GENERALIZED QUADRANGLES OF ORDER (S, T) WITH $|S - T| = 2$

by

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ABSTRACT

The subject of this thesis is generalized quadrangles (GQ) whose parameters differ by two. The roles of regular points, lines and ovoids are examined extensively. Such regularities allow for connections between various GQ.

Affine planes associated with $GQ(q+1,q-1)$ are shown to be isomorphic to those associated with $GQ(q,q)$. Using a new idea, the grid-like axiom, the affine planes associated with certain regular ovoids are shown to be isomorphic. While regular ovoids can be used to obtain numerous other ovoids, the fan containing a pivotal ovoid is shown to be unique. Moreover if a fan contains more than one regular ovoid it is shown to contain every regular ovoid of the GQ.

A new characterization of the $GQ(q+1,q-1)$ arising from a q-arc is provided. After considering the characterizations due to De Soete and Thas and to Payne, the standard coordinatization of the GQ is introduced. By examining the coordinatizing field permutations, Payne's sixth characterization axiom is shown to be equivalent to having desarguesian planes arise from the pivotal ovoids.

Collineations of the known $GQ(q+1, q-1)$ are discussed. The main new result here is that in only one case are there collineations which are not induced by semilinear maps of the underlying vector space.

An appendix is provided as a brief introduction to projective geometry with a list of the known hyperovals.

This abstract accurately represents the content of the candidate's thesis. I recommend its publication.

Signed ______________________
Stanley E. Payne

DEDICATION

This thesis is dedicated to my mother Janet, my sister Michelle, and my wife Adriana.

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CONTENTS

Chapter

1. Introduction and review

1.1 Preface

While countless articles have been published on the topic of generalized quadrangles, relatively few self-contained references in this area are available. One notable exception is *Finite generalized quadrangles* by Payne and Thas which was published in 1984. Since then there has been somewhat of an explosion of interest in GQ. It now appears that due to the great proliferation of work in this field no suitable sequel would be fathomable. So it seems that a better task might be collecting much of what is known about a particular type of GQ and providing a sort of specialized reference manual. This is the motivation for what follows with the focus on generalized quadrangles whose parameters differ by 2. The purpose of this thesis is to gather together in one place much of the relevant work to date involving these GQ, including work which originated with this author. Portions of the material here were previously reported with Payne in [PM98].

Chapter one provides an introduction to generalized quadrangles with some basic definitions and small examples. In chapter two the known examples are presented as are properties of spreads/ovoids in these examples. In particular the hyperbolic lines in the pivotal ovoids are seen to be related to affine lines and planar arcs.

Chapter three deals primarily with regular ovoids of $GQ(q+1,q-1)$ and the corresponding affine planes. In addition to the known result that pivotal ovoids correspond to regular points, the following material is new:

(1) The planes corresponding to ovoids are shown to be isomorphic, and sometimes identical, to planes associated with GQ of order q in theorems 3.4.4 and 3.4.5.

(2) The Grid-like Axiom is introduced in section 3.5.

(3) Every regular ovoid is shown to be pivotal for exactly one fan in theorem 3.5.4.

(4) Planes associated with ovoids of a grid-like fan are shown to be isomorphic in theorem 3.5.5.

Chapter four begins with an elaboration of the proof by Thas and van Maldeghem showing that no non-classical $GQ(q+1,q-1)$ can have all points regular. Next regular pairs of points are examined leading to a new classification of regular ovoids.

In chapter five, a new characterization of the $GQ(q+1,q-1)$ arising from a q-arc is provided. After considering the characterizations due to De Soete and Thas and to Payne, the standard coordinatization of the GQ is introduced. By examining the coordinatizing field permutations, Payne's sixth characterization axiom is shown to be equivalent to having desarguesian planes arise from the pivotal ovoids.

Collineations of the known $GQ(q+1, q-1)$ are discussed in chapter six. The main new result here is that in only one case are there collineations of the quadrangle which are not collineations of projective three-space. The chapter concludes with a clarification of remarks made in [PM98] regarding findings of [GJS94]. In Chapter Seven, two previously known alternative constructions are provided: one due to Payne and the other to De Bruyn.

An appendix is provided as a brief introduction to projective geometry with a list of the known hyperovals.

While no work can be completely self-contained, an attempt has been made to keep the reader from having to consult an unwieldy number of references to understand the material here. Most terms not defined in this thesis can be found in either [Dem68] or [PT84]. A nice overview of generalized quadrangles is given in [Tha95] as well as [Pay96], the former of which concerns generalized n-gons with n not necessarily equal to 4.

1.2 Historical background

The study of finite generalized quadrangles (GQ) is a relatively young branch of discrete mathematics. J. Tits first introduced the notion of a generalized polygon in 1959 [Tit59]. For the next decade some progress was made in this field, particularly in the study of what are now termed the classical generalized quadrangles (see for example [Dem68] and [FH64]); then from the late 1960's through the early 1980's many geometers and algebraists began looking more deeply at this subject. A plethora of new results and a good number of new examples were discovered.

As the focus of this thesis is on $GQ(s,t)$ where s and t differ by two, some background on these GQ may be in order. (The roles of s and t are described in the next section.) In the late 1960's and early 1970's new constructions for $GQ(s, s+2)$ were given by Ahrens and Szekeres in [AS69] and independently by Hall in [Hal71]. This inspired Payne's work in [Pay71b] which gave new $GQ(s, s-2)$.

With these constructions in hand, two questions naturally arise: What are the collineations of these GQ, and what are the defining characteristics of these GQ? Much work has been done in an attempt to answer these questions. These two questions serve as a motivation for much of this thesis.

1.3 Definitions, examples, and observations

We begin with some basic definitions and then look at a few small examples. Let $\mathcal{P}$ and $\mathcal{B}$ be two non-empty sets, called points and lines, with an incidence relation $\mathcal{I}$ such that there are positive integers s and t satisfying

G1) Each point is incident with $t+1$ lines; any two points are mutually incident with at most one line.

G2) Each line is incident with $s+1$ points; any two lines are mutually incident with at most one point.

G3) Given a line L and a point x not incident with L there is a unique point y and a unique line M such that $x \mathcal{I} M \mathcal{I} y \mathcal{I} L$.

Such a collection $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is called a **generalized quadrangle of order** (s, t) written $GQ(s, t)$; when $s = t$ the GQ is said to have order s. The **dual** of a $GQ(s, t)$ is the $GQ(t, s)$ obtained by interchanging the roles of points and lines. Any theorem or definition given for a GQ can be dualized by interchanging points and lines. It will be assumed that whenever a definition or theorem is given, its dual has also been given.

Two points incident with a common line are **collinear** and two lines incident with a common point are **concurrent**. $x \sim y$ means that x and y are either collinear if x and y are points or concurrent if x and y are lines.

If X is a set of points (respectively, lines) of $\mathcal{S}$, then $X^{\perp}$ denotes the set of all points collinear (resp., lines concurrent) with everything in X; $X^\perp$ is called the **trace** of X. If $X = \{x\}$ is a singleton set, it is common to write $X^\perp$ as $x^\perp$. The **span** of X, written $X^{\perp\perp}$, is the set of all points collinear (resp., lines concurrent) with all of $X^\perp$. By convention, $x \in x^\perp$. It is worth noting that $(X^{\perp\perp})^\perp = X^\perp$.

Given any two noncollinear points x and y, $\{x, y\}^{\perp\perp}$ is called the **hyperbolic line** through x and y. Dually the **hyperbolic point** on two nonconcurrent lines L and M is the span $\{L, M\}^{\perp\perp}$.

Often GQ under examination here will be related to projective geometries. To avoid confusion, the symbol $< x, y >$ will be used for the projective space spanned by x and y whereas xy will indicate the line of the GQ containing x and y. Sometimes the term $\mathcal{S}$-line will be used to refer to a line of $\mathcal{S}$.

Straightforward arguments demonstrate the following:

- For $x \in \mathcal{P}$, $|x^\perp| = st + s + 1$.
- For $L \in \mathcal{B}$, $|L^\perp| = st + t + 1$.
- $|\mathcal{P}| = (s + 1)(st + 1)$.
- $|\mathcal{B}| = (t + 1)(st + 1)$.
- For two noncollinear points x, y, $|\{x, y\}^\perp| = t + 1$

and $2 \leq |\{x, y\}^{\perp\perp}| \leq t + 1$.

- For two nonconcurrent lines L, M, $|\{L, M\}^{\perp}| = s + 1$

and $2 \leq |\{L, M\}^{\perp\perp}| \leq s + 1$.

Two noncollinear points form a **regular pair** provided their span attains the upper bound, $t + 1$; two collinear points are defined to form a regular pair. An individual point is a **regular point** provided it forms a regular pair with every other point. Dually, $\{L, M\}$ is a **regular pair of lines** provided either $\{L, M\}^{\perp\perp} = s + 1$ or L and M are concurrent; a **regular line** is one which forms a regular pair with every other line. A point is **coregular** provided all lines incident with it are regular.

The following three observations will be useful when working with regularity.

Observation 1.3.1 $\{x_1, x_2\}$ is regular if and only if whenever $\{y_1, y_2\} \subseteq \{x_1, x_2\}^{\perp}$, then $\{y_1, y_2\}^{\perp} = \{x_1, x_2\}^{\perp\perp}$.

Proof: If $x_1 \sim x_2$, then clearly the observation holds as $\{x_1, x_2\}^{\perp} = \{x_1, x_2\}^{\perp\perp}$. Now assume x_1 and x_2 are not collinear. Suppose $\{x_1, x_2\}$ is regular and let $\{x_1, x_2\}^{\perp\perp} = \{x_1, \ldots, x_{t+1}\}$. If $\{y_1, y_2\} \subseteq \{x_1, x_2\}^{\perp}$, then each x_i is in $\{y_1, y_2\}^{\perp}$. Because $|\{y_1, y_2\}^{\perp}| = t + 1$, $\{y_1, y_2\}^{\perp} = \{x_1, x_2\}^{\perp\perp}$.

Now suppose $\{y_1, y_2\}^{\perp} = \{x_1, x_2\}^{\perp\perp}$ whenever $\{y_1, y_2\} \subseteq \{x_1, x_2\}^{\perp}$. This forces $|\{x_1, x_2\}^{\perp\perp}| = t + 1$ in which case $\{x_1, x_2\}$ is regular. ■

Observation 1.3.2 If $\{x_1, x_2\}$ is regular then $z_1, z_2 \in \{x_1, x_2\}^{\perp\perp}$ if and only if $\{z_1, z_2\}^{\perp\perp} = \{x_1, x_2\}^{\perp\perp}$.

