mathbf{1} & 1 & 0 & 1 & 1 & 1 & 1 \\ \mathbf{1} & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 1 & 1 \\ 0 & 1 & 0 & \mathbf{1} & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & \mathbf{1} & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 1 & \mathbf{1} & 0 & 0 \end{array}\right)$$

It is easy to see that this is a tournament matrix. Further, one can verify that the vector $(111111-1-1-1)^\top$ is a null vector for this matrix, so it is singular. Finally, the bolded ones form a set of 9 isolated ones, and so the matrix has full Boolean rank. We generalize this example by giving a method for constructing tournament matrices with full Boolean rank in Theorem 6.22, and showing how to choose singular matrices from this class in Theorem 6.23. The following three lemmas will let us consider the tournament matrix in our construction in block form to show it has a full set of isolated ones, and hence full Boolean rank.

Lemma 6.19 *Let M be a matrix which has block form $\begin{pmatrix} A & B \\ O & C \end{pmatrix}$. If A and C have a full set of isolated ones, then M has a full set of isolated ones.*

Proof: Let S_A and S_C denote the sets of isolated ones in A and C respectively. We claim $S_A \cup S_C$ is a set of isolated ones in M. Pick two entries $M_{i,j}, M_{k,l} \in S_A \cup S_C$. If $M_{i,j}$ and $M_{k,l}$ are both in S_A or both in S_C, then they occur in different rows and columns, and do not appear in a submatrix of the form $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ by choice of S_A and S_C. So, without loss of generality, assume

$M_{i,j} \in S_A$ and $M_{k,l} \in S_C$. Then they obviously do not occur in the same row or column, and since the lower left block is O, they occur together in a 2×2 submatrix which is a submatrix of $\begin{pmatrix} 1\,1 \\ 0\,1 \end{pmatrix}$. Thus, $S_A \cup S_C$ forms a set of isolated ones, and since S_A and S_C are full sets of isolated ones for A and C, $S_A \cup S_C$ is a full set of isolated ones for M. ∎

Lemma 6.20 *Pick k odd, and let P be the permutation matrix which corresponds to the permutation $i \mapsto i+1 \pmod{k}$. Then the matrix $M = \sum_{i=1}^{\frac{k-1}{2}} P^i$ has full Boolean rank. Further, the entries from P in M form a set of k isolated ones.*

Proof: Pick $M_{i,i+1}$ and $M_{j,j+1}$ (arithmetic modulo k with congruences classes from $\{1, \ldots, k\}$). Obviously these entries are not in the same row or column. Now, by our construction of M, $M_{c,d} = 1$ if and only if $d \neq c$ and $(d-c)$ modulo k is less than or equal to $\frac{k-1}{2}$. Since exactly one of $(j+1-i)$ and $(i+1-j)$ modulo k is less than or equal to $\frac{k-1}{2}$, we have that the 2×2 principal submatrix containing $M_{i,i+1}$ and $M_{j,j+1}$ must be $\begin{pmatrix} 1\,1 \\ 0\,1 \end{pmatrix}$ or its transpose. Thus the entries from P form a set of k isolated ones in M. ∎

Lemma 6.21 *Pick odd k, and let $S \subseteq \{1, 2, \ldots, k\}$. Let P be the $k \times k$ permutation matrix which corresponds to the permutation $i \mapsto i+1 \pmod{k}$. Let $A = O_k + \sum_{i \in S} P^i$, and $B = \sum_{i=1}^{\frac{k-1}{2}} P^i$. Let M be the $2k \times 2k$ tournament matrix which has block form $\begin{pmatrix} B & A \\ A^c & B^\top \end{pmatrix}$. Then the entries from P in the upper left block and from $P^\top$ in the lower right block form a set of $2k$ isolated ones in M.*

Proof: By Lemma 6.20, then entries from P in B and $P^\top$ in $B^\top$ form a set of isolated ones in B and $B^\top$ respectively. We now show that these together form a set of isolated ones in M. Choose $M_{i,i+1}$ with $1 \leq i \leq k$ (arithmetic modulo k with congruence classes from $\{1, \ldots, k\}$). If this entry appears in a submatrix of the form $\begin{pmatrix} 1\,1 \\ 1\,1 \end{pmatrix}$ with some $M_{j,j-1}$ with $k+1 \leq j \leq 2k$ (arithmetic modulo k with congruence classes from $\{k+1, \ldots, 2k\}$), then the other entries of this submatrix must be $M_{j,i+1} = A^c_{j-k,i+1}$ and $M_{i,j-1} = A_{i,j-k-1}$. So, $A^c_{j-k,i+1} = 1$ and $A^c_{j-k-1,i} = 0$. By our construction of A, $A^c_{j-k-1,i} = 0$ implies that $A^c_{j-k,i+1} = 0$ (arithmetic modulo k with congruence classes from $\{1, \ldots, k\}$), a contradiction. Thus, the entries from P in B and $P^\top$ in $B^\top$ form a set of $2k$ isolated ones. ■

We now use the previous three lemmas to construct a class of tournament matrices with full Boolean rank. Following this result, we will show, in Theorem 6.23, how to set up a problem for finding singular matrices of the form constructed in Theorem 6.22. We then show how this problem, in a specific instance, is equivalent to a network flow problem. We finish the section by showing that all of our formulations of this network flow problem have solutions, and hence there exist an infinite number of singular tournament matrices with full Boolean rank.

Theorem 6.22 *Pick $m \in \mathbb{Z}$ and odd $k \in \mathbb{Z}$. For $1 \leq i < j \leq m$, and $i + j$ odd, let $S_{i,j} \subseteq \{1, 2, \ldots, k\}$ (we will allow $S_{i,j}$ to be empty). Let P be the $k \times k$ permutation matrix which corresponds to the permutation $i \mapsto i + 1 \pmod{k}$. For $1 \leq i < j \leq m$, define the matrix $X_{i,j}$ by $X_{i,j} = O_k + \sum_{l \in S_{i,j}} P^l$. Define the matrix A by $A = \sum_{j=1}^{\frac{k-1}{2}} P^j$. Define the $mk \times mk$ matrix M to have the $m \times m$*

block form M' given by,

$$M'_{i,j} = \begin{cases} A & \textit{if } i = j \textit{ and } i \textit{ is odd,} \\ A^\top & \textit{if } i = j \textit{ and } i \textit{ is even,} \\ J_k & \textit{if } i + j \textit{ is even and } i < j, \\ O_k & \textit{if } i + j \textit{ is even and } j < i, \\ X_{i,j} & \textit{if } i + j \textit{ is odd and } i < j, \\ X^c_{j,i} & \textit{if } i + j \textit{ is odd and } j < i. \end{cases}$$

Then M has full Boolean rank.

Proof: We show M has a set of mk isolated ones, and hence full Boolean rank. Let S be the set of entries formed by taking the entries of P in the i, i block of M for i odd, and the entries of $P^\top$ in the i, i block of M for i even. Now, if two entries of S occur in the same block, then by Lemma 6.20, they are not in the same row or column, and do not appear in a 2×2 submatrix of all ones. For entries in different blocks we consider two cases. First suppose the two entries come from blocks whose positions on the block diagonal have the same parity. Then the entries are contained in the upper left and lower right blocks of a block submatrix of the form $\begin{pmatrix} A & J \\ O & A \end{pmatrix}$. So, by Lemma 6.19, they do not appear in the same row or column, and do not occur together in a 2×2 submatrix of all ones. Now suppose the entries come from blocks whose positions on the block diagonal have different parity. Then they occur in the upper left and lower right blocks of a block submatrix of the form $\begin{pmatrix} A & X \\ X^c & A^\top \end{pmatrix}$, where $X = O_k + \sum_{j \in S} P^j$ for some $S \subseteq \{1, \ldots, k\}$. So, by Lemma 6.21, the entries do not occur in the same row or column, or in a 2×2 submatrix of all ones. Thus S forms a set of mk isolated ones and M has full Boolean rank. ∎

Let us construct an example of the matrix M in Theorem 6.22 when $m =$

$k = 7$. In this case, P is the permutation matrix,

$$P = \begin{pmatrix} 0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,0\,0\,1 \\ 1\,0\,0\,0\,0\,0\,0 \end{pmatrix}.$$

Choose the sets $S_{i,j}$ as follows. Let $S_{1,2} = \{1, 2, 4\}$, $S_{2,3} = \{1\}$, $S_{1,4} = S_{3,4} = \{1, 2, 3, 5, 7\}$, $S_{1,6} = S_{2,5} = S_{2,7} = \{1, 2, 3, 4\}$, and $S_{3,6} = S_{4,5} = S_{4,7} = S_{5,6} = S_{6,7} = \emptyset$. (Note, we have many choices for each $S_{i,j}$, we just use these for the example.) So our blocks will be

- $A = P + P^2 + P^3$,
- $X_{1,2} = P + P^2 + P^4$,
- $X_{2,3} = P$,
- $X_{1,4} = X_{3,4} = P + P^2 + P^3 + P^5 + I$,
- $X_{1,6} = X_{2,5} = X_{2,7} = P + P^2 + P^3 + P^4$, and
- $X_{3,6} = X_{4,5} = X_{4,7} = X_{5,6} = X_{6,7} = O_k$,

and our matrix will have block form

$$M = \begin{pmatrix} A & X_{1,2} & J & X_{1,4} & J & X_{1,6} & J \\ X_{1,2}^c & A^\top & X_{2,3} & J & X_{2,5} & J & X_{2,7} \\ O & X_{2,3}^c & A & X_{3,4} & J & X_{3,6} & J \\ X_{1,4}^c & O & X_{3,4}^c & A^\top & X_{4,5} & J & X_{4,7} \\ O & X_{2,5}^c & O & X_{4,5}^c & A & X_{5,6} & J \\ X_{1,6}^c & O & X_{3,6}^c & O & X_{5,6}^c & A^\top & X_{6,7} \\ O & X_{2,7}^c & O & X_{4,7}^c & O & X_{6,7}^c & A \end{pmatrix}. \tag{6.2}$$

Note that the matrix at the beginning of this section is also an example of our construction in which $m = k = 3$, and $S_{1,2} = S_{2,3} = \{1, 2\}$. We now show how to use our construction to find singular tournament matrices with full Boolean rank.