Proof: The proof is similar to that of observation 1.3.1 and is left to the reader. ■

Observation 1.3.3 A point x_1 is regular if and only if every pair of points in $x_1^{\perp}$ is regular.

Proof: Assume x_1 is regular. Let $y_1, y_2 \in x_1^{\perp},\ x_2 \in \{y_1, y_2\}^{\perp} \backslash \{x_1\}$. By Observation 1.3.1, $\{y_1, y_2\}^{\perp} = \{x_1, x_2\}^{\perp}$. Hence $\{y_1, y_2\}^{\perp\perp} = \{x_1, x_2\}^{\perp}$. Therefore $\{y_1, y_2\}$ is regular.

Conversely, assume every pair of points in $x_1^{\perp}$ is regular. Let x_2 be some point other than x_1 and let $\{y_1, y_2\} \subseteq \{x_1, x_2\}^{\perp}$. By Observation 1.3.1, as $\{x_1, x_2\} \subseteq \{y_1, y_2\}^{\perp}$, $\{x_1, x_2\}$ is regular. Therefore x_1 is regular. ■

In the future, the proofs of numbered observations will be left to the reader.

An **ovoid** is a collection of $st + 1$ pairwise noncollinear points. For $k \leq st + 1$, a **k-cap** is a set of k pairwise non-collinear points. A **spread** is a collection of $st + 1$ pairwise nonconcurrent lines.

A set of ovoids which partitions the point-set is a **fan**. A set of spreads which partitions the line-set is called a **packing**. An ovoid (resp. spread) is said to be **regular** if the span of any pair of its points (resp. lines) is of maximum size and is contained in the ovoid (resp. spread).

A $GQ(s,t)$ with s or t equal to 1 is called **thin**; otherwise the GQ is **thick**. A smallest thick GQ would have parameters $s = t = 2$. Such a GQ is constructed below. For the most part, the GQ examined in this thesis will be thick. However, it is sometimes instructive to begin with small, thin examples.

Example 1.3.4 The 4×4 Grid:

Let $\mathcal{B}$ be the lines of a 4×4 grid and let $\mathcal{P}$ be the points of intersection on the grid. Observe that this gives a $GQ(3,1)$.

The next example is the dual of the first.

Example 1.3.5 $K_{4,4}$ (The Complete Balanced Bipartite Graph on 8 Nodes):

Let $\mathcal{P}$ be the node set of $K_{4,4}$ and let $\mathcal{B}$ be the edge set. This gives a $GQ(1,3)$.

In general for $n > 1$, any $n \times n$ grid is a $GQ(n-1,1)$ whose dual is $K_{n,n}$ (sometimes called a **dual grid**) which is a $GQ(1,n-1)$. However the case $n = 4$ is of particular interest here. To see why, first consider the smallest thick GQ.

Example 1.3.6 $GQ(2,2)$:

Let $S = \{1, 2, \ldots, 6\}$ and let $\mathcal{P} = \{\ \{i,j\} | i,j \in S, i \neq j\}$ be the set of **duads**, i.e. subsets of size 2, from S . A **syntheme** is a triple of disjoint members of $\mathcal{P}$. For example $\{\{1,2\},\{3,4\},\{5,6\}\}$ is a syntheme. Let $\mathcal{B}$ be the set of all synthemes formed from the duads in $\mathcal{P}$. Straightforward verification shows that $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \in)$ is a GQ of order 2.

Sometimes it is helpful to view a particularly GQ from more than one point of view.

The $GQ(2,2)$ may also be viewed differently using algebra. Let Z_2 be the group of order 2 and set $G = Z_2 \times Z_2 \times Z_2$. For $i \in \{1,2,3\}$ let e_i be the element with a 1 in the ith position and zeros elsewhere, let e be the identity element, and let $j = \sum e_i$. Construct the following subgroups of G: $A_i = \{e, e_i\}$, $A_i^* = \{e, e_i, j, e_i + j\}$. Write $\mathcal{F} = \{A_1, A_2, A_3\}$ and $\mathcal{F}^* = \{A_1^*, A_2^*, A_3^*\}$.

Let the points of $\mathcal{S}$ be the elements of G, cosets of subgroups in $\mathcal{F}^*$, and the set $\mathcal{F}$. Let the lines of $\mathcal{S}$ be the subgroups in F together with their cosets. If incidence is given by containment and inclusion, then $\mathcal{S}$ is isomorphic to example 1.3.6.

Similarly, examples 1.3.4 and 1.3.5 may be viewed from an algebraic point of view. Before this is done a relationship between the three examples is established. At first glance examples 1.3.4 and 1.3.5 may seem unrelated to 1.3.6. However, there is a very elegant connection. Let $\mathcal{S}$ be a $GQ(2,2)$ with a point x, and let P_x be the points of $\mathcal{S}$ which are not in $x^{\perp}$. Let B_1 be the lines of $\mathcal{S}$ which do not contain x (note that each of these lines has a point removed from it). For each $y \in P_x$, let L_y be a new line which joins y to the unique remaining point of $\{x, y\}^{\perp\perp}$, and let B_2 be the set of all such L_y. Let $B_x = B_1 \cup B_2$. Then P_x and B_x form the point and line sets (respectively) of a $GQ(1, 3)$.

Now return to $\mathcal{S}$ and let L be any line of $\mathcal{S}$. We create a similarly derived structure as follows. Let B_L be the lines of $\mathcal{S}$ which are not in $L^\perp$; let P_1 be the points of $\mathcal{S}$ which are not on L. For each $M \in B_L$, let x_M be a new point incident with the unique remaining line of $\{L, M\}^{\perp\perp}$. Define P_2 to be the set of all such x_M with $P_L = P_1 \cup P_2$. The resulting structure with point set P_L and line set B_L is a $GQ(3, 1)$.

This relationship between the three GQ is not coincidental; in fact it lies at the heart of this thesis. For this reason this process is explained in more detail here. Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be any $GQ(q, q)$ with a regular point x. Let $\mathcal{P}_x = \mathcal{P} - x^\perp$. Let $\mathcal{B}_x = \mathcal{B}_x^1 \cup \mathcal{B}_x^2$ where $\mathcal{B}_x^1$ is the set of lines which are not incident with x, and $\mathcal{B}_x^2 = \{\ \{x, y\}^{\perp\perp} | y \in \mathcal{P} - x^\perp\}$, i.e. $\mathcal{B}_x^2$ is the set of hyperbolic lines through x. If the incidence $\mathcal{I}_x$ is given by that of $\mathcal{I}$ and by containment, it is easy to verify that the resulting structure $\mathcal{S}_x = (\mathcal{P}_x, \mathcal{B}_x, \mathcal{I}_x)$ has q points on each line, and each point of $\mathcal{S}_x$ is on $q + 2$ points. That G3 holds is shown below in cases.

(1) Let $L \in \mathcal{B}_x^1$ and let p be a point of $\mathcal{P}_x$ not on L. In this case p and L

are also elements of $\mathcal{S}$. There is a unique line $M \in \mathcal{B}$ incident with p and a point z of L.

(a) If $z \in \mathcal{P}_x$, then M is the unique line of $\mathcal{B}_x^1$ collinear in $\mathcal{S}_x$ with p and a point of L. If some hyperbolic line of $\mathcal{B}_x^2$ contined p and a point w of L, then w and p would both be collinear in $\mathcal{S}$ with a point of $x^\perp$ which would give a triangle in $\mathcal{S}$. Hence M is the unique line of $\mathcal{B}_x$ through p and a point of L.

(b) If $z \notin \mathcal{P}_x$, then $z \in \{x, p\}^\perp$. As x is regular, every line through z contains a unique point of $\{x, p\}^{\perp\perp}$. Specifically, L contains a unique point w of $\{x, p\}^{\perp\perp}$. The hyperbolic line $\{x, p\}^{\perp\perp}$ is the unique line of $\mathcal{B}_x$ containing p and a point of L.

(2) Let $L \in \mathcal{B}_x^2$ and let p be a point of $\mathcal{P}_x$ not on L. Counting the number of points in $\mathcal{S}_x$ gives $|\mathcal{P}_x| = |\mathcal{P}| - |x^\perp| = (q+1)(q^2+1) - (q+1)q - 1 = q^3$. In $\mathcal{S}$ there are q points in $L \backslash \{x\}$. Each of these points is on $q+1$ lines. As no point of $\mathcal{P}_x$ can be on more than one such line, these lines cover $q(q+1)(q-1) = q^3$ points of $\mathcal{P}_x \backslash L$; i.e. these lines partition the points of $\mathcal{P}_x \backslash L$. Hence there is a unique line $M \in \mathcal{B}_x^1$ which is incident with p and a point of L.

Because $p \notin L$, the regularity of x implies $\{x, p\}^{\perp\perp}$ has no point of $\mathcal{P}_x$ in common with L. Therefore M is the unique line of $\mathcal{B}_x$ which is incident with p and a point of L.

This shows that $\mathcal{S}_x$ is a $GQ(q-1,q+1)$. Most GQ constructions presented here are similarly straightforward to verify. For this reason it will be often left to the reader to check that a given incidence structure satisfies G1, G2, and G3.

The process of forming $\mathcal{S}_x$ from $\mathcal{S}$ is called **expanding about the regular point** x. $\mathcal{S}_x$ is sometimes written $P(\mathcal{S},x)$.

Dually, if L is a regular line of $\mathcal{S}$, replacing x with L and interchanging the roles of points and lines gives a $GQ(q+1,q-1)$ written either as $\mathcal{S}_L$ or $P(\mathcal{S},L)$. Naturally this process is called **expanding about the regular line** L.

If these ideas are interpreted algebraicly, observe that the 4×4 dual grid can be constructed using groups. Let $\mathcal{F}^+ = \mathcal{F} \cup \{A_4\}$ where $A_4 = \{e, j\}$. The points of the dual grid can be viewed as points of G; the lines can be viewed as cosets of subgroups in $\mathcal{F}^+$.

The 4×4 grid can be constructed similarly. Let $\mathcal{F}' = \{A_1, A_2, A_3A_1, A_3A_2\}$. The points of the grid are the elements of G and the cosets of subgroups in $\mathcal{F}'$. Section 7.1 of this thesis explores this interpretation further.

In some respects expansion about regular points and lines is the GQ-analogue to the relationship between projective and affine planes. If π is a projective plane, a line L and all of the points on L can be removed from π to form the corresponding affine plane. Similarly, a point p and all of the lines through p can be removed from π to form the corresponding dual affine plane.

We know that to every projective plane of order n there corresponds an affine plane of order n. Every known $GQ(q,q)$ has at least one regular point or one regular line, and hence gives constructions of $GQ(q-1,q+1)$ or $GQ(q+1,q-1)$. Moreover, every known GQ whose parameters differ by 2 arise by the method of expansion. However, expanding about two different regular points may produce to two non-isomorphic GQ.

Recall that every known projective plane has prime power order. Similarly in every known example of GQ with parameters $(s,s),(s-1,s+1)$, or $(s+1,s-1)$, s is a power of a prime. This may seem unusual at first reading, however further examination shows a strong relationship between projective geometries and GQ.

We close this chapter with some special constructions of planes relating to GQ. Readers unfamiliar with projective and affine planes are directed to the appendix. Let $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathcal{I})$ be a $GQ(q,q)$ with a regular point p incident with a regular line L.

Construction 1.3.7 The projective plane $\pi^*(p) = (\ \mathcal{P}^*(p), \mathcal{L}^*(p)\ )$:

Let $\mathcal{P}^*(p) = p^\perp$. Let $\mathcal{L}^*(p) = \{\{x,y\}^{\perp\perp} | x,y \in p^\perp\}$, i.e. $\mathcal{L}^*(p)$ consists of the

$\mathcal{S}$-lines and hyperbolic lines determined by pairs of points in $p^\perp$. Incidence is given by containment. Clearly $|\mathcal{P}^*(p)| = q^2 + q + 1$ and each member of $\mathcal{L}^*(p)$ contains $q+1$ points. Because any pair of points in $\mathcal{P}^*(p)$ determines a unique member of $\mathcal{L}^*(p)$, $\pi^*(p)$ is a projective plane of order q.

Construction 1.3.8 The affine plane $\pi(p) = (\ \mathcal{P}(p), \mathcal{L}(p)\ )$:

Let $\mathcal{L}(p) = \mathcal{L}^*(p) \backslash \{L\}$, and let $\mathcal{P}(p)$ be the points of $\mathcal{P}(p)$ with the points of L removed. This makes $\pi(p)$ an affine plane of order q.

Construction 1.3.9 The projective plane $\pi^*(L) = (\ \mathcal{P}^*(L), \mathcal{L}^*(L)\ )$:

Let $\mathcal{P}^*(L) = \{\ \{M, N\}^{\perp\perp} | M, N \in L^\perp\}$ and $\mathcal{L}^*(L) = L^\perp$. Incidence is given by containment. This is a projective plane of order q.

Construction 1.3.10 The affine plane $\pi^*(L) = (\ \mathcal{P}^*(L), \mathcal{L}^*(L)\ )$:

Remove L and all of its points to form the affine plane $\pi^*(L) = (\ \mathcal{P}^*(L), \mathcal{L}^*(L)\ )$.

The projective planes constructed in 1.3.7 and 1.3.9 are well known. The related affine planes of constructions 1.3.8 and 1.3.10 will be revisited in section 3.4.

2. Known examples

In this chapter a number of GQ will be constructed from objects in projective 3-space. For these constructions some notation for the projective objects is first established. See the appendix for more on projective planes, projective 3-space, ovals, and oval permutations. Let $F = GF(q)$ and let α be an oval permutation of F. Let $A = (0,1,0,0), B = (0,0,1,0), \Omega = \{w_s = (1, s, s^\alpha, 0) | s \in F\} \cup \{A\}$; thus Ω is an oval in the plane $\Pi_\infty = [0,0,0,1]^T$ embedded in $PG(3,q) = \mathcal{G}$. Let $\Omega^+ = \Omega \cup \{B\}$ and let $\Omega^- = \Omega \backslash \{A\}$. If q is even then B is the nucleus of Ω.

2.1 Two classical examples: $W(q)$ and $Q(4,q)$

For two classical examples we visit polar spaces. Let ν be a symplectic polarity of $\mathcal{G}$. Hence for a point $x \in \mathcal{G}, x^\nu$ is a plane containing x. For example let ν be given by the alternating form $g(u,v) = uMv^T$ where M is the skew-symmetric matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

The lines of $\mathcal{G}$ which are fixed by ν are called **totally isotropic** lines. The points of $\mathcal{G}$ together with the totally isotropic lines of $\mathcal{G}$ under ν form a $GQ(q,q)$ written $W(q)$ in which every point is regular. (The polarity ν is suppressed in the notation because all such GQ are isomorphic.)

Now let $f : F^5 \rightarrow F$ be some non-degenerate quadratic form and let Q be the associated quadric. This means $b(x,y) = f(x+y) - f(x) - f(y)$ is a symmetric bilinear form, and $Q = \{x \in PG(4,q) | f(x) = 0\}$. That f is non-degenerate means that for each x there is at least one y such that $f(x+y) \neq f(x) + f(y)$. The points of Q together with the projective lines whose points are all in Q also form a $GQ(q,q)$, written $Q(4,q)$. The two GQ, $W(q)$ and $Q(4,q)$, are point-line duals of one another. They are self-dual exactly when q is even [PT84].

2.2 A construction by Tits

The first non-classical example we will examine is one due to Tits. It was first reported in [Dem68]. In [Pay85a] Payne modified Tit's description slightly. It is this modified description which is presented here.

Define an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ where points are of the following two types:

i) Points of $\mathcal{G} \backslash \Pi_\infty$ and

ii) Planes of $\mathcal{G}$ which contain either exactly 1 or $q+1$ points of Ω.

The lines of $\mathcal{S}$ are the lines of $\mathcal{G}$ which meet Ω in exactly one point. Incidence is given by that of $\mathcal{G}$. Let $w_0, \ldots, w_q$ be the points of Ω with the respective tangent lines labeled $L_0, \ldots, L_q$. Verification that G1, G2, and G3 hold follows from well known counts from projective geometry. These are outlined below.

If p is a type i) point of $\mathcal{P}$, then the lines of $\mathcal{B}$ through p are exactly the $q+1$ projective lines $< p, w_k >, 0 \leq k \leq q+1$. If p is the plane Π_∞, then the lines of $\mathcal{B}$ incident with p are the $q+1$ tangents to Ω. Finally if p is any other type ii) point, then in $\mathcal{G}$, p contains exactly one point of Ω, say w_j. The lines of $\mathcal{B}$ incident with p are the $q+1$ projective lines through w_j in the plane p.

If $L = L_k$ is a tangent to Ω in Π_∞, then the points of $\mathcal{P}$ incident with Π_∞ are the $q+1$ projective planes through L_k. If L is a line which meets Π_∞ in the point w_j, then the type i) points incident with L are the q projective points of $L \backslash \{w_j\}$, and the unique type ii) point incident with L is the plane through L and L_j.

Let L be a line of $\mathcal{B}$, let p_1 be a type i) point not incident with L, and let p_2 be a type ii) point not incident with L. Projectively, let L meet Ω at w_j, and let $M =< p_1, w_j >$. As M and L meet in $\mathcal{S}$ at the type ii) point $< L, M >$, M is the unique line of $\mathcal{B}$ incident with p_1 and a point of L. If L is a tangent line to Ω, let M be the line of p_2 which is also tangent to Ω; in this case L and M are both incident in $\mathcal{S}$ with the point Π_∞. If L is not a tangent line then in $\mathcal{G}$, L meets Π_∞ at a point w_j. If $p_2 = \Pi_\infty$ then let $M = L_j$; otherwise if $p_2 \cap \Omega = L_k$, let y be the projective point common to L and p_2 and let $M =< y, w_k >$. In either case M is the unique line of $\mathcal{B}$ incident with p_2 and a point of L.

With G1, G2, and G3 satisfied, this construction is shown to yield a $GQ(q,q)$ which is often written $T_2(\Omega)$ or $\mathcal{S}(\Omega)$. When q is odd, this construction does not provide new quadrangles; however when q even and with $2 \neq \alpha \neq 2^{-1}$, $T(\Omega)$ is different from $W(q)$ and $Q(4,q)$. Assume for the remainder of this section that q is even. In this case B is the nucleus of Ω.

Proposition 2.2.1 Π_∞ is a regular point of $T_2(\Omega)$.

Proof: Let x be a point of $T_2(\Omega)$ not $\mathcal{S}$-collinear with Π_∞, in which case x is a projective point. Observe that $\{\Pi_\infty, x\}^\perp$ is the set of planes through $< x, B >$, and $\{\Pi_\infty, x\}^{\perp\perp}$ is the set of q projective points on $< x, B >$ other than B together with Π_∞. Thus $\{\Pi_\infty, x\}^{\perp\perp}$ is as large as possible. ■

Proposition 2.2.2 The lines of Π_∞ which are tangent to Ω are regular lines of $T_2(\Omega)$.

Proof: Let L a line of Π_∞ tangent to Ω at p and let M be a line not coincident in $T_2(\Omega)$ with L; this means M is a projective line meeting Ω at some point $r \neq p$. Observe that $\{L, M\}^{\perp}$ is the set of q lines spanned by p and a point of M other than r together with the line $< B, r >$; and $\{L, M\}^{\perp\perp}$ is the q lines through r in $< M, p >$ other than $< r, p >$ together with L. ■

2.3 Constructions by Ahrens and Szekeres

Ahrens and Szekeres provided two examples of non-classical GQ which we describe here. For q odd the original description of $AS(q)$ is as follows: Points of $AS(q)$ are the points of affine three-space $AG(3, q)$. Lines of $AS(q)$ are the following curves of $AG(3, q)$:

(i) $x = \sigma,\ y = a,\ z = b,$

(ii) $x = a,\ y = \sigma,\ z = b,$

(iii) $x = c\sigma^2 - b\sigma + a,\ y = -2c\sigma + b,\ z = \sigma.$

Payne has given the following alternative construction of this $GQ(q-1, q+1)$: Choose any point x of $W(q)$ (recall x is regular in $W(q)$). Let $\mathcal{P}_x$ be the points of $W(q)$ which are not contained in a totally isotropic line through x. Let $\mathcal{B}_x = \mathcal{B}_1 \cup \mathcal{B}_2$ where $\mathcal{B}_1$ is the set of totally isotropic lines of $W(q)$ and $\mathcal{B}_2 = \{\{x, y\}^{\perp\perp} | y \in \mathcal{P}_x\}$. Incidence is given by containment. Notice that this is the process of expanding $W(q)$ about x which gives the $GQ(q-1, q+1)$

written $P(W(q), x)$. This construction works regardless of the parity of q. For the remainder of this thesis the case where q is even will be of particular interest.