Theorem 6.23 *Choose $m \in \mathbb{Z}$ and odd $k \in \mathbb{Z}$. For $1 \le i < j \le m$, $i + j$ odd, let $x_{i,j}$ be an integer between 0 and k. Let M be the $m \times m$ matrix defined as follows.*

$$M_{i,j} = \begin{cases} \frac{k-1}{2} & \text{if } i = j, \\ k & \text{if } i + j \text{ is even and } i < j, \\ 0 & \text{if } i + j \text{ is even and } j < i, \\ x_{i,j} & \text{if } i + j \text{ is odd and } i < j, \\ k - x_{j,i} & \text{if } i + j \text{ is odd and } j < i. \end{cases}$$

Construct an $mk \times mk$ tournament matrix T as in Theorem 6.22 by choosing each $S_{i,j}$ such that $|S_{i,j}| = x_{i,j}$. If M is singular, then T is a singular tournament matrix with full Boolean rank.

Proof: By Theorem 6.22, T has full Boolean rank. We show T is singular if M is singular. Suppose $M\mathbf{v} = \mathbf{0}$, for some $\mathbf{v}$. Let $v_1, v_2, \ldots, v_m$ denote the entries of $\mathbf{v}$. Let $\mathbf{v}_i$ be the k-vector whose every entry is v_i. Let $\overline{\mathbf{v}}$ be the mk-vector defined by $\overline{\mathbf{v}}_i = (\mathbf{v}_1^\top, \mathbf{v}_2^\top, \ldots, \mathbf{v}_m^\top)^\top$. Then,

$$\begin{aligned} T\overline{\mathbf{v}} &= \begin{pmatrix} A & X_{1,2} & J & X_{1,3} \cdots \\ X_{1,2}^c & A^\top & X_{2,3} & J & \cdots \\ O & X_{2,3}^c & A & X_{3,4} \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \overline{\mathbf{v}} \\ &= \begin{pmatrix} A\mathbf{v}_1 + X_{1,2}\mathbf{v}_2 + J\mathbf{v}_3 + X_{1,3}\mathbf{v}_4 + \cdots \\ X_{1,2}^c\mathbf{v}_1 + A^\top\mathbf{v}_2 + X_{2,3}\mathbf{v}_2 + J\mathbf{v}_4 + \cdots \\ O\mathbf{v}_1 + X_{2,3}^c\mathbf{v}_2 + A\mathbf{v}_3 + X_{3,4}\mathbf{v}_4 + \cdots \\ \vdots \end{pmatrix}. \end{aligned}$$

Now, $A\mathbf{v}_1 + X_{1,2}\mathbf{v}_2 + J\mathbf{v}_3 + X_{1,3}\mathbf{v}_4 + \cdots$ is a k-vector whose every entry is $\frac{k-1}{2}v_1 + x_{1,2}v_2 + kv_3 + x_{1,3}v_4 + \cdots$, which is the first entry of $M\mathbf{v}$. Similarly, every entry of $X^c_{1,2}\mathbf{v}_1+A^\top\mathbf{v}_2+X_{2,3}\mathbf{v}_2+J\mathbf{v}_4+\cdots$ is $(k-x_{1,2})v_1+\frac{k-1}{2}v_2+x_{2,3}v_3+kv_4+\cdots$, the second entry of $M\mathbf{v}$. Every entry of $O\mathbf{v}_1 + X^c_{2,3}\mathbf{v}_2 + A\mathbf{v}_3 + X_{3,4}\mathbf{v}_4 + \cdots$ is $0v_1 + (k - x_{2,3})v_2 + \frac{k-1}{2}v_3 + x_{3,4}v_4 + \cdots$, the third entry of $M\mathbf{v}$, and so on. So, since $M\mathbf{v} = \mathbf{0}$, Each of the above is $\mathbf{0}$, and so $T\mathbf{v} = \mathbf{0}$, and T is singular. ■

We now examine a particular example of our result. Consider the 7×7 matrix

$$\begin{pmatrix} 3\,3\,7\,5\,7\,4\,7 \\ 4\,3\,1\,7\,4\,7\,4 \\ 0\,6\,3\,5\,7\,0\,7 \\ 2\,0\,2\,3\,0\,7\,0 \\ 0\,3\,0\,7\,3\,0\,7 \\ 3\,0\,7\,0\,7\,3\,0 \\ 0\,3\,0\,7\,0\,7\,3 \end{pmatrix}.$$

This fits the description of the matrix in the statement of Theorem 6.23, and one can verify that it has a null vector of $(1111-1-1-1)^\top$. So, from this singular matrix we can construct a singular tournament matrix with full Boolean rank of order 49. This matrix has $x_{1,2} = 3$, $x_{2,3} = 1$, $x_{3,4} = 5$, $x_{4,5} = x_5 = x_{6,7} = 0$, $x_{1,4} = 5$, $x_{2,5} = 4$, $x_{3,6} = x_{4,7} = 0$, and $x_{1,6} = x_{2,7} = 4$. So, we choose subsets $S_{1,2}, S_{1,4}, S_{1,6}, S_{2,3}\dots$ from $\{1,\dots,7\}$ such that $|S_{i,j}| = x_{i,j}$. Note, that the sets we chose in the example above meet these conditions. Thus, the matrix M in equation (6.2) is an example of a matrix constructed by Theorem 6.23. That is, it is a 49×49 tournament matrix with full Boolean rank, and a null vector whose first 28 entries are 1 and whose last 21 entries are -1. We now show how to set up the problem of choosing the $x_{i,j}$ for the construction in Theorem 6.23 as a network flows problem, but to do so we need to select a null vector.

Null vectors for tournament matrices have been studied by Maybee and Pullman in [40] and Shader in [57]. These authors give several necessary conditions for a vector to be the null vector of a tournament matrix. One of the best known of these is a condition of Shader [57], which states that a null vector $\mathbf{v}$ must satisfy $\mathbf{v}^\top \mathbf{v} = \mathbf{v}^\top \mathbf{1}$. (A proof of this result can be found in appendix A.) In [57], Shader shows that the k^2-vector whose first $\binom{k+1}{2}$ entries are 1 and whose last $\binom{k-1}{2}$ entries are -1 is the null vector of some tournament matrix.

Choose odd $m = k$, and let $\mathbf{v}$ be the k-vector whose first $\frac{k+1}{2}$ entries are 1 and last $\frac{k-1}{2}$ entries are -1. Treating the $x_{i,j}$ in the matrix M of Theorem 6.23 as variables, we will show how to solve the equation by $M\mathbf{v} = \mathbf{0}$ for each $x_{i,j}$ in $\{x_{i,j} : 1 \leq i < j \leq k, i+j \text{ odd}\}$. Then the matrix T constructed by Theorem 6.23 will be a $k^2 \times k^2$ tournament matrix with full Boolean rank and the k^2-null vector constructed by Shader.

We outline the process of our problem by examining the case $k = 5$. So, with M as constructed in Theorem 6.23, and $\mathbf{v}$ the 5-vector $\mathbf{v} = (1\ 1\ 1\ -1\ -1)^\top$, we are looking to solve $M\mathbf{v} = \mathbf{0}$, equivalently

$$\begin{pmatrix} 2 & x_{1,2} & 5 & x_{1,4} & 5 \\ 5 - x_{1,2} & 2 & x_{2,3} & 5 & x_{2,5} \\ 0 & 5 - x_{2,3} & 2 & x_{3,4} & 5 \\ 5 - x_{1,4} & 0 & 5 - x_{3,4} & 2 & x_{4,5} \\ 0 & 5 - x_{2,5} & 0 & 5 - x_{4,5} & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

for the $x_{i,j}$. This gives us the following system of linear equations,

$$\begin{cases} 2 + x_{1,2} + 5 - x_{1,4} - 5 = 0 \\ 5 - x_{1,2} + 2 + x_{2,3} - 5 - x_{2,5} = 0 \\ 0 + 5 - x_{2,3} + 2 - x_{3,4} - 5 = 0 \\ 5 - x_{1,4} + 0 + 5 - x_{3,4} - 2 - x_{4,5} = 0 \\ 0 + 5 - x_{2,5} + 0 - (5 - x_{4,5}) - 2 = 0 \end{cases}.$$

This reduces to

$$\begin{cases} x_{1,2} - x_{1,4} = -2 \\ -x_{1,2} + x_{2,3} - x_{2,5} = -2 \\ -x_{2,3} - x_{3,4} = -2 \\ -x_{3,4} - x_{4,5} - x_{1,4} = -8 \\ x_{4,5} - x_{2,5} = 2 \end{cases}. \tag{6.3}$$

Let D be the digraph with vertex set $\{1,2,3,4,5\}$ with $i \to j$ if and only if $i < j$ and $i+j$ is odd. The adjacency matrix of D is

$$\begin{pmatrix} 0\,1\,0\,1\,0 \\ 0\,0\,1\,0\,1 \\ 0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,1 \\ 0\,0\,0\,0\,0 \end{pmatrix}.$$

Note that the ones in the adjacency matrix of D are in the same positions as the $x_{i,j}$ positions of M. Let E denote the incidence matrix of D. So,

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{pmatrix}.$$

Let D_1 be the 5×5 diagonal matrix with ones in the first 3 diagonal positions and -1s in the last 2, and D_2 be the 6×6 diagonal matrix with ones in the first, second and fourth positions and -1s in the third, fifth and sixth positions. Consider the matrix

$$D_1ED_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

A quick inspection shows that equation (6.3) is equivalent to the matrix equation

$$D_1ED_2\mathbf{x} = \mathbf{b}$$

where $\mathbf{x} = (x_{1,2}, x_{2,3}, x_{3,4}, x_{4,5}, x_{1,4}, x_{2,5})^\top$ and $\mathbf{b} = (-2,-2,-2,-8,\ 2)^\top$. Noting that D_1 is its own inverse we get the following matrix equation

$$ED_2\mathbf{x} = D_1\mathbf{b} = (-2,-2,-2,\ 8,-2)^\top. \tag{6.4}$$

Multiplying E by D_2 will multiply the third, fourth and fifth columns of E by -1. As an incidence matrix, this corresponds to reversing the orientation of arcs $(3,4)$, $(1,4)$ and $(2,5)$ in D. Let D' be the digraph obtained from D by reversing these arcs. Then ED_2 is the incidence matrix of D'

A solution to equation (6.4) will provide us with $x_{i,j}$ for the construction in Theorem 6.23, provided that each $x_{i,j}$ is an integer between 0 and 5. So, we are looking for a solution to the following problem,

$$\text{Determine } \mathbf{x} = (x_{1,2}, x_{2,3}, \ldots, x_{2,5})^\top$$
$$\text{subject to}$$
$$ED_2\mathbf{x} = \mathbf{b},$$
$$x_{i,j} \in \mathbb{Z} \text{ and } 0 \leq x_{i,j} \leq 5.$$

It is a well known result of network flows that a network flow with integer constraints has an integer solution (see [1]). So, since ED_2 is the incidence matrix of D', this problem is the matrix formulation of the feasible flow problem below. A *feasible flow* on a network is one which satisfies the constraints of the network. The network N in this problem can be seen in Figure 6.2.

Find a feasible flow for the following network: $N = (V(N), A(N))$ with $V(N) = \{1,2,3,4,5\}$ and $(i,j) \in A(N)$ if and only if $i+j$ is odd and one of the following is true, $i < j \leq 3$, $3 \leq i < j$ or $j \leq 3 < i$. Assign to each arc (i,j) of N a capacity of $u_{i,j} = 5$, and the give the vertices supplies $b(1) = b(2) = b(3) = b(5) = -2$ and $b(4) = 8$.

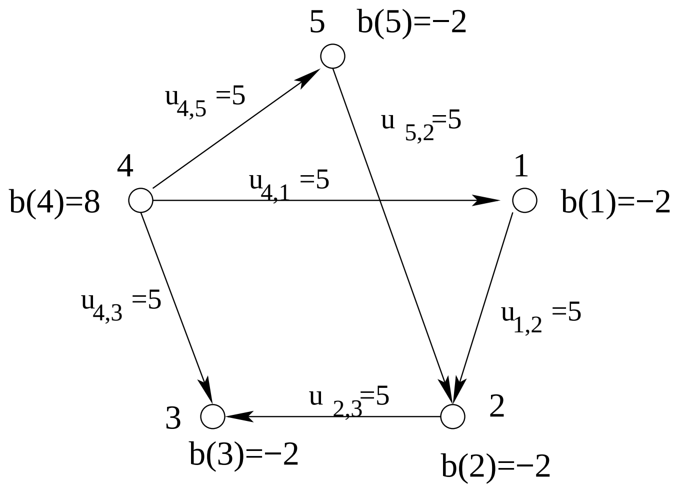

Figure 6.2: The network in our feasible flow problem

In Theorem 6.26 we show how to set up our problem of finding $\{x_{i,j} : 1 \leq i < j \leq m, i + j \text{ odd}\}$ for the construction in Theorem 6.23 as a feasible flow problem for any odd k, and $m = k$. To do so, we deal with calculations for the variables and constants separately in Lemmas 6.24 and 6.25 respectively. Once we have shown the desired equivalence for all odd k, we present two algorithms which solve our feasible flow problem. We use two algorithms, for as it turns out we need to consider separate networks for $k \equiv 1 \pmod 4$ and $k \equiv 3 \pmod 4$. Lemmas 6.28 and 6.29 show that our algorithms do in fact construct feasible flows for our two networks. This leads to our main result that there exists a $k^2 \times k^2$ singular tournament matrix with full Boolean rank for all odd $k \geq 3$.

We note that our method of construction lets us generalize our class quite easily. Our two algorithms will construct an infinite class of the desired tournament matrices. However, our algorithms take advantage of the equivalence of the feasible flow problem to a maximum flow problem on an augmented network, and there are several algorithms which can solve a maximum flow problem. For example, we could use the labeling algorithm, the preflow push algorithm, and the network simplex algorithm. Each of these algorithms can be found in [1]. Since we know the desired solution exists, running any of these algorithms to solve our feasible flow problem should result in different feasible flows. From any of these flows we have several choices for the sets $S_{i,j}$, and hence this method allows us to create several of our desired tournament matrices for a given order.

Lemma 6.24 *Choose a positive odd integer k. Let $\overline{X}$ be the $k \times k$ matrix with*

$$\overline{X}_{i,j} = \begin{cases} 0 & \text{if } i+j \text{ is even,} \\ x_{i,j} & \text{if } i+j \text{ is odd and } i<j, \\ -x_{j,i} & \text{if } i+j \text{ is odd and } j<i, \end{cases}$$

where each $x_{i,j}$ is an indeterminate. Let $\mathbf{v}$ be the k-vector whose first $\frac{k+1}{2}$ entries are 1 and whose last $\frac{k-1}{2}$ entries are -1. Let D be the digraph with $V(D) = \{1,2,\ldots,k\}$ and $(i,j) \in A(D)$ if and only if $i<j$ and $i+j$ is odd. Let $n = (\frac{k+1}{2})k - (\frac{k+1}{2})^2$ and let E be the $k \times n$ incidence matrix of D. Let D_1 be the $k \times k$ diagonal matrix whose first $\frac{k+1}{2}$ diagonal entries are 1 and whose last $\frac{k-1}{2}$ diagonal entries are -1. Let D_2 be the $n \times n$ diagonal matrix whose diagonal entries are ± 1 with the m,m entry of D_2 equal to -1 if and only if the m^{th} arc of D (as listed in E) is of the form (i,j) where $i < \frac{k+3}{2} \leq j$. Let $\mathbf{x}$ be the n-vector whose entries are $x_{i,j}$ listed in the same order as the arcs (i,j) are listed in E. Then

$$\overline{X}\mathbf{v} = D_1 E D_2 \mathbf{x}.$$

Proof: Pick i and define the sets $S_1, S_2, \ldots, S_6$ by

- $S_1 = \{j : 2 \nmid (i+j) \text{ and } 1 \leq j < i \leq \frac{k+1}{2}\}$,
- $S_2 = \{j : 2 \nmid (i+j) \text{ and } 1 \leq i < j \leq \frac{k+1}{2}\}$,
- $S_3 = \{j : 2 \nmid (i+j) \text{ and } 1 \leq i \leq \frac{k+1}{2} < j\}$,
- $S_4 = \{j : 2 \nmid (i+j) \text{ and } 1 \leq j < \frac{k+3}{2} \leq i \leq k\}$,
- $S_5 = \{j : 2 \nmid (i+j) \text{ and } \frac{k+3}{2} \leq j < i \leq k\}$, and
- $S_6 = \{j : 2 \nmid (i+j) \text{ and } \frac{k+3}{2} \leq i < j \leq k\}$.

We show the i^{th} entry of $\overline{X}\mathbf{v}$ is equal to the i^{th} entry of $D_1ED_2\mathbf{x}$ in two cases. First, consider the i^{th} entry of $\overline{X}\mathbf{v}$. For $i \leq \frac{k+1}{2}$, this is

$$-\sum_{j\in S_1} x_{j,i} + \sum_{j\in S_2} x_{i,j} - \sum_{j\in S_3} x_{i,j},$$

and for $i \geq \frac{k+3}{2}$, this is

$$-\sum_{j\in S_4} x_{j,i} + \sum_{j\in S_5} x_{j,i} - \sum_{j\in S_6} x_{i,j}.$$

Now consider the i^{th} entry of $D_1ED_2\mathbf{x}$. First assume $i \leq \frac{k+1}{2}$, so D_1 does not affect the entry. So, this is the i^{th} entry of $ED_2\mathbf{x}$. Since E is an incidence matrix, we are summing over the vertices in $O_D(i)$ and subtracting the vertices in $I_D(i)$. However, by our construction of D_2, we need to flip the sign of our entries for $j \geq \frac{k+3}{2}$. So, the i^{th} entry of $ED_2\mathbf{x}$ is

$$\sum_{\{j\in O_D(i): j\leq \frac{k+1}{2}\}} x_{i,j} - \sum_{\{j\in I_D(i): j\leq \frac{k+1}{2}\}} x_{j,i} - \sum_{\{j\in O_D(i): j> \frac{k+1}{2}\}} x_{i,j} + \sum_{\{j\in I_D(i): j> \frac{k+1}{2}\}} x_{j,i}.$$

By our construction of D, $j \in O_D(i)$ if $i < j$ and $i+j$ is odd, and $j \in I_D(i)$ if $j < i$ and $i + j$ is odd. So, in the above equation, the last summand is

0, and $S_2 = \{j \in O_D(i) : j \leq \frac{k+1}{2}\}$, $S_1 = \{j \in I_D(i) : j \leq \frac{k+1}{2}\}$, and $S_3 = \{j \in O_D(i) : j \geq \frac{k+3}{2}\}$. So, the sum is

$$\sum_{j \in S_2} x_{i,j} - \sum_{j \in S_1} x_{j,i} - \sum_{j \in S_3} x_{i,j},$$

which is equal to the i^{th} entry of $\overline{X}\mathbf{v}$.