For q even this next construction also yields a $GQ(q-1, q+1)$. It is a variation on the construction of $T_2(\Omega)$ due to Ahrens and Szekeres [AS69] and independently to Hall [Hal71]. Let $\mathcal{S}^+ = (\mathcal{P}^+, \mathcal{B}^+, \mathcal{I}^+)$ be the incidence structure in which $\mathcal{P}^+$ consists of the type i) points of $\mathcal{P}$, and $\mathcal{B}^+$ is the set of lines of $\mathcal{G}$ which meet Π_∞ in a unique point of Ω^+. It follows that $\mathcal{S}^+$ is a $GQ(q-1, q+1)$ using the incidence of $\mathcal{G}$. Such a construction is written $T_2^*(\Omega^+)$ or sometimes simply $\mathcal{S}(\Omega^+)$. Notice that $T_2^*(\Omega^+) = P(T_2(\Omega), \Pi_\infty)$. The following observations are immediate.

Observation 2.3.1 For each $x \in \Omega^+$, the lines of $\mathcal{G} \backslash \Pi_\infty$ through x form a spread.

Observation 2.3.2 The set of all such spreads formed above is a packing $\mathcal{M}$.

Further examination reveals

Proposition 2.3.3 If L_1, L_2 are in a spread of $\mathcal{M}$, then $\{L_1, L_2\}$ is regular.

Proof: Let $L_1 \cap L_2 = y$. L_1, L_2 determine a plane Π meeting Π_∞ in a line L through y. This line meets some other point $x \neq y$ of Ω^+. $\{L_1, L_2\}^\perp$ is then the set of q lines of $\Pi \backslash \{L\}$ through x. Hence $\{L_1, L_2\}^{\perp\perp}$ is the set of q lines in $\Pi \backslash \{L\}$ through y. ■

This packing in fact has a stronger property than just having regular pairs within spreads. Observe that every pair of lines from a spread S of $\mathcal{M}$ has its trace contained in some other spread of $\mathcal{M}$ and has its span back in S. Such a spread S is said to be **pivotal** for the packing $\mathcal{M}$.

Proposition 2.3.4 If L_1, L_2 are in different spreads of $\mathcal{M}$, then $\{L_1, L_2\}^{\perp\perp} = \{L_1, L_2\}$

Proof: Assume S_1 is a spread of $\mathcal{M}$ containing L_1 and assume S_2 is a different spread of $\mathcal{M}$ containing L_2. Let $L \in \{L_1, L_2\}^{\perp\perp} \backslash \{L_1, L_2\}$. First suppose $\{L_1, L_2\}^{\perp}$ contained two lines M_1, M_2 from some spread S in $\mathcal{M}$. Then as S is pivotal L_1, L_2 are contained in a spread of $\mathcal{M}$, a contradiction. So $\{L_1, L_2\}^{\perp}$ has one member in each spread of $\mathcal{M} \backslash \{S_1, S_2\}$. As spreads contain only non-concurrent lines, L is in either S_1 or S_2. But as $\{L_1, L_2\}$ is regular, $\{L_1, L_2\}^{\perp\perp} = \{L_1, L\}^{\perp\perp} = \{L_2, L\}^{\perp\perp}$. Hence $\{L_1, L_2\}$ is contained in either S_1 or S_2, a contradiction. ■

Observe that nothing in the proof above relied on the specific construction of the quadrangle, rather it relied only on having a GQ with a packing of pivotal spreads.

2.4 A construction by Payne

This section provides a construction of $GQ(q+1, q-1)$ for q even due to Payne [Pay71b, Pay72b, Pay85a] which also uses the hyperoval Ω^+. When Ω^+ does not contain a conic, the GQ constructed here are actually different from the duals of those constructed by Ahrens and Szekeres.

Let $\mathcal{O}_{AB}$ (respectively $\mathcal{O}_{BA}$) be the set of planes in $\mathcal{G}$ which meet A but not B (respectively B but not A). Finally let $\mathcal{P}^- = \mathcal{P}^+ \cup \mathcal{O}_{AB} \cup \mathcal{O}_{BA}$ and $\mathcal{B}^-$ be the set of lines in $\mathcal{G} \backslash \Pi_\infty$ which contain a point of Ω^-. Again incidence is that of $\mathcal{G}$. This GQ is often denoted $\mathcal{S}(\Omega^-)$. Observe that $\mathcal{S}(\Omega^-) = P(T_2(\Omega), < A, B >)$.

Any two planes of $\mathcal{O}_{AB}$ meet in a projective line. Such a line cannot be a line of $\mathcal{S}(\Omega^-)$. Since $\mathcal{O}_{AB}$ contains q^2 planes, $\mathcal{O}_{AB}$ is an ovoid. Similarly $\mathcal{O}_{BA}$ is an ovoid.

If L is a line of $PG(3, q)$ the set of planes through L is called the **dual line in** $PG(3, q)$. If one of these planes is removed, the result is called a **dual line in** $AG(3, q)$. For any two points $x_0, y_0 \in \Pi_\infty$, let $\mathcal{P}_{x_0,y_0}$ be the dual line in $AG(3, q)$ consisting of the q planes other than Π_∞ which contain the line $< x_0, y_0 >$. In particular observe that $\mathcal{P}_{AB} = \{\Pi_i = [1, 0, 0, i]^T | i \in F\}$. Observe that no projective line of Π_i is a line of $\mathcal{S}(\Omega^-)$. Hence each Π_i contains q^2 points of $\mathcal{S}(\Omega^-)$ which are pairwise noncollinear. Thus each Π_i is an ovoid of $\mathcal{S}(\Omega^-)$.

Observation 2.4.1 $\mathcal{M} = \{\mathcal{O}_{AB}, \mathcal{O}_{BA}\} \cup \mathcal{P}_{A,B}$ is a fan of ovoids.

Each Π_i has q^2+q+1 projective points. q^2 of these are pairwise non-collinear points of the quadrangle, and the remaining $q+1$ projective points are not part of the GQ. Furthermore each Π_i has q^2+q+1 projective lines and no quadrangle lines. Consequently, the Π_i will be viewed both as projective and quadrangle objects, where the context will dictate meaning.

The dual notion of a pivotal spread is a **pivotal ovoid**; i.e. an ovoid $\mathcal{O}$ is pivotal for a fan $\mathcal{M}$ provided every pair of points in $\mathcal{O}$ is regular, every pair in $\mathcal{O}$ has its trace in some other ovoid of $\mathcal{M}$, and every pair in $\mathcal{O}$ has its span in $\mathcal{O}$. This leads to the next proposition.

Proposition 2.4.2 The ovoids $\mathcal{O}_{AB}, \mathcal{O}_{BA}$ are pivotal for $\mathcal{M}$. Moreover the trace and span of any pair within one of these ovoids are affine lines or dual lines in $AG(3,q)$.

Proof: Choose w_s to be a projective point of Ω^-, and let $P_1, P_2 \in \mathcal{P}_{A,w_s}$. As points of $\mathcal{S}$, P_1 and P_2 are in the ovoid $\mathcal{O}_{AB}$. Now if Q is any of the q planes of $\mathcal{P}_{B,w_s}$, then Q meets P_1 in a projective line through w_s; i.e., as points of $\mathcal{S}$, Q and P_1 are collinear by a line of $\mathcal{S}$. Likewise Q and P_2 are collinear by a line of $\mathcal{S}$. Thus $Q \in \{P_1, P_2\}^\perp$. Hence $\{P_1, P_2\}^\perp = \mathcal{P}_{B,w_s} \subset \mathcal{O}_{BA}$, and $\{P_1, P_2\}^{\perp\perp} = \mathcal{P}_{A,w_s} \subset \mathcal{O}_{AB}$.

Now let $P_1, P_2 \in \mathcal{O}_{AB}$ such that $P_1 \cap \Pi_\infty =< A, w_s >\neq P_2 \cap \Pi_\infty =< A, w_t >$. Let $P_1 \cap P_2 = \lambda_{1,2} = \{A, x_1, \ldots, x_q\}$. Then $\{P_1, P_2\}^\perp = \{x_1, \ldots, x_q\} \subset< \lambda_{1,2}, B >\in P_{AB}$ and $\{P_1, P_2\}^{\perp\perp} = \{< \lambda_{1,2}, w_i > | w_i \in \Omega^-\} \subset \mathcal{O}_{AB}$. Thus we see that $\mathcal{O}_{AB}$ (and by a similar argument, $\mathcal{O}_{BA}$) is a pivotal ovoid. ■

One might be tempted to think that just as the packing constructed in the previous section consisted of pivotal spreads, the fan constructed here consists entirely of pivotal ovoids. However this is not always the case. Up until now, no conditions on the oval permutation α were made. At this point, the type of oval we start with will determine certain properties of the associated GQ.

Proposition 2.4.3 If $< \alpha >= Aut(F)$, then all ovoids of $\mathcal{M}$ are pivotal. Moreover the trace and span of pairs within one of these ovoids are either affine lines, dual lines in $AG(3, q)$, or planar arcs isomorphic to Ω^- which complete to hyperovals with A and B.

Proof: $\mathcal{O}_{AB}, \mathcal{O}_{BA}$ were considered in the previous proposition. Now consider each of the Π_i. Let $x_1, x_2 \in \Pi_a \in P_{AB}$; let $x_0 =< x_1, x_2 > \cap \Pi_\infty$.

If $x_0 = A$, then $\{x_1, x_2\}^\perp = \{< A, x_1, w_i > | w_i \in \Omega^-\} \subset \mathcal{O}_{AB}$ and $\{x_1, x_2\}^{\perp\perp} =< x_1, x_2 > \setminus \{A\} \subset \Pi_a$. Similar results hold when $x_0 = B$.

Now consider the situation where $x_0 \in < x_1, x_2 > \setminus \{A, B\}$. Let $x_1 = (a, b, c, 1)$, $x_2 = (a, e, f, 1)$. In this case $x_0 = (0, y, 1, 0)$ where $y = (b+e)(c+f)^{-1} \neq 0$ and hence $b \neq e$. For each $w_s \in \Omega^-$ there is a unique other point $w_{\hat{s}}$ of Ω^- on $< x_0, w_s >$. Let $\lambda_1 =< x_1, w_s >$; let $\lambda_2 =< x_2, w_{\hat{s}} >$ and observe $\lambda_1 \cap \lambda_2 = (a+k, b+ks, c+ks^\alpha, 1) = (a+h, e+h\hat{s}, f+h\hat{s}^\alpha, 1)$ for some $h, k \neq 0$. Matching up first coordinates gives $h = k$. Matching second coordinates gives $k = (b+e)(s+\hat{s})^{-1}$, and matching third coordinates gives $k = (c+f)(s+\hat{s})^{-\alpha}$. From this we have $(b+e)(s+\hat{s})^{-1} = (c+f)(s+\hat{s})^{-\alpha}$ implies $(s+\hat{s}) = [(b+e)(c+f)^{-1}]^{\frac{1}{1-\alpha}}$ (we use the fact that α generates $Aut(F)$). Plugging this in above we get $k = (b+e)(s+\hat{s})^{-1} = (b+e)([(b+e)(c+f)^{-1}]^{\frac{1}{1-\alpha}})^{-1} = (b+e)^{\frac{\alpha}{\alpha-1}}(c+f)^{\frac{1}{1-\alpha}}$.

Now k is written in terms of the coordinates of x_1 and x_2; hence we can write $\{x_1, x_2\}^\perp = \{(a+k, b+ks, c+ks^\alpha, 1) | s \in F\} = \Psi^-$ where k is as given above. It is easy to see that Ψ^- is a q-arc in Π_{a+k} (note that k is necessarily different from zero). Moreover if we let $\Psi^+ = \Psi^- \cup \{A, B\}$ we have Ψ^+ is projectively equivalent to Ω^+. To see this let ψ be the transformation given by $(t, u, v, w)^\psi = (t, u, v, w) \begin{bmatrix} a+k & b & c & 1 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Specifically $A^\psi = A, B^\psi = B$, and $(1, s, s^\alpha, 0)^\psi = (a+k, b+ks, c+ks^\alpha, 1)$.

Now let $\Sigma^- = \{p_r = (a, b+(b+e)r, c+(c+f)r^\alpha, 1) | r \in F\}$. For a fixed p_r, consider the line formed by p_r and some $(a+k, b+ks, c+ks^\alpha, 1) \in \{x_1, x_2\}^\perp$. Such a line would intersect Π_∞ in $(1, (b+e)^{\frac{1}{1-\alpha}}(c+f)^{\frac{1}{\alpha-1}}r + s, k^{-1}(c+f)r^\alpha + s^\alpha, 0)$. But $[(b+e)^{\frac{1}{1-\alpha}}(c+f)^{\frac{1}{\alpha-1}}r + s]^\alpha = k^{-1}(c+f)r^\alpha + s^\alpha$. This point is on Ω^- and hence this line is a GQ line. In other words each point of Σ^- is collinear in the quadrangle with each point of $\{x_1, x_2\}^\perp$. Since Σ^- has size q we see that in fact $\Sigma^- = \{x_1, x_2\}^{\perp\perp}$. It is easy to see that Σ is a q-arc in Π_a which forces Π_a to be a pivotal ovoid.

Note also that $\Sigma^+ = \Sigma^- \cup \{A, B\}$ is projectively equivalent to Ω^+. This equivalence is given by σ where

$$(t, u, v, w)^\sigma = (t, u, v, w) \begin{bmatrix} a & b & c & 1 \\ 0 & b+e & 0 & 0 \\ 0 & 0 & c+f & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

. Clearly σ maps Ω^+ to Σ^+ with A and B fixed.

Therefore each $\Pi_i \in \mathcal{P}_{A,B}$ is pivotal. Hence $\mathcal{M}$ is a fan with every ovoid pivotal. ∎

Examining pairs of points from different ovoids gives the following.

Proposition 2.4.4 If all ovoids of $\mathcal{M}$ are pivotal then for x_1, x_2 in different ovoids, $\{x_1, x_2\}^{\perp\perp} = \{x_1, x_2\}$.

Proof: The proof is similar to that of proposition 2.3.4, with "line" and "spread" replaced by "point" and "ovoid". ■

As with proposition 2.3.4, note that the proof depends not on the particular construction but only on the existence of a fan with all members pivotal. This idea of recognizing properties of GQ independent of their constructions will be useful in their classification and characterization.

3. Regular ovoids in $\mathcal{S}$

In this chapter regular ovoids are examined extensively. The main theme in this chapter is the relationship between regular ovoids and affine planes. Along the way, a method of deriving a GQ of order q from one of order $(q+1, q-1)$ is presented.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a $GQ(q+1, q-1)$ with q even. Let $\mathcal{O}_\infty$ be an ovoid of $\mathcal{S}$, i.e. $\mathcal{O}_\infty$ is a set of $1 + (q+1)(q-1) = q^2$ points of $\mathcal{S}$, no two on a line. This means that each line of $\mathcal{S}$ contains exactly one point of $\mathcal{O}_\infty$. Also assume that $\mathcal{O}_\infty$ is regular: for all $x, y \in \mathcal{O}_\infty, \{x, y\}$ is a regular pair and $\{x, y\}^{\perp\perp}$ is a set of q points all in $\mathcal{O}_\infty$.

3.1 Affine planes and ovoids

This section begins by noting that the points and hyperbolic lines of the regular ovoid $\mathcal{O}_\infty$ can be viewed as the points and lines of an affine plane. The parallel classes of this affine plane are used to construct the remaining ovoids of a fan. Examining these ovoids from the affine point of view will help in determining other characteristics of the GQ.

Construct an affine plane $\pi(\mathcal{O}_\infty)$ as follows. Take the q^2 points of

$\mathcal{O}_\infty$ to be the points of the $\pi(\mathcal{O}_\infty)$. For x and y in $\mathcal{O}_\infty$, let the hyperbolic line $\{x,y\}^{\perp\perp}$ be a line of $\pi(\mathcal{O}_\infty)$. Let incidnce be given by containment. Because $\mathcal{O}_\infty$ is regular, $|\{x,y\}^{\perp\perp}| = q$. Furthermore by observation 1.3.2, each pair of points in $\pi(\mathcal{O}_\infty)$ is contained in a unique line of $\pi(\mathcal{O}_\infty)$. This makes $\pi(\mathcal{O}_\infty)$ an affine plane (see the appendix).

Let $T_1, T_2, \ldots, T_q$ be the pairwise disjoint lines of one parallel class of $\pi(\mathcal{O}_\infty)$. The points of these lines account for all the points of the ovoid. Hence $T_1 \cup T_2 \cup \cdots \cup T_q = \mathcal{O}_\infty$. The proposition below shows how a related ovoid can be constructed from parallel classes of $\pi(\mathcal{O}_\infty)$.

Proposition 3.1.1 $T_1^\perp \cup T_2^\perp \cup \cdots \cup T_q^\perp$ is an ovoid of $\mathcal{S}$.

Proof: Let $T_i = \{x_i, y_i\}^{\perp\perp}$. Recall that $T_i^\perp$ is the set of points collinear with everything in T_i.

Let $z_1 \in T_1^\perp = \{x_1, y_1\}^\perp$ and $z_2 \in T_2^\perp = \{x_2, y_2\}^\perp$. If $z_2 \in T_1^\perp$ then $y_1 \in T_2$. (To see this observe that the q points of $z_2^\perp \cap \mathcal{O}_\infty$ are exactly the q points of T_2.) This is a contradiction because T_1 is parallel to T_2. Thus for all $i \neq j, T_i^\perp \cap T_j^\perp = \emptyset$ which implies $|T_1^\perp \cup \cdots \cup T_q^\perp| = q^2$.

If $z_1' \in T_1^\perp$, then $z_1 \not\sim z_1'$, otherwise a triangle in $\mathcal{S}$ is formed. Now suppose $z_1 \sim z_2$; then $z_1 \in \{x_1, z_2\}^\perp$. Since z_1 is collinear with each of the q non-collinear points of T_1, z_2 must be on one of the lines from z_1 to a point of T_1, say $z_2 \in y_1 z_1$. Since each of the q lines through z_2 contains exactly one point of $T_2 = \{x_2, y_2\}^\perp$, the line y_1z_1 must be such a line. This means there is a $\tilde{z} \in y_1z_1$ such that $\tilde{z} \in T_2$, giving two points of $\mathcal{O}_\infty, y_1$ and $\tilde{z}$, which are collinear in $\mathcal{S}$. But this cannot be as $\mathcal{O}_\infty$ is an ovoid.

This shows that no two points of $T_1^\perp \cup \cdots \cup T_q^\perp$ are collinear in $\mathcal{S}$. Therefore $T_1^\perp \cup \cdots \cup T_q^\perp$ is an ovoid of $\mathcal{S}$. ■

This shows that for each parallel class of $\pi(\mathcal{O}_\infty)$ there is a corresponding ovoid. These ovoids will be of primary interest. However as an aside observe that in fact, many more ovoids can be constructed.

Proposition 3.1.2 If $T_i' \in \{T_i, T_i^\perp\}, 1 \leq i \leq q$, then $\bigcup_{i=1}^q T_i'$ is an ovoid.

Proof: For distinct i and j, no point of T_i is collinear with a point of T_j. Likewise by proposition 3.1.1, no point of $T_i^\perp$ is collinear with a point of $T_j^\perp$. Assume $T_1' = T_1$ and $T_2' = T_2^\perp$. Let $y_1 \in T_1, z_1 \in T_1^\perp, z_2 \in T_2^\perp$. If $z_2 \in T_1$, then $z_2 \sim z_1$. As shown above, this cannot happen. Hence $T_1 \cap T_2^\perp = \emptyset$, which implies $|T_1' \cup \cdots \cup T_q'| = q^2$.

We need to show that no point of T_1 is collinear with a point of $T_2^\perp$. For contradiction, suppose $y_1 \sim z_2$. As above, since every line through z_2 contains a point of T_2, there is some $\tilde{z} \in y_1z_2$ with $\tilde{z} \in T_2$. But then $\tilde{z}$ and y_1 are two points of $\mathcal{O}_\infty$ which are collinear in $\mathcal{S}$. This contradicts $\mathcal{O}_\infty$ is an ovoid of $\mathcal{S}$. Therefore no two points of $\bigcup_{i=1}^q T_i'$ are collinear. ■

Let $E_0, E_1, \ldots E_q$ be the $q+1$ parallel classes of lines of $\pi(\mathcal{O}_\infty)$. Then for $0 \leq i \leq q$, $\mathcal{O}_i = \bigcup\{T^\perp : T \in E_i\}$ is an ovoid as shown above, and $\mathcal{M} = \{\mathcal{O}_\infty, \mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_q\}$ is a fan of ovoids. As $\mathcal{O}_\infty$ is regular, it follows that $\mathcal{O}_\infty$ is pivotal for $\mathcal{M}$. By construction each $\mathcal{O}_i$ is partitioned into hyperbolic lines of size q whose traces partition $\mathcal{O}_\infty$, $0 \leq i \leq q$. These ovoids may or may not also be regular; in some cases they are all regular. The final theorem of chapter 4 will help to classify the regular ovoids.

3.2 Intersection of traces with ovoids of $\mathcal{M}$

In this section two propositions from [Pay85a] are reviewed. These propositions indicate how the traces of point pairs intersect the ovoids of $\mathcal{M}$. They will be used in examing regular pairs in Chapter 4 and in characterizing GQ in Chapter 5.