Now assume $i \geq \frac{k+3}{2}$. Then the i^{th} entry of $D_1 E D_2 \mathbf{x}$ is the negative of the i^{th} entry of $ED_2\mathbf{x}$. First, note that i^{th} entry of ED_2 in this case is

$$\sum_{\{j \in O_D(i) : j < \frac{k+3}{2}\}} x_{i,j} + \sum_{\{j \in I_D(i) : j < \frac{k+3}{2}\}} x_{j,i} + \sum_{\{j \in O_D(i) : j \geq \frac{k+3}{2}\}} x_{i,j} - \sum_{\{j \in I_D(i) : j \geq \frac{k+3}{2}\}} x_{j,i}.$$

So, the i^{th} entry of $D_1 E D_2 \mathbf{x}$ is

$$- \sum_{\{j \in O_D(i) : j < \frac{k+3}{2}\}} x_{i,j} - \sum_{\{j \in I_D(i) : j < \frac{k+3}{2}\}} x_{j,i} - \sum_{\{j \in O_D(i) : j \geq \frac{k+3}{2}\}} x_{i,j} + \sum_{\{j \in I_D(i) : j \geq \frac{k+3}{2}\}} x_{j,i}.$$

By our construction of D and since $i \geq \frac{k+3}{2}$, the first sum in this equation is 0, and $S_4 = \{j \in I_D(i) : j \leq \frac{k+1}{2}\}$, $S_6 = \{j \in O_D(i) : j \geq \frac{k+3}{2}\}$, and $S_5 = \{j \in I_D(i) : j \geq \frac{k+3}{2}\}$. Thus, the total sum is

$$- \sum_{j \in S_4} x_{j,i} - \sum_{j \in S_6} x_{i,j} + \sum_{j \in S_5} x_{j,i}.$$

Thus, $\overline{X}\mathbf{v} = D_1 E D_2 \mathbf{x}$, as desired. ■

Lemma 6.25 *Choose an odd integer $k \geq 3$. Let S be the set of even integers between 1 and k. Let T be the rotational k-tournament with symbol S, and let A be the adjacency matrix of T. Let $M' = kA + \frac{k-1}{2}I$. Let $\mathbf{v}$ be the k-vector whose first $\frac{k+1}{2}$ entries are 1 and last $\frac{k-1}{2}$ entries are -1. Then, for $k \equiv 1 \pmod 4$,*

$$M'\mathbf{v} = \left(\frac{k-1}{2}, \frac{k-1}{2}, \ldots, \frac{k-1}{2} \,\middle|\, \frac{3k+1}{2}, -\frac{k-1}{2}, \ldots, \frac{3k+1}{2}, -\frac{k-1}{2} \right)^\top,$$

and for $k \equiv 3 \pmod 4$

$$M'\mathbf{v} = \left(-\frac{k+1}{2}, \frac{3k-1}{2}, \ldots, -\frac{k+1}{2}, \frac{3k-1}{2} \,\middle|\, \frac{k+1}{2}, \frac{k+1}{2}, \ldots, \frac{k+1}{2} \right)^{\top}.$$

In both cases, the bar separates the first $\frac{k+1}{2}$ *entries from the last* $\frac{k-1}{2}$.

Proof: The i^{th} entry of $M'\mathbf{v}$ is

$$\sum_{j=1}^{\frac{k+1}{2}} M'_{i,j} - \sum_{j=\frac{k+3}{2}}^{k} M'_{i,j}. \tag{6.5}$$

To calculate these values we first count the number of ones occurring in various blocks of A. We do so since every value of M' off of the diagonal is a k corresponding to a 1 in A. We make our counts by first considering the subtournaments W and W' of T whose adjacency matrices are the principle submatrices of A induced on the first $\frac{k+1}{2}$ rows and columns, and induced on the last $\frac{k-1}{2}$ rows and columns respectively. We then count the number of ones in the remaining two blocks by counting the number of odd and even positive integers less than $\frac{k+3}{2}$ and between $\frac{k+3}{2}$ and k. For all these counts we need to consider the cases $k \equiv 1 \pmod 4$ and $k \equiv 3 \pmod 4$ separately.

First consider the subtournament W of T induced on the first $\frac{k+1}{2}$ vertices. If $k \equiv 1 \pmod 4$, then $\frac{k+1}{2}$ is odd, and so $j - i \pmod{\frac{k+1}{2}}$ is even if and only if $j - i \pmod k$ is even. So, in this case, W is regular, and $d^+_W(i) = \frac{k-1}{4}$ for each i. When $k \equiv 3 \pmod 4$, $\frac{k+3}{2}$ is odd, and so the subtournament induced on the first $\frac{k+3}{2}$ vertices is regular. So, the subtournament induced on the first $\frac{k+1}{2}$ vertices, W, is near regular, and by examining the inset of the $\frac{k+3}{2}^{rd}$ vertex we see that $d^+_W(i) = \frac{k+1}{4}$ if i is even and $d^+_W(i) = \frac{k+1}{4} - 1$ if i is odd.

By applying the vertex transitivity of T, in particular using the automorphism $\phi : x \mapsto x - \frac{k+1}{2}$, we may use a similar argument to show that the subtournament W' induced on the last $\frac{k-1}{2}$ vertices, $\frac{k+3}{2}, \frac{k+5}{2}, \ldots, k$, is regular when $k \equiv 3 \pmod 4$, and $d^+_{W'}(i) = \frac{k-3}{4}$ for each i. Also, vertex transitivity and a similar argument will show that W' is near regular when $k \equiv 1 \pmod 4$, and will have $d^+_{W'}(i) = \frac{k-1}{4}$ for i odd, and $d^+_{W'}(i) = \frac{k-1}{4} - 1$ for i even.

Now suppose $k \equiv 1 \pmod 4$, and let $u = \frac{k-1}{4}$. The number of even integers j with $\frac{k+3}{2} \leq j \leq k$ is $|\{2u+2, 2u+4, \ldots, 4u\}| = \frac{4u-(2u+2)}{2} + 1 = u - 1 + 1 = u = \frac{k-1}{4}$. Similarly, the number of odd integers j with $\frac{k+3}{2} \leq j \leq k$ is $|\{2u+3, 2u+5, \ldots, 4u+1\}| = \frac{4u+1-(2u+3)}{2} + 1 = u - 1 + 1 = u = \frac{k-1}{4}$. The number of positive even $j \leq \frac{k+1}{2}$ is $|\{2, 4, \ldots, 2u\}| = u = \frac{k-1}{4}$, and the number of positive odd $j \leq \frac{k+1}{2}$ is $|\{1, 3, \ldots, 2u+1\}| = u + 1 = \frac{k+3}{4}$.

Now, suppose $k \equiv 3 \pmod 4$, and let $u = \frac{k-3}{4}$. The number of even integers j with $\frac{k+3}{2} \leq j \leq k$ is $|\{2u+4, 2u+6, \ldots, 4u+2\}| = \frac{4u+2-(2u+4)}{2} + 1 = u = \frac{k-3}{4}$. The number of odd integers j with $\frac{k+3}{2} \leq j \leq k$ is $|\{2u+3, 2u+5, \ldots, 4u+3\}| = \frac{4u+3-(2u+3)}{2} + 1 = u + 1 = \frac{k-3}{4} + 1 = \frac{k+1}{4}$. The number of positive even $j \leq \frac{k+1}{2}$ is $|\{2, 4, \ldots, 2u+2\}| = u + 1 = \frac{k+1}{4}$, and the number of positive odd $j \leq \frac{k+1}{2}$ is $|\{1, 3, \ldots, 2u+1\}| = u + 1 = \frac{k+1}{4}$.

We now calculate the values in $M'\mathbf{v}$. First, assume $k \equiv 1 \pmod 4$. If $i \leq \frac{k+1}{2}$, then the i^{th} entry of $M'\mathbf{v}$, the value in equation (6.5), is

$$\left(\frac{k-1}{2} + \sum_{j \in O_W(i)} k\right) - \sum_{\{\frac{k+3}{2} \leq j \leq k : 2|(j-i)\}} k = \frac{k-1}{2} + k\left(\frac{k-1}{4}\right) - k\left(\frac{k-1}{4}\right)$$
$$= \frac{k-1}{2}.$$

If $i \geq \frac{k+3}{2}$ and i is even, then the i^{th} entry of $M'\mathbf{v}$ is

$$\sum_{\{j \leq \frac{k+1}{2} : 2 \nmid j\}} k - \left(\frac{k-1}{2} + \sum_{j \in O_{W'}(i)} k \right) = k\left(\frac{k+3}{4}\right) - \left(\frac{k-1}{2} + k\left(\frac{k-1}{4} - 1\right)\right)$$
$$= \frac{k^2 + 3k - k^2 + k}{4} - \frac{k-1}{2} + k$$
$$= 2k - \frac{k-1}{2}$$
$$= \frac{3k+1}{2}.$$

If $i \geq \frac{k+3}{2}$ and i is odd, then the i^{th} entry of $M'\mathbf{v}$ is

$$\sum_{\{j \leq \frac{k+1}{2} : 2 | j\}} k - \left(\frac{k-1}{2} + \sum_{j \in O_{W'}(i)} k \right) = k\left(\frac{k-1}{4}\right) - \left(\frac{k-1}{2} + k\left(\frac{k-1}{4}\right)\right)$$
$$= -\frac{k-1}{2}.$$

So the result holds for $k \equiv 1 \pmod 4$.

Now assume that $k \equiv 3 \pmod 4$. First assume $i \geq \frac{k+3}{4}$. Then the i^{th} entry of $M'\mathbf{v}$, the value in equation (6.5), is

$$\sum_{\{j \leq \frac{k+1}{2} : 2 \nmid (j-i)\}} k - \left(\frac{k-1}{2} + \sum_{j \in O_{W'}(i)} k \right) = k\left(\frac{k+1}{4}\right) - \left(\frac{k-1}{2} - k\left(\frac{k-3}{4}\right)\right)$$
$$= \frac{k^2 + k - k^2 + 3k}{4} - \frac{k-1}{2}$$
$$= k - \frac{k-1}{2}$$
$$= \frac{k+1}{2}.$$

If $i \leq \frac{k+1}{2}$ and i is even, then the i^{th} entry of $M'\mathbf{v}$ is

$$\begin{aligned}\left(\frac{k-1}{2} + \sum_{j \in O_W(i)} k\right) - \sum_{\{\frac{k+3}{2} \leq j \leq k : 2 | j\}} k &= \frac{k-1}{2} + k\left(\frac{k+1}{4}\right) - k\left(\frac{k-3}{4}\right) \\ &= \frac{k-1}{2} + k \\ &= \frac{3k-1}{2}.\end{aligned}$$

If $i \leq \frac{k+1}{2}$ and i is odd, then the i^{th} entry of $M'\mathbf{v}$ is

$$\begin{aligned}\left(\frac{k-1}{2} + \sum_{j \in O_W(i)} k\right) - \sum_{\{\frac{k+3}{2} \leq j \leq k : 2 \nmid j\}} k &= \frac{k-1}{2} + k\left(\frac{k+1}{4} - 1\right) - k\left(\frac{k+1}{4}\right) \\ &= \frac{k-1}{2} - k \\ &= -\frac{k+1}{2}.\end{aligned}$$

So, the result holds for $k \equiv 3 \pmod 4$. ■

With the calculations made in Lemmas 6.24 and 6.25, we are now ready to show how our problem of finding appropriate $x_{i,j}$ for the construction in Theorem 6.23 is equivalent to a feasible flow problem on a particular network. For $k \equiv 1 \pmod 4$ and $k \equiv 3 \pmod 4$ our problem requires two different networks defined on the same digraph. We define these networks, N_1 and N_2, below.

Choose $k \geq 3$. Let D be the digraph with $V(D) = \{1, 2, \ldots, k\}$ and $A(D) = \{(i,j) : 2 \nmid (i+j), i < j \leq \frac{k+1}{2}\} \cup \{(i,j) : 2 \nmid (i+j), \frac{k+3}{2} \leq i < j\} \cup \{(i,j) : 2 \nmid (i+j), j \leq \frac{k+1}{2} < i\}$. Form the network N_1 (for $k \equiv 1 \pmod 4$) from D by giving each arc of N_1 a capacity of $u_{i,j} = k$, and assigning the following supplies, $b(i)$, to the vertices. For $i = 1, \ldots, \frac{k+1}{2}$, set $b(i) = -\frac{k-1}{2}$. For $i = \frac{k+3}{2}, \frac{k+7}{2}, \ldots, k$, set

$b(i) = \frac{3k+1}{2}$. For $i = \frac{k+5}{2}, \frac{k+9}{2}, \ldots, k-1$ set $b(i) = -\frac{k-1}{2}$. Form the network N_2 (for $k \equiv 3 \pmod 4$) from D by giving each arc of N_2 a capacity of $u_{i,j} = k$, and assigning the following supplies to the vertices. For $i = \frac{k+3}{2}, \frac{k+5}{2}, \ldots, k-1, k$, set $b(i) = \frac{k+1}{2}$. For $i = 1, 3, \ldots, \frac{k-1}{2}$ set $b(i) = \frac{k+1}{2}$, and for $i = 2, 4, \ldots, \frac{k+1}{2}$ set $b(i) = -\frac{3k-1}{2}$.

Theorem 6.26 *Choose odd $k \geq 3$. Let $\mathbf{v}$ be the k-vector whose first $\frac{k+1}{2}$ entries are 1 and whose last $\frac{k-1}{2}$ entries are -1. There exist integers $x_{i,j}$ with $0 \leq x_{i,j} \leq k$, so that the $k \times k$ matrix M defined by*

$$M_{i,j} = \begin{cases} \frac{k-1}{2} & \text{if } i = j, \\ k & \text{if } i+j \text{ is even and } i < j, \\ 0 & \text{if } i+j \text{ is even and } j < i, \\ x_{i,j} & \text{if } i+j \text{ is odd and } i < j, \\ k - x_{j,i} & \text{if } i+j \text{ is odd and } j < i, \end{cases}$$

has $\mathbf{v}$ as a null vector if and only if there is a feasible flow on N_1 for $k \equiv 1 \pmod 4$, and a feasible flow on N_2 for $k \equiv 3 \pmod 4$.

Proof: Let $\overline{X}$ be as defined in Lemma 6.24, and let M' be the matrix defined in Lemma 6.25. Then $M = M' + \overline{X}$. Let D' be the digraph with $V(D') = \{1, 2, \ldots, k\}$ and $(i,j) \in A(D')$ if and only if $i < j$ and $i + j$ is odd. Let $n = (\frac{k+1}{2})k - (\frac{k+1}{2})^2$ and let E' be the $k \times n$ incidence matrix of D'. Let D_1 be the $k \times k$ diagonal matrix whose first $\frac{k+1}{2}$ diagonal entries are 1 and whose last $\frac{k-1}{2}$ diagonal entries are -1. Let D_2 be the $n \times n$ diagonal matrix whose diagonal entries are ± 1 with the m, m entry of D_2 equal to -1 if and only if the m^{th} arc of D' (as listed in E') is of the form (i,j) where $i < \frac{k+3}{2} \leq j$. Let $\mathbf{x}$ be the n-vector whose entries are $x_{i,j}$ listed in the same order as the arcs (i,j) are listed in E. So, by Lemmas 6.24 and 6.25, $M\mathbf{v} = D_1 E' D_2 \mathbf{x} + M'\mathbf{v}$.

Now, $M\mathbf{v} = \mathbf{0}$ if and only if $D_1E'D_2\mathbf{x} = -M'\mathbf{v}$. We show that this second equation, together with some added constraints, is equivalent to a feasible flow problem on N_1 or N_2. First, note D_1 is its own inverse, so we can rewrite the above equation as $E'D_2\mathbf{x} = -D_1M'\mathbf{v}$. Also, right multiplication of E' by D_2 reverses the orientation of the arcs (i,j) of D' with $i < \frac{k+1}{2} \leq j$. This means that $E'D_2$ is the incidence matrix of the digraph D in our networks N_1 and N_2. Let E be the incidence matrix of D. So, $M\mathbf{v} = \mathbf{0}$ if and only if $E\mathbf{x} = -D_1M'\mathbf{v}$. Now, assume that $k \equiv 1 \pmod 4$. So, $M'\mathbf{v} = (\frac{k-1}{2}, \frac{k-1}{2}, \ldots, \frac{k-1}{2} | \frac{3k+1}{2}, -\frac{k-1}{2}, \ldots, \frac{3k+1}{2}, -\frac{k-1}{2})^\top$. So, $-D_1M'\mathbf{v} = (-\frac{k-1}{2}, -\frac{k-1}{2}, \ldots, -\frac{k-1}{2} | \frac{3k+1}{2}, -\frac{k-1}{2}, \ldots, \frac{3k+1}{2}, -\frac{k-1}{2})^\top$. So, when $k \equiv 1 \pmod 4$, $M\mathbf{v} = \mathbf{0}$ if and only if $E\mathbf{x}$ is equal to

$$\left(-\frac{k-1}{2}, -\frac{k-1}{2}, \ldots, -\frac{k-1}{2} \,\middle|\, \frac{3k+1}{2}, -\frac{k-1}{2}, \ldots, \frac{3k+1}{2}, -\frac{k-1}{2}\right)^\top.$$

Similarly, when $k \equiv 3 \pmod 4$,

$$-D_1M'\mathbf{v} = \left(\frac{k+1}{2}, -\frac{3k-1}{2}, \ldots, \frac{k+1}{2}, -\frac{3k-1}{2} \,\middle|\, \frac{k+1}{2}, \frac{k+1}{2}, \ldots, \frac{k+1}{2}\right)^\top,$$

and so $M\mathbf{v} = \mathbf{0}$ if and only if

$$E\mathbf{x} = \left(\frac{k+1}{2}, -\frac{3k-1}{2}, \ldots, \frac{k+1}{2}, -\frac{3k-1}{2} \,\middle|\, \frac{k+1}{2}, \frac{k+1}{2}, \ldots, \frac{k+1}{2}\right)^\top.$$

With the added constraint that $x_{i,j} \in \mathbb{Z}$ and $0 \leq x_{i,j} \leq k$ for each i, j, these are the matrix formulations for feasible flow problems on the networks N_1 and N_2 respectively. So, if such an M exists, we can solve these feasible flow problems. Further, since a network flow problem with integer constraints has an integer solution if and only if it has a solution, there exists such an M if the corresponding feasible flow problem has a solution. ■

A feasible flow problem on a network can be solved by solving a maximum flow problem on a particular augmented network. Let N be the network on which we are trying to solve the feasible flow problem and create the augmented network N' as follows. Set $V(N') = V(N) \cup \{s, t\}$. For each $i \in V(N)$ with $b(i) > 0$, add the arc (s, i) with capacity $u_{s,i} = b(i)$. For each $i \in V(N)$ with $b(i) < 0$ add the arc (i, t) with capacity $u_{i,t} = -b(i)$. Given a maximum flow on N' which saturates every new arc, the flow restricted to N solves the feasible flow problem. We give two algorithms which solve the feasible flow problem on N_1 and N_2 by solving the maximum flow problem on the corresponding augmented networks. First we need the following definitions.