Put $I = \{0, 1, \ldots, q\}$; $\tilde{I} = I \cup \{\infty\}$.

Proposition 3.2.1 Let $b, d \in \mathcal{O}_j$, with $j \in I, b \neq d$. If $\{b,d\}^\perp \cap \mathcal{O}_\infty \neq \emptyset$ then $\{b,d\}^\perp \subseteq \mathcal{O}_\infty$.

Proof: Let $b, d \in \mathcal{O}_j, b \neq d$. Suppose $a \in \{b,d\}^\perp \cap \mathcal{O}_\infty$ and $c \in \{b,d\}^\perp$ with $c \notin \mathcal{O}_\infty$. As each line meets each ovoid of $\mathcal{M}$ exactly once, let $a' = bc \cap \mathcal{O}_\infty$ and let f be the unique point of ad collinear with a'. As

$a, a' \in \mathcal{O}_\infty$ (which is pivotal) and $b \in \{a, a'\}^\perp$ then all of $\{a, a'\}^\perp$ is contained in $\mathcal{O}_j$; specifically $f \in \mathcal{O}_j$. But d is assumed to be in $\mathcal{O}_j$, a contradiction as $d \sim f$.

■

Proposition 3.2.2 Let $\mathcal{O}_\infty, \mathcal{O}_i, \mathcal{O}_j$ be distinct ovoids of $\mathcal{M}$.

i) $x \in \mathcal{O}_i$, $y \in \mathcal{O}_j$, and $x \not\sim y$ implies $|\{x, y\}^\perp \cap \mathcal{O}_\infty| = 1$.

ii) $a \in \mathcal{O}_\infty$, $b \in \mathcal{O}_i$, and $a \not\sim b$ implies $|\{a, b\}^\perp \cap \mathcal{O}_j| = 1$;

Proof: For each $a \in \mathcal{O}_\infty$, $a^\perp$ contains $q(q-1)$ noncollinear pairs $(z, w) \in \mathcal{O}_i \times \mathcal{O}_j$. As $\mathcal{O}_\infty$ is pivotal, if a and a' are distinct points in $\mathcal{O}_\infty$, then $a^\perp$ and $a'^\perp$ can never contain the same noncollinear pair $(z, w) \in \mathcal{O}_i \times \mathcal{O}_j$. Thus there are $(|\mathcal{O}_\infty|)q(q-1) = q^3(q-1)$ such pairs as a runs over the elements of $\mathcal{O}_\infty$.

On the other hand, counting the number of such pairs shows that there are q^2 choices for $z \in \mathcal{O}_i$ and $q^2 - q$ choices for $w \in \mathcal{O}_j$ with $z \not\sim w$. This gives $q^3(q-1)$ noncollinear pairs $(z, w) \in \mathcal{O}_i \times \mathcal{O}_j$, which proves part i).

For a fixed $a \in \mathcal{O}_\infty, x \in \mathcal{O}_i$, $\{a, x\}^\perp$ has exactly q points. If any two of these were in the same ovoid of $\mathcal{M}$ then the previous proposition would be contradicted. This proves ii). ■

3.3 Constricting about a regular ovoid

At this point additional GQ can be constructed. Having a regular ovoid $\mathcal{O}_\infty$ in $\mathcal{S}$ allows us to form a new GQ, $\mathcal{S}_\infty$, of order q via a method called **constriction about** $\mathcal{O}_\infty$. This is done by defining the points, lines and incidence of $\mathcal{S}_\infty$ as follows:

Points of $\mathcal{S}_\infty$: $\mathcal{P}_\infty = \mathcal{P} \backslash \mathcal{O}_\infty$ together with the symbols $(\mathcal{O}_0), \ldots, (\mathcal{O}_q)$

Lines of $\mathcal{S}_\infty$: $\mathcal{B}_\infty = \mathcal{B} \cup \{T^\perp : T \text{ is a line of } \pi(\mathcal{O}_\infty)\} \cup \{L_\infty\}$

Incidence of $\mathcal{S}_\infty$: A line of B is incident with a point of $\mathcal{P}_\infty$ provided the two were incident in $\mathcal{S}$. If T is a hyperbolic line of $\mathcal{O}_\infty$, then $T^\perp$ is incident with the q points it contains in $\mathcal{S}$ and $T^\perp \sim (\mathcal{O}_i)$ provided $\mathcal{O}_i$ is the (unique) ovoid from $\mathcal{M}$ containing $T^\perp$. Finally L_∞ is incident with each of the $q+1$ points $(\mathcal{O}_i)$, $0 \leq i \leq q$.

In chapter 1, it is was noted that the process of expanding about a regular line to form a $GQ(q+1, q-1)$ from a $GQ(q, q)$ was analogous to forming an affine plane from a projective plane. Similarly, the process of constricting about a regular ovoid in a $GQ(q+1, q-1)$ to form a $GQ(q, q)$ is analogous to forming a projective plane by adding in a line to an affine plane.

3.4 Regularity in $\mathcal{S}_\infty$ and planar isomorphisms

Here the regularity of lines and points of $\mathcal{S}_\infty$ is examined. First a regular line is found; next regular points are found in terms of pivotal ovoids from $\mathcal{S}$.

Proposition 3.4.1 L_∞ is a regular line of $\mathcal{S}_\infty$.

Proof: For this proof, incidences from both $GQ, \mathcal{S}$ and $\mathcal{S}_\infty$, are considered. Let L_i, L_j, L_k be distinct lines of $\mathcal{S}_\infty$ meeting L_∞ at distinct points $(\mathcal{O}_i), (\mathcal{O}_j), (\mathcal{O}_k)$ respectively. In $\mathcal{S}$, $L_i = T_i^\perp, L_j = T_j^\perp$ where T_i and T_j are affine lines in distinct parallel classes of $\pi(\mathcal{O}_\infty)$. By the dual of observation 1.3.3 it suffices to show that $\{L_i, L_j\}$ is a regular pair.

Let $L \in \{L_i, L_j\}^\perp$; in $\mathcal{S}$, L intersects $\mathcal{O}_\infty$ in some point x. Hence x is collinear with all of $T_i^\perp$ and with all of $T_j^\perp$. Let R_x be the set of lines through x thought of as lines of $\mathcal{S}_\infty$; then $R_x \subset \{L_i, L_j\}^\perp$. In fact as $|R_x| = q = |\{L_i, L_j\}^\perp - \{L_\infty\}|$, $\{L_i, L_j\}^\perp = R_x \cup \{L_\infty\}$. If $L_k \sim L$ then by similar argument L is concurrent everything in R_x; i.e. any line concurrent with the pair $\{L_\infty, L\} \subset \{L_i, L_j\}^\perp$ is concurrent with all of $\{L_i, L_j\}^\perp$. From the dual of observation 1.3.1, this means that $\{L_i, L_j\}$ is a regular pair and hence L_∞ is regular. ■

For the next proposition the following lemma is needed.

Lemma 3.4.2 In any $GQ(s,t)$, let L and M be distinct lines through z such that each point on $L-\{z\}$ forms a regular pair with each point on $M-\{z\}$. In this case z is itself a regular point.

Proof: For any $x \in L-\{z\}$, $y \in M-\{z\}$ observe that $|\{x,y\}^{\perp} - \{z\}| = t$. There are s choices for x and s choices for y, and hence there are s^2 traces $\{x,y\}^{\perp}$ containing z. This accounts for s^2t points not collinear with z. On the other hand, the total number of points of the GQ outside of $z^{\perp}$ is $(s+1)(st+1) - s(t+1) - 1 = s^2t$. Now every point not collinear with z has been accounted for in these traces. As $\{x,y\}$ is assumed to be regular, every pair from $\{x,y\}^{\perp}$ must be regular. Specifically z forms a regular pair with every point it is not collinear with, i.e. z is regular. Thus any noncollinear pair in $z^{\perp}$ is regular. ∎

We will now consider the relationship between regular points in $\mathcal{S}_{\infty}$ and pivotal ovoids in $\mathcal{S}$.

Proposition 3.4.3 $(\mathcal{O}_i)$ is a regular point of $\mathcal{S}_{\infty}$ if and only if $\mathcal{O}_i$ is pivotal for the fan $\mathcal{M}$.

Proof: 1) (*($\mathcal{O}_i$) is regular implies $\mathcal{O}_i$ is pivotal for $\mathcal{M}$.*)

In what follows assume that x and y are points of $\mathcal{S}_{\infty}$ which were also points of $\mathcal{S}$; i.e. they are not points of the form $(\mathcal{O}_j)$.

$\{x, y\}$ is a set of non-collinear points of $(\mathcal{O}_i)^\perp$ in $\mathcal{S}_\infty$ if and only if $x, y \in \mathcal{O}_i$ and x and y are in distinct traces, $T_1^\perp, T_2^\perp$ of the hyperbolic lines, say T_1 and T_2, of $\mathcal{O}_\infty$ such that $T_1^\perp, T_2^\perp \subseteq \mathcal{O}_i$ and $T_1 \cap T_2 = \emptyset$.

$(\mathcal{O}_i)$ is a regular point of $\mathcal{S}_\infty$ implies $\{x, y\}$ is a regular pair in $\mathcal{S}_\infty$, and hence $\{x, y,\}^{\perp\perp} = \{x_1 = x, x_2 = y, x_3, x_4, \ldots, x_{q+1}\}$. Each point of $\{x, y\}^{\perp\perp}$ is collinear with each point of $\{x, y\}^\perp$. Each line through $(\mathcal{O}_i)$ of the form $T_j^\perp$ contains a unique point of $\{x, y\}^{\perp\perp}$. As there are only q such lines, the remaining point must be some $(\mathcal{O}_j)$. This gives $\{x, y\}^\perp = \{y_1, y_2, \ldots, y_q, y_{q+1} = (\mathcal{O}_i)\}$ and $\{x, y\}^{\perp\perp} = \{x_1, x_2, \ldots, x_q, x_{q+1}\}$.

Thus $y_1, y_2, \ldots, y_q$ are all collinear with $(\mathcal{O}_j)$ in $\mathcal{S}_\infty$ which means they are all contained in the ovoid $\mathcal{O}_j$ in $\mathcal{S}$. This shows that if $(\mathcal{O}_i)$ is a regular point of $\mathcal{S}_\infty$, any two points contained in $\mathcal{O}_i$ (in $\mathcal{S}$) have their trace in an ovoid $\mathcal{O}_j$ of $\mathcal{M}$ and have their span completely contained in $\mathcal{O}_i$. This means $\mathcal{O}_i$ is pivotal for $\mathcal{M}$.

2) (*$\mathcal{O}_i$ is pivotal for $\mathcal{M}$ in $\mathcal{S}$ implies $(\mathcal{O}_i)$ is a regular point of $\mathcal{S}_\infty$.*)

Assume $\mathcal{O}_i$ is pivotal for $\mathcal{M}$. Let x_1, x_2 be collinear with $(\mathcal{O}_i)$ such that $x_1 \not\sim x_2$ and x_1, x_2 not on L_∞. Then $x_1, x_2 \in \mathcal{O}_i$, and if $\mathcal{O}_i = T_1^\perp \cup \cdots \cup T_q^\perp$ with $\mathcal{O}_\infty = T_1 \cup \cdots \cup T_q$, without loss of generality assume $x_1 \in T_1, x_2 \in T_2$. In $\mathcal{S}$, since $\mathcal{O}_i$ is pivotal, $\{x_1, x_2\}^{\perp\perp} = \{x_1, \ldots, x_q\} \subseteq \mathcal{O}_i$ and $\{x_1, x_2\}^\perp = \{y_1, \ldots, y_q\} \subseteq \mathcal{O}_j$ for some $\mathcal{O}_j \in \mathcal{M}$. Then for $1 \leq i, j \leq q$, $x_i \sim y_j$

in $\mathcal{S}$ and hence also in $\mathcal{S}_\infty$. Back in $\mathcal{S}_\infty, \{x_1, x_2\}^\perp = \{y_1, \ldots, y_q, (\mathcal{O}_i)\}$ and $\{x_1, x_2\}^{\perp\perp} = \{x_1, \ldots, x_q, (\mathcal{O}_j)\}$. Therefore $\{x_1, x_2\}$ is a regular pair of points in $\mathcal{S}_\infty$. By lemma 3.4.2, $\{x_1, (\mathcal{O}_k)\}$ is regular for all $(\mathcal{O}_k)$ on L_∞. Therefore $(\mathcal{O}_i)$ is a regular point of $\mathcal{S}_\infty$. ∎

In addition to the relationship between regular points of $\mathcal{S}_\infty$ and pivotal ovoids in $\mathcal{S}$, there is also an interesting relationship between planes associated with $\mathcal{S}_\infty$ and planes associated with $\mathcal{S}$. Here, too, regularity plays a key role.

For the moment, consider the plane $\pi(\mathcal{O}_\infty)$. Let $\mathcal{E}_0 = \{T_1, T_2, \ldots, T_q\}$ and $\mathcal{E}_1 = \{R_1, R_2, \ldots, R_q\}$ be two distinct parallel classes of lines in $\pi(\mathcal{O}_\infty)$.

Each point of $\pi(\mathcal{O}_\infty)$ is on a unique hyperbolic line T_i and a unique hyperbolic line R_j, hence these points may be labeled as $x_{i,j} = T_i \cap R_j$.

$T_i^\perp = \{x_{i,j}, x_{i,h}\}^\perp$ for any $h \neq j$, and $R_j^\perp = \{x_{i,j}, x_{k,j}\}^\perp$ for any $k \neq i$. In what follows the notation is abused somewhat, with the "$\perp$" notation used for whichever GQ is convenient. The context should make it clear what the notation means. To aid in this, when $T_i^\perp$ and $R_j^\perp$ are viewed as lines of $\mathcal{S}_\infty$, we will write $L_i = T_i^\perp$ and $M_j = R_j^\perp$.

Traces and spans of lines in $\mathcal{S}_\infty$ can be determined as follows. In the GQ $\mathcal{S}_\infty$, $\{L_i, M_j\}^\perp$ is the set of lines of $\mathcal{S}$ which contain $x_{i,j}$, together with the line L_∞. Every line of $\{L_i, M_j\}^{\perp\perp}$ can be viewed as the trace in $\mathcal{S}$ of a hyperbolic line of $\mathcal{O}_\infty$ which contains $x_{i,j}$.

From the regularity of L_∞ in $\mathcal{S}_\infty$, there is an affine plane $\pi(L_\infty)$ which arises as in construction 1.3.10. The point set here can be viewed as $\{\{L_i, M_j\}^{\perp\perp} : 1 \leq i, j \leq q\}$; $\mathcal{L}(L_\infty) = \{H^\perp : H$ is a hyperbolic line of $\mathcal{O}_\infty\}$. In this setting, if $H^\perp$ is an affine line of $\pi(L_\infty)$, the affine points incident with $H^\perp$ are the hyperbolic points in $\mathcal{S}_\infty$ formed by the line $H^\perp$ and some line of $\mathcal{S}_\infty$ concurrent with L_∞ and disjoint from $H^\perp$.

Theorem 3.4.4 The affine planes $\pi(\mathcal{O}_\infty)$ and $\pi(L_\infty)$ are isomorphic.

Proof: Define ψ from $\mathcal{P}(\mathcal{O}_\infty)$ to $\mathcal{P}(L_\infty)$ by $\psi(x_{i,j}) = \{T_i^\perp, R_j^\perp\}^{\perp\perp}$. To see that ψ preserves collinearity, first suppose that $l = T_i = \{x_{i,1}, \ldots, x_{i,q}\}$. In which case $\psi(l) = \{\{T_i^\perp, R_1^\perp\}^{\perp\perp}, \{T_i^\perp, R_2^\perp\}^{\perp\perp}, \ldots, \{T_i^\perp, R_q^\perp\}^{\perp\perp}\}$, i.e., exactly the set of points in $\mathcal{P}(L_\infty)$ on the line $T_i^\perp$. Now suppose that $l \notin \mathcal{E}_0$, say $l = \{x_{1,r_1}, x_{2,r_2}, \ldots, x_{q,r_q}\}$. $\psi(l)$ is then $\{\psi(x_{1,r_1}), \psi(x_{2,r_2}), \ldots, \psi(x_{q,r_q})\}$ which equals $\{\{T_1^\perp, R_{r_1}^\perp\}^{\perp\perp}, \{T_2^\perp, R_{r_2}^\perp\}^{\perp\perp}, \ldots, \{T_q^\perp, R_{r_q}^\perp\}^{\perp\perp}\}$ which in turn is equal to $\{\{T_1^\perp, l^\perp\}^{\perp\perp}, \{T_2^\perp, l^\perp\}^{\perp\perp}, \ldots, \{T_q^\perp, l^\perp\}^{\perp\perp}\}$. But this is exactly the set of points in $\mathcal{P}(L_\infty)$ on the line $l^\perp \in \mathcal{L}(L_\infty)$. Thus ψ maps lines to lines. Thus ψ is an isomorphism between $\pi(\mathcal{O}_\infty)$ and $\pi(L_\infty)$. ■

Now suppose that the ovoid $\mathcal{O}_0$ is also pivotal for the fan $\mathcal{M}$, so that the point $(\mathcal{O}_0)$ of L_∞ is regular in $\mathcal{S}_\infty$.

From the regularity of the point $(\mathcal{O}_0)$ in $\mathcal{S}_\infty$ there is an affine plane $\pi(\ (\mathcal{O}_0)\ )$ which arises as in construction 1.3.8. This new affine plane has point set $\mathcal{P}(\ (\mathcal{O}_0)\ ) = \{x : x \in \mathcal{O}_0\}$ and line set $\mathcal{L}(\ (\mathcal{O}_0)\ ) = \{\{x, y\}^{\perp\perp} : x, y \in \mathcal{O}_0\}$. But these are exactly the point– and line sets of $\pi(\mathcal{O}_0)$, and the incidences are the same. This proves the following theorem.

Theorem 3.4.5 The two affine planes $\pi(\ (\mathcal{O}_0)\ )$ and $\pi(\mathcal{O}_0)$ are identical (not merely isomorphic).

In the case under consideration here, i.e., that both $\mathcal{O}_\infty$ and $\mathcal{O}_0$ are pivotal for $\mathcal{M}$ in $\mathcal{S}$, Payne has shown in [Pay72a], [Pay77] and Chapter 12 of [PT84] that $\mathcal{S}_\infty$ is an **amalgamation of planes** (see references for more details). In light of theorems 3.4.4 and 3.4.5 the following is an immediate consequence

Theorem 3.4.6 $\mathcal{S}_\infty$ is the amalgamation of two desarguesian projective planes if and only if the two affine planes $\pi(\mathcal{O}_\infty)$ and $\pi(\mathcal{O}_0)$ are desarguesian.

This theorem will be utilized in a characterization given later.

3.5 Grid-like Fans

Next consider the associated planes in the case where at least three of the ovoids of $\mathcal{M}$ are pivotal. To answer the question, "Are these planes isomorphic", a new axiom is introduced and examined. Let $\mathcal{S}$ be a $GQ(q+1, q-1)$ with a fan $\mathcal{M}$.

Grid-Like Axiom: Let $\mathcal{O}_i, \mathcal{O}_j, \mathcal{O}_k$ be any triple of distinct pivotal ovoids for $\mathcal{M}$ with $x_1, x_2 \in \mathcal{O}_i,\ y_1, y_2 \in \mathcal{O}_j,\ z_1, z_2 \in \mathcal{O}_k$ such that $\{x_1, x_2\}^\perp \not\subset \mathcal{O}_j \cup \mathcal{O}_k$ and $\{y_1, y_2\}^\perp \not\subset \mathcal{O}_i \cup \mathcal{O}_k$. If x_1, y_1, z_1 lie on a line and if x_2, y_2, z_2 lie on a line, then every line meeting $\{x_1, x_2\}^{\perp\perp}$ and $\{y_1, y_2\}^{\perp\perp}$ also meets $\{z_1, z_2\}^{\perp\perp}$.

Any fan with at least pivotal ovoids satisfying this axiom will be called a **grid-like fan**. In section 2.4, $P(T_2(\Omega), < A, B >)$ was shown to have a fan $\mathcal{M} = \{\mathcal{O}_{AB}, \mathcal{O}_{BA}\} \cup \mathcal{P}_{A,B}$ with every ovoid pivotal provided Ω was a translation oval, i.e. $< \alpha >= Aut(F)$. Additionally, the following theorem holds.

Theorem 3.5.1 $\mathcal{M} = \{\mathcal{O}_{AB}, \mathcal{O}_{BA}\} \cup \mathcal{P}_{A,B}$ is a grid-like fan.