Let N be a network, and $\mathbf{x}$ a flow on N. We define the *residual network* R of N to be the network obtained from N by deleting an arc (i, j) if $x_{i,j} = u_{i,j}$, and in which every arc has an updated capacity of $r_{i,j} = u_{i,j} - x_{i,j}$. The capacities in a residual network are called *residual capacities.* In the following algorithms we begin by creating the residual network for a flow of $\mathbf{0}$, pushing a certain amount of flow along a given path, and then updating the residual network.

Algorithm 1

- Create the augmented network N_1' for N_1 and the residual network R of N_1' with flow $\mathbf{0}$.
- Order the vertices of N_1 by $s, \frac{k+3}{2}, \frac{k+5}{2}, \ldots, k, 1, 2, 3, \ldots, \frac{k+1}{2}, t$.
- Choose an s,t-path, P, of length 3 from s to t with vertices chosen preferentially from our ordering.

- Augment $\min\{r_{i,j} : (i,j) \in P\}$ units of flow along P.
- Update the residual network R.
- Repeat until R contains no s,t-paths of length 3.
- Choose an s,t-path, P, of length 4 of the form $s, x, y, (y+1), t$, where $x \geq \frac{k+3}{2} > y$ and x and y are chosen preferentially from our ordering.
- Augment $\min\{r_{i,j} : (i,j) \in P\}$ units of flow along P.
- Update the residual network.
- Continue until no s,t-paths of this form exist in R.

To prove Lemmas 6.28 and 6.29 we need to introduce the concept of an s,t-cut in a network. Let N be a network and s,t distinct vertices in N. An s,t*-cut* in N is a set of arcs S so that the there is no path from s to t in the network obtained from N by removing the arcs in S. The *capacity* of an s,t-cut, S is defined to be $\sum_{(i,j)\in S} u_{i,j}$. A *minimum* s,t*-cut* is one with minimum capacity. One of the best known results of network flows is the "Max-flow Min-cut" theorem, which we state below. Proofs of this result can be found in [1] and [26].

Theorem 6.27 [26] *Let N be a network, and $s,t \in V(N)$. The size of a maximum flow from s to t is equal to the size of a minimum s,t-cut.*

Lemma 6.28 *Algorithm 1 produces a maximum flow from s to t in the augmented network N_1' of N_1 of value $\frac{3k^2-2k-1}{8}$.*

Proof: We claim that algorithm 1 terminates without saturating any arcs of the form (i, j) where $i, j \notin \{s, t\}$. So, if the algorithm terminates with this property, then it did so by saturating every arc of the form (s, i) or every arc of the form (i, t). Since $\sum_{i \in O(s)} u_{s,i} = \frac{3k^2-2k-1}{8}$, and $\{(s, i) : i \in O(s)\}$ is an s, t-cut for N_1', this will imply we have a maximum flow of value $\frac{3k^2-2k-1}{8}$.

Choose an arc of the form (x, y) with $x \geq \frac{k+3}{2}$. This arc is used at most once in a path of length 3 chosen by algorithm 1, otherwise we would have used the path s, x, y, t twice. This contradicts the fact that algorithm 1 will augment $\min\{r_{s,x}, r_{x,y}, r_{y,t}\}$ along this path and saturate one of its arcs. This arc will be used at most once in a path of length 4 chosen by the algorithm. To see this, consider a similar argument as before. Once we have augmented $\min\{r_{s,x}, r_{x,y}, r_{y,y+1}, r_{y+1,t}\}$ units of flow along the path $s, x, y, (y+1), t$, we must saturate one of the arcs, and remove it from R. This is the only path of length 4 containing (x, y) that algorithm 1 will choose, and this path will not exist in the residual network once we augment along the path. Since $\min\{r_{s,x}, r_{x,y}, r_{y,t}\} \leq \frac{k-1}{2}$ and $\min\{r_{s,x}, r_{x,y}, r_{y,y+1}, r_{y+1,t}\} \leq \frac{k-1}{2}$, we augment a maximum of $2\frac{k-1}{2} < k = u_{x,y}$ units of flow on (x, y).

Now consider an arc of the form $(y, y+1)$ for $y \leq \frac{k+1}{2}$. Any flow pushed along this arc by algorithm 1 will be pushed along a path of the form $s, x, y, (y+1), t$. Since we can push at most $\frac{k-1}{2}$ units of flow on $(y+1, t)$, we will push at most $\frac{k-1}{2}$ units on $(y, y+1)$. No other arcs will be chosen by algorithm 1. This proves our claim and hence the result. ■

Algorithm 2

- Create the augmented network N_2' for N_2, and the residual network R of N_2' with flow $\mathbf{0}$.
- Choose a path $s, i, (i+1), t$ where $i \leq \frac{k+1}{2}$ and i is odd, and augment $\frac{k+1}{2}$ units of flow along this path.
- Update the residual network R.
- Repeat until R contains no paths of this form.
- Choose a path $s, (\frac{k+1}{2}+i), 1, (i+2), t$, for $i = 2, 4, \ldots, \frac{k-3}{2}$, and augment $\frac{k+1}{2}$ units of flow along this path.
- Update the residual network.
- Continue until no paths of this form exist in R.
- Choose a path s, i, j, t in R where $i \geq \frac{k+3}{2}$ and i odd, and augment $\frac{k+1}{2}$ units of flow along this path.
- Update the residual network.
- Continue until no paths of this form exist in R.

Lemma 6.29 *Algorithm 2 produces a maximum flow from s to t in the augmented network N_2' of N_2 of value $\frac{3k^2+2k-1}{8}$.*

Proof: We claim that algorithm 2 terminates without saturating any arcs of the form (i, j) where $i, j \notin \{s, t\}$. Since each arc of the form (s, i), $i \in O(s)$, appears in some path used by algorithm 2, and $u_{s,i} = \frac{k+1}{2}$ for each $i \in O(s)$, our claim implies we will have pushed $\sum_{i \in O(s)} u_{s,i} = (\frac{k+1}{2})(\frac{k-1}{2} + \frac{k+1}{4}) = \frac{3k^2+2k-1}{8}$

units of flow from s to t. Since $\{(s,i) : i \in O(s)\}$ is an s,t-cut in N_2', this flow will be maximum.

Since $\frac{k+1}{2} < k$, we prove our claim by showing each arc (i,j) with $i,j \notin \{s,t\}$ is used at most once by algorithm 2. Pick an arc (i,j) with $i,j, \notin \{s,t\}$. First suppose $i \leq \frac{k+1}{2}$. If $i \geq 2$, and (i,j) is selected by the algorithm, then it will appear in a path of the form $s, i, (i+1), t$. So $j = i+1$, and it is easy to see that for distinct i we have distinct j. If $i = 1$, then the algorithm will select the arcs (i,j) for each even $j \leq \frac{k+1}{2}$. Each of these are also, obviously, distinct. So, an arc (i,j) with $i \leq \frac{k+1}{2}$ is used at most once. Now assume that $i \geq \frac{k+3}{2}$. If i is even, the the algorithm selects the arc $(i,1)$ in a path of the form $s, i, 1, (i - \frac{k+1}{2} + 2), t$. Such arcs are all distinct, since each i is distinct. If i is odd, then the algorithm selects exactly one arc (i,j) for each i, and these arcs will also all be distinct, as each begins with a distinct i. Thus algorithm 2 uses each arc (i,j) with $i,j \notin \{s,t\}$ at most once. Thus algorithm 2 never saturates an arc (i,j) with $i,j \notin \{s,t\}$. This proves our claim and hence the result. ■

One can verify that the sum of the supplies in N_1 is equal to the sum of the demands, which is equal to $\frac{3k^2-2k-1}{8}$, and that the sum of the supplies in N_2 is equal to the sum of the demands, which is equal to $\frac{3k^2+2k-1}{8}$. So, Lemma 6.28 shows that for any $k \equiv 1 \pmod 4$, N_1 has a feasible flow, and Lemma 6.29 shows that for $k \equiv 3 \pmod 4$, N_2 has a feasible flow. So, Theorems 6.23 and 6.26 together with Lemmas 6.28 and 6.29 give us our main result.

Theorem 6.30 *If $k \geq 3$ is odd, then there exists a $k^2 \times k^2$ singular tournament matrix with full Boolean rank.*

Appendix A. Extra Proofs

A.1 Normal tournament matrices

In this section we provide a proof that a tournament matrix is normal if and only if it is regular. We denote by M^* the complex conjugate transpose of M.

Theorem A.1 *An $n \times n$ tournament matrix is normal if and only if it is a regular tournament matrix.*

Proof: Let M be and $n \times n$ regular tournament matrix, and let T be the n-tournament whose adjacency matrix is M. Then $(MM^\top)_{i,j} = |O(i) \cap O(j)| = |O(i)| + |O(j)| - |O(i) \cup O(j)| = \frac{n-1}{2} + \frac{n-1}{2} - |O(i) \cup O(j)| = (n-1) - |V(T) - (I[i] \cap I[j])| = (n-1) - (n - |I[i] \cap I[j]|) = -1 + (|I(i) \cap I(j)| + 1) = |I(i) \cap I(j)| = (M^\top M)_{i,j}$. So M is normal.

Now, assume M is an $n \times n$ normal tournament matrix. By the complex spectral theorem, this is equivalent to M having an orthonormal basis $\mathcal{B}$ of eigenvectors in $\mathbb{C}^n$. Let $\mathbf{x}$ and $\mathbf{y}$ be distinct vectors in $\mathcal{B}$, associated with (not necessarily distinct) eigenvalues λ and μ. Then $0 = 0 + 0 = \mu \mathbf{x}^* \mathbf{y} + \overline{\lambda} \mathbf{x}^* \mathbf{y} = \mathbf{x}^* \mu \mathbf{y} + (\lambda \mathbf{x})^* \mathbf{y} = \mathbf{x}^*(M\mathbf{y}) + (M\mathbf{x})^* \mathbf{y} = \mathbf{x}^* M \mathbf{y} + \mathbf{x}^* M^\top \mathbf{y} = \mathbf{x}^*(M + M^\top)\mathbf{y} = \mathbf{x}^*(J - I)\mathbf{y} = \mathbf{x}^* J \mathbf{y} - \mathbf{x}^* I \mathbf{y} = \mathbf{x}^* \mathbf{1} \mathbf{1}^* \mathbf{y} - \mathbf{x}^* \mathbf{y} = (\mathbf{x}^* \mathbf{1})(\mathbf{1}^* \mathbf{y})$. So, $(\mathbf{x}^* \mathbf{1}) = 0$, or $(\mathbf{1}^* \mathbf{y}) = 0$. Since it cannot be that all of the vectors in $\mathcal{B}$ are orthogonal to $\mathbf{1}$, we must have some $\mathbf{v} \in \mathcal{B}$ such that $\mathbf{v}^* \mathbf{1} \neq 0$. By applying the above equation to $\mathbf{v}$ and $\mathbf{x}$ for all $\mathbf{x} \neq \mathbf{v}$ in $\mathcal{B}$ we see that $\mathbf{x}^* \mathbf{1} = 0$ for all $\mathbf{x} \neq \mathbf{v}$ in $\mathcal{B}$. This implies that the span of $\mathcal{B} - \{\mathbf{v}\}$ is the orthogonal complement of $\mathbf{1}$. So, since $\mathbf{v}$ is orthogonal to every element of $\mathcal{B} - \{\mathbf{v}\}$, $\mathbf{v}$ must be a scalar multiple of $\mathbf{1}$.

So, $\mathbf{1}$ is an eigenvector of M, which implies that the row sums of M must all be equal, and hence M is a regular tournament matrix. ■

A.2 Two new proofs of the real rank of a tournament matrix

In this section we provide two new proofs that the real rank of an $n \times n$ tournament matrix is n or $n-1$. The first utilizes the Lemma A.2 of Shader [57]. The second uses a result on the rank of the complement of a matrix due to Brualdi, Manber and Ross [13]. This together with a result on the eigenvalues of a tournament matrix taken from Brauer and Gentry [9], give us our second proof. We note that the second proof also gives us the result of Shader [57], that a tournament matrix is singular if and only if $\mathbf{1}$ is not in its column space. For completeness, we will provide proofs of cited results as well.

Lemma A.2 [57] *Let M be an $n \times n$ singular tournament matrix and $\mathbf{x}$ a vector in null(M). Then $(\mathbf{x}^\top \mathbf{1})^2 = \mathbf{x}^\top \mathbf{x}$.*

Proof: Since

$$
\begin{aligned}
0 &= \mathbf{0x} + \mathbf{x}^\top \mathbf{0} \\
&= (M\mathbf{x})^\top \mathbf{x} + \mathbf{x}^\top M\mathbf{x} \\
&= \mathbf{x}^\top M^\top \mathbf{x} + \mathbf{x}^\top M\mathbf{x} \\
&= \mathbf{x}^\top (M^\top + M)\mathbf{x} \\
&= \mathbf{x}^\top (J - I)\mathbf{x} \\
&= \mathbf{x}^\top J\mathbf{x} - \mathbf{x}^\top I\mathbf{x} \\
&= \mathbf{x}^\top \mathbf{1}\mathbf{1}^\top \mathbf{x} - \mathbf{x}^\top \mathbf{x} \\
&= (\mathbf{x}^\top \mathbf{1})^2 - \mathbf{x}^\top \mathbf{x},
\end{aligned}
$$

$(\mathbf{x}^\top \mathbf{1})^2 = \mathbf{x}^\top \mathbf{x}$. ■

We now give our first new proof.

Theorem A.3 *Let M be an $n \times n$ tournament matrix. Then the rank of M is at least $n-1$.*

Proof: Suppose, to the contrary, that the rank of M is less than $n-1$. So, the dimension of null(M) is at least 2. Choose orthonormal vectors $\mathbf{x}$ and $\mathbf{y}$ in null(M). Then,

$$\begin{aligned}
0 &= \mathbf{0}\mathbf{y} + \mathbf{x}^\top \mathbf{0} \\
&= \mathbf{x}^\top M^\top \mathbf{y} + \mathbf{x}^\top M \mathbf{y} \\
&= \mathbf{x}^\top (M^\top + M)\mathbf{y} \\
&= \mathbf{x}^\top (J - I)\mathbf{y} \\
&= \mathbf{x}^\top J \mathbf{y} - \mathbf{x}^\top I \mathbf{y} \\
&= \mathbf{x}^\top \mathbf{1}\mathbf{1}^\top \mathbf{y} - x^\top y \\
&= (\mathbf{x}^\top \mathbf{1})(\mathbf{y}^\top \mathbf{1}) - 0 \\
&= (\pm\sqrt{\mathbf{x}^\top \mathbf{x}})(\pm\sqrt{\mathbf{y}^\top \mathbf{y}}) \\
&= (\pm\sqrt{1})(\pm\sqrt{1}) \\
&= \pm 1,
\end{aligned}$$

where the eighth equality comes from Lemma A.2. However, $0 = \pm 1$ is an obvious contradiction, and so the rank of M is no less than $n-1$. ■

Lemma A.4 [13] Let A be an $m \times n$ $(0,1)$-matrix. Then $r_{\mathbb{R}}(A)$ and $r_{\mathbb{R}}(A^c)$ differ by at most 1, and $r_{\mathbb{R}}(A) = r_{\mathbb{R}}(A^c)$ if and only if the following condition holds:

1 is a linear combination of the columns of A if and only if **1** is a linear combination of the columns of A^c.

Proof: Note $A^c = J_{m,n} - A$, and $A = J_{m,n} - A^c$. Suppose A has real rank r. So, A can be written as $A = X_1 + X_2 + \cdots X_r$ where each X_i is a rank one matrix. So, $A^c = J_{m,n} - X_1 - X_2 - \cdots - X_r$. Thus, $r_{\mathbb{R}}(A^c) \leq r_{\mathbb{R}}(A) + 1$. Similarly, $r_{\mathbb{R}}(A) \leq r_{\mathbb{R}}(A^c) + 1$. So, $r_{\mathbb{R}}(A^c) - 1 \leq r_{\mathbb{R}}(A) \leq r_{\mathbb{R}}(A^c) + 1$ as desired. For equality recall that elementary row and column operations do not change the rank of a matrix. So, if **1** is a linear combination of the columns of A and of A^c, then by adding **1** to each column of $-A$ we see that $r_{\mathbb{R}}(A) = r_{\mathbb{R}}(-A) = r_{\mathbb{R}}(J_{m,n} - A) = r_{\mathbb{R}}(A^c)$. Similarly, if **1** is a linear combination of the columns of A^c, $r_{\mathbb{R}}(A^c) = r_{\mathbb{R}}(A)$. Now if **1** is not a linear combination of the columns of A, then the span of the columns of $A^c = J - A$ contains each column of A, and **1**. Since **1** is not a linear combination of the columns of A, this span has dimension at least one greater than the column space of A. So, $r_{\mathbb{R}}(A) < r_{\mathbb{R}}(A^c)$. Similarly, if **1** is not a linear combination of the columns of A^c, then $r_{\mathbb{R}}(A^c) < r_{\mathbb{R}}(A)$. ■

For a complex number z we denote the real part of z by $Re(z)$.

Lemma A.5 [9] Let M be a tournament matrix of order n, and let λ be an eigenvalue of M with an associated eigenvector $\mathbf{v}$. Then

$$-\frac{1}{2} \leq Re(\lambda) \leq \frac{n-1}{2}$$

with $Re(\lambda) = -\frac{1}{2}$ if and only if $\mathbf{1}^*\mathbf{v} = 0$ and $Re(\lambda) = \frac{n-1}{2}$ only if M is the adjacency matrix of a regular tournament.

Proof: Since M is a tournament matrix $M + M^\top = J - I$, so $\mathbf{v}^*(M + M^\top)\mathbf{v} = \mathbf{v}^*(J - I)\mathbf{v}$. Now,

$$\mathbf{v}^* M\mathbf{v} + \mathbf{v}^* M^\top \mathbf{v} = \mathbf{v}^*(M\mathbf{v}) + (M\mathbf{v})^*\mathbf{v} = \mathbf{v}^*\lambda\mathbf{v} + \overline{\lambda}\mathbf{v}^*\mathbf{v} = 2Re(\lambda)|\mathbf{v}|^2.$$

Also,

$$\mathbf{v}^* J\mathbf{v} - \mathbf{v}^* I\mathbf{v} = \mathbf{v}^*\mathbf{1}\mathbf{1}^*\mathbf{v} - \mathbf{v}^*\mathbf{v} = (\mathbf{1}^*\mathbf{v})^*(\mathbf{1}^*\mathbf{v}) - \mathbf{v}^*\mathbf{v} = |\mathbf{1}^*\mathbf{v}|^2 - |\mathbf{v}|^2.$$

So, $2Re(\lambda)|\mathbf{v}|^2 = |\mathbf{1}^*\mathbf{v}|^2 - |\mathbf{v}|^2$, and so $(2Re(\lambda) + 1)|\mathbf{v}|^2 = |\mathbf{1}^*\mathbf{v}|^2$. Hence,

$$Re(\lambda) = \frac{|\mathbf{1}^*\mathbf{v}|^2}{2|\mathbf{v}|^2} - \frac{1}{2}.$$

Since $\frac{|\mathbf{1}^*\mathbf{v}|^2}{2|\mathbf{v}|^2} \geq 0$,

$$Re(\lambda) \geq -\frac{1}{2},$$

as desired. Notice also that because $\frac{|\mathbf{1}^*\mathbf{v}|^2}{2|\mathbf{v}|^2} \geq 0$, this statement holds with equality when $\frac{|\mathbf{1}^*\mathbf{v}|^2}{2|\mathbf{v}|^2} = 0$. Thus, this holds with equality exactly when $|\mathbf{1}^*\mathbf{v}|^2 = 0$, i.e. when $\mathbf{1}^*\mathbf{v} = 0$.

To show that $Re(\lambda) \leq \frac{n-1}{2}$, recall from the previous paragraph that $(2Re(\lambda) + 1)|\mathbf{v}|^2 = |\mathbf{1}^*\mathbf{v}|^2$. By the Cauchy-Schwarz Inequality, $|\mathbf{1}^*\mathbf{v}| \leq |\mathbf{1}||\mathbf{v}|$, with equality if and only if $\mathbf{v}$ is a scalar multiple of $\mathbf{1}$. So $|\mathbf{1}^*\mathbf{v}|^2 = |\mathbf{1}|^2|\mathbf{v}|^2$. Further, $|\mathbf{1}|^2 = \sum_{i=1}^n 1 = n$. So,

$$(2Re(\lambda) + 1)|\mathbf{v}|^2 \leq n|\mathbf{v}|^2.$$

Thus,

$$Re(\lambda) \leq \frac{n-1}{2}.$$

To see equality holds only when M is the adjacency matrix of a regular tournament, recall that equality holds only when $\mathbf{v}$ is a scalar multiple of $\mathbf{1}$. This implies that $\mathbf{1}$ is an eigenvector of M. Thus, $M\mathbf{1} = a\mathbf{1}$ for some scalar a. Now, the i^{th} entry of $M\mathbf{1}$ is $\sum_{j=1}^{n} M_{i,j}$. So, the row sum of each row of M must be equal, and so M must be the adjacency matrix of a regular tournament. ■

Lemma A.6 *If M is a tournament matrix, then M^c is non-singular.*

Proof: Let M be an $n \times n$ tournament matrix. So, $M + M^\top = J_n - I_n$. Thus, $M^c = M^\top + I_n$. Now, if M^c were singular, there would exist some $\mathbf{x}$ such that $(M^\top + I)\mathbf{x} = \mathbf{0}$, and hence $\mathbf{x}$ would be an eigenvector of $M^\top$ with eigenvalue -1. Since $M^\top$ is a tournament matrix, this contradicts Lemma A.5. So, M^c is non-singular. ■

Theorem A.7 *Let M be an $n \times n$ tournament matrix. Then $r_{\mathbb{R}}(M) \geq n - 1$, and $r_{\mathbb{R}}(M) = n$ if and only if $\mathbf{1}$ is in the column space of M.*

Proof: By Lemma A.4 and Lemma A.6, $r_{\mathbb{R}}(M) \geq r_{\mathbb{R}}(M^c) - 1 = n - 1$. Further, by Lemma A.4, since M^c is non-singular, $r_{\mathbb{R}}(M) = r_{\mathbb{R}}(M^c) = n$ if and only if $\mathbf{1}$ is in the column space of M. ■

Appendix B. C++ Code

The following is the C++ code used for the search in chapter 3. It was designed by the author and Carey Jenkins. It was parallelized, for use on the departments Beowulf cluster, and implemented by Carey Jenkins.

```
#include <iostream.h>
#include "mpi.h"
#include <stdio.h>

int pattern[19] = {0,1,0,0,1,1,1,1,0,1,0,1,0,0,0,0,1,1,0};

// first: row, second: s, third: entry
int possible_rows[19][512][19];

void sprinkle_first_row (int rows[19][512][19], int temp_row[19],
int passed_entry, int &s, int &max_ints);

void find_em (int rows[19][512][19], int &so_far,
int partial[19]);

int main(int argc, char** argv) {

MPI_Init(&argc, &argv);
```

```
int rank;
MPI_Comm_rank(MPI_COMM_WORLD, &rank);

//generate all sprinkles of row 1
// a vector storing partially constructed sprinkle
int temp_row[19];
// init this vector
for (int i = 0; i <= 18; i++)
temp_row[i] = 0;
// to start recursive first_row sprinkle generation
int root_entry = -1;
// this corresponds to the second index of possible rows
int stew = -1;
// only 9 integers will be placed in temp_row;
// this var counts them int max_ints = 0;
sprinkle_first_row(possible_rows, temp_row,
root_entry, stew, max_ints);

/*
//output sprinkles of row 1
cout << endl << endl;
for (int s = 0; s <= 511; s++) {
cout << s << ":  ";
```

```
for (int entry = 0; entry <= 18; entry++)
cout << possible_rows[0][s][entry];
cout << endl;
}
*/

//cycle all sprinkles of row 1 to
//generate sprinkles of remaining rows
for (int row = 1; row <= 18; row++)
for (int s = 0; s <= 511; s++)
for (int entry = 0; entry <= 18; entry++)
possible_rows[row][s][entry] =
possible_rows[0][s][(entry + row) % 19];

/*
//for a fixed s, let's view cyclic shifts of rows
for (int row = 0; row <= 18; row++) {
for (int entry = 0; entry <= 18; entry++)
cout << possible_rows[row][1][entry];
cout << endl;
}
*/

//search for desired matrices
```

```
// the number of rows succesfully constructed
int rows_so_far;
// contains the matrix's partial construction
//information via s only
int partial[19];
partial[0] = 0;
int dot;
//only use 32 processors
if (rank >= 0 && rank <= 31)
for (int S = rank * 16; S < (rank + 1) * 16; S++) {
partial[1] = S;
rows_so_far = 2;
//check dot product of row 1 and row 2
dot = 0;
for (int a = 0; a <= 18; a++)
dot = dot + possible_rows[0][0][a] * possible_rows[1][S][a];
//a little feedback please
if (dot == 0) {
printf("Processor %d is now working on S = %d.\n", rank, S);
find_em(possible_rows, rows_so_far, partial);
}
}
MPI_Finalize();
}
```

```
void sprinkle_first_row(int rows[19][512][19],
int temp_row[19], int passed_entry, int &s, int &max_ints) {

for (int entry = passed_entry + 1; entry <= 18; entry++)
if (pattern[entry] == 1) {
max_ints++;
temp_row[entry] = 1;
if (entry < 18)
sprinkle_first_row(possible_rows, temp_row, entry, s, max_ints);
if (max_ints == 9) {
s++;
for (int e = 0; e <= 18; e++)
rows[0][s][e] = temp_row[e];
}
temp_row[entry] = -1;
if (entry < 18)
sprinkle_first_row(possible_rows, temp_row, entry, s, max_ints);
if (max_ints == 9) {
s++;
for (int e = 0; e <= 18; e++)
rows[0][s][e] = temp_row[e];
}
max_ints--;
```

```
}
}

void find_em (int rows[19][512][19], int &so_far,
int partial[19]) {

//the passed variable so_far represents the
//number of rows that have been appended
//to partial before entering the recursive node.
//note that if so_far = 3, then only
//entries 0, 1, and 2 of partial are current

int dot_out;

//some feedback
if (so_far > 7) {
cout << "so_far: " << so_far << endl;
for (int i = 0; i < so_far; i++) {
cout << "s: " << partial[i] << ": ";
for (int j = 0; j <= 18; j++)
cout << rows[i][partial[i]][j] << "   ";
cout << endl << endl;
}
}
```

```
//if we've reached 19, we're successful and show output
if (so_far == 19) {
cout << endl << endl << "Hey we found a matrix!!!" << endl;
for (int i = 0; i <= 18; i++) { // rows correspond to i
for (int j = 0 ; j <= 18; j++) // entries correspond to j
cout << rows[i][partial[i]][j] << "   ";
cout << endl;
}
}

else

//try each sprinkling of row being considered
//the row under consideration is rows[so_far]
for (int s = 0; s <= 511; s++) {

//init dot product here so that
//if there are no rows so far, first s appended
dot_out = 0;

//check dot product to previous rows
for (int r = 0; r < so_far; r++) {
for (int a = 0; a <= 18; a++)
```

```
dot_out = dot_out + rows[r][partial[r]][a] * rows[so_far][s][a];
if (dot_out != 0)
break;
}

//if dot product is zero, append successful rows and look deeper
if (dot_out == 0) {
partial[so_far] = s;
so_far++;
find_em(rows, so_far, partial);
so_far--;
}
}
}
```

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