Proof: Let $P_1, P_2 \in \mathcal{O}_{AB}$, and let $Q_1, Q_2 \in \mathcal{O}_{BA}$. Let $x_1, x_2 \in \Pi_a,\ y_1, y_2 \in \Pi_{a+m},\ z_1, z_2 \in \Pi_{a+n}$ with $m, n, 0$ all distinct such that x_1, y_1, z_1, P_1, and Q_1 are all collinear in $\mathcal{S}$ by λ_1, and x_2, y_2, z_2, P_2, Q_2 are all collinear in $\mathcal{S}$ by λ_2. $P_1 =< A, \lambda_1 >, Q_1 =< B, \lambda_1 >, P_2 =< A, \lambda_2 >$, and $Q_2 =< B, \lambda_2 >$. Let $x_0 =< x_1, x_2 > \cap \Pi_\infty,\ y_0 =< y_1, y_2 > \cap \Pi_\infty,\ z_0 =< z_1, z_2 > \cap \Pi_\infty$. Let

$w_s = \lambda_1 \cap \Pi_\infty,\ w_t = \lambda_2 \cap \Pi_\infty$.

Construct the hyperbolix lines $H_1 = \{x_1, x_2\}^{\perp\perp}, H_2 = \{y_1, y_2\}^{\perp\perp}, H_3 = \{z_1, z_2\}^{\perp\perp}, H_4 = \{P_1, P_2\}^{\perp\perp}$, and $H_5 = \{Q_1, Q_2\}^{\perp\perp}$. Let $\mathcal{H} = \{H_1, \ldots, H_5\}$, To show that $\mathcal{M}$ is grid-like, we need to show that any line intersecting two hyperbolic lines of $\mathcal{H}$ must intersect every line of $\mathcal{H}$. To do this, we must consider the various ways x_0, y_0, and z_0 could be related to each other and to A and B. The following lemma will assist in this.

Lemma 3.5.2 If $x_0 \neq y_0$, then in fact $x_0 \neq z_0 \neq y_0$.

Proof of lemma: Suppose for contradiction that $y_0 = z_0$. Let Π_* be the plane $< y_0, y_1, z_1 >$. Observe the following implications. $z_1, y_1 \in \Pi_*$ implies $x_1, w_s \in \Pi_*; z_1, z_0 \in \Pi_*$ implies $z_2 \in \Pi_*; z_2, y_2 \in \Pi_*$ implies $x_2, w_t \in \Pi_*; x_1, x_2 \in \Pi_*$ implies $x_0 \in \Pi_*$. Finally $x_0, y_0 \in \Pi_*$ implies $A, B \in \Pi_*$. Thus $\Pi_\infty \cap \Pi_*$ is line containing A, B, w_s, w_t, a contradiction. Hence $y_0 \neq z_0$, and similarly $x_0 \neq z_0$. This proves the lemma, and return now to the proof of the theorem.

Consider the following cases and subcases for x_0, y_0, and z_0:

(1) $x_0 = A$ with

(a) $x_0 = y_0 = z_0$, or

(b) x_0, y_0, z_0 are distinct members of $< A, B >$.

(2) $x_0 = B$ (But results here will be like those above with A and B interchanged.)

(3) $x_0 = y_0 = z_0 \in < A, B > \setminus \{A, B\}$ with

(a) $w_s = w_t$, or

(b) $w_s \neq w_t$.

Case 1: $x_0 = A$. Let $x_1 = (a, b, c, 1), x_2 = (a, b + e, c, 1), y_1 = (a + m, b + ms, c + ms^\alpha, 1), y_2 = (a + m, b + e + mt, c + mt^\alpha, 1), z_1 = (a + n, b + ns, c + ns^\alpha, 1), y_2 = (a + n, b + e + nt, c + nt^\alpha, 1)$.

First consider $A = x_0 = y_0 = z_0$ (this is true if and only if $s = t$). If $x_3 \in \{x_1, x_2\}^{\perp\perp}, y_3 \in \{y_1, y_2\}^{\perp\perp}$ with $x_3 \sim y_3$, then there exists some $d \in F$ such that $x_3 = (a, b + d, c, 1)$ and $y_3 = (a + m, b + d + ms, c + ms^\alpha, 1)$. Let $z_3 = (a + n, b + d + ns, c + ns^\alpha, 1)$. Observe that $z_3 \in \{z_1, z_2\}^{\perp\perp}$ and $z_3 \in < x_3, w_s > = < x_3, y_3 >$. This means that any $\mathcal{S} - \mathit{line}$ meeting $\{x_1, x_2\}^{\perp\perp}$ and $\{y_1, y_2\}^{\perp\perp}$ must also meet $\{z_1, z_2\}^{\perp\perp}$. Also, observe that $P_1 = P_2 \in \{x_1, x_2\}^{\perp}$, and that $< B, x_3, y_3 > \in \{Q_1, Q_2\}^{\perp\perp}$.

Now consider $x_0 = A, y_0, z_0$ as three distinct members of $< A, B >$, and let $x_1, x_2, y_1, y_2, z_1, z_2$ be as above. Without loss of generality, consider $y_0 = B$, Then $y_2 = (a + m, b + ms, c + ms^\alpha + h, 1)$ for some h. Comparing this to y_2 as written above observe $t = em^{-1} + s$ and $h = m^{1-\alpha}e^\alpha$. So in fact $y_2 = (a + m, b + ms, c + ms^\alpha + m^{1-\alpha}e^\alpha, 1)$, and also $z_2 = (a + n, b + e, n(em^{-1} +$

$s), c + n(em^{-1} + s)^\alpha, 1)$.

Again choose $x_3 \in \{x_1, x_2\}^{\perp\perp}, y_3 \in \{y_1, y_2\}^{\perp\perp}$ with $x_3 \sim y_3$. Thus $x_3 = (a, b + d, c, 1), y_3 = (a + m, b + ms, c + ms^\alpha + \hat{h}, 1)$ such that $< x_3, y_3 >$ intersects Π_∞ at a point $(m, d + ms, \hat{h} + ms^\alpha, 0) \in \Omega^-$. Dividing through by m gives $(dm^{-1} + s)^\alpha = s^\alpha + \hat{h}m^{-1}$ which forces $\hat{h} = d^\alpha m^{1-\alpha}$. In fact $y_3 = (a + m, b + ms, c + ms^\alpha + d^\alpha m^{1-\alpha}, 1)$, and $< x_3, y_3 > \cap \Pi_\infty = (1, s + dm^{-1}, s^\alpha + d^\alpha m^{-\alpha}, 0)$. From this it follows that $< x_3, y_3 > \cap \Pi_{a+n} = (a + n, b + d + n(s + dm^{-1}), c + n(s^\alpha + d^\alpha m^{-\alpha}), 1)$. Call this point z_3.

Now observe that $z_3 \in \{z_1, z_2\}^{\perp\perp} = \{(a + n, b + ns, c + ns^\alpha, 1), (a + n, b + ns + (1 + nm^{-1})e, c + ns^\alpha + nm^{-\alpha}e^\alpha, 1)\}^{\perp\perp} = \{(a + n, b + ns + (1 + nm^{-1})er, c + ns^\alpha + nm^{-\alpha}e^\alpha r^\alpha, 1) | r \in F\}$ (to see this let $r = de^{-1}$). Any $\mathcal{S} - line$ meeting $\{x_1, x_2\}^{\perp\perp}$ and $\{y_1, y_2\}^{\perp\perp}$ must also meet $\{z_1, z_2\}^{\perp\perp}$. Observe that $< A, x_3, y_3 > \in \{P_1, P_2\}^{\perp\perp}$ and $< B, x_3, y_3 > \in \{Q_1, Q_2\}^{\perp\perp}$.

As was noted above, case 2 is similar to case 1.

Case 3: $x_0 = y_0 = z_0 = (0, y, 1, 0), y \neq 0$. Let $x_1 = (a, b, c, 1), x_2 = (a, e, f, 1)$ which forces $y = (b + e)(c + f)^{-1}$. Let $y_1 = (a + m, b + ms, c + ms^\alpha, 1)$. Consider $w_s = w_t$; thus $y_2 = (a + m, e + ms, f + ms^\alpha, 1), z_1 = (a + n, b + ns, c + ns^\alpha, 1)$, and $z_2 = (a + n, e + ns, f + ns^\alpha, 1)$. Again let $x_3 \in \{x_1, x_2\}^{\perp\perp}, y_3 \in \{y_1, y_2\}^{\perp\perp}$ with $x_3 \sim y_3$. $x_3 = (a, b + (b + e)r, c + (c + f)r^\alpha, 1), r \neq 0, 1; y_3 = (a + m, b + ms + (b + e)v, c + ms^\alpha + (c + f)v^\alpha, 1)$ for some v.

$x_3 \sim y_3$ implies $(1, s+m^{-1}(b+e)(r+v), s^\alpha+m^{-1}(c+f)(r+v)^\alpha, 0) \in \Omega^-$ which in turn implies that $[s+m^{-1}(b+e)(r+v)]^\alpha = s^\alpha + m^{-1}(c+f)(r+v)^\alpha$. Note that if $r \neq v$ then $m = (c+f)y^{\frac{\alpha}{\alpha-1}}$ in which case $< x_2, y_1 > \cap \Pi_\infty = (1, y^{\frac{1}{1-\alpha}} + s, y^{\frac{\alpha}{1-\alpha}} + s^\alpha, 0) \in \Omega^-$. But this means that $x_2 \sim y_1$ and therefore $y_1 \in \{x_1, x_2\}^\perp$.

The only case needing to be considered is when $r = v$. But in this case $< x_3, y_3 > \cap \Pi_\infty = w_s$, and hence $< x_3, y_3 > \cap \Pi_3 = (a+n, b+ns+(b+e)r, c+ns^\alpha+(c+f)r^\alpha, 1) \in \{z_1, z_2\}^{\perp\perp}$. Once again, any $\mathcal{S}-line$ meeting $\{x_1, x_2\}^{\perp\perp}$ and $\{y_1, y_2\}^{\perp\perp}$ must also meet $\{z_1, z_2\}^{\perp\perp}$. Also note that $< A, x_3, y_3 > \; = \; < A, x_3, w_s > \in \{P_1, P_2\}^{\perp\perp}$ and likewise $< B, x_3, y_3 > \in \{Q_1, Q_2\}^{\perp\perp}$.

Finally consider $w_s \neq w_t$. Since $w_t \in < x_0, x_1, y_1 > \cap \Pi_\infty$, then $w_t \in < w_s, x_0 >$. Thus $w_t = (1, s + y^{\frac{1}{1-\alpha}}, s^\alpha + y^{\frac{\alpha}{1-\alpha}}, 0)$, i.e, $t = s + y^{\frac{1}{1-\alpha}}$. From this we see that$y_2 = (a+m, e+ms+my^{\frac{1}{1-\alpha}}, f+ms^\alpha+my^{\frac{\alpha}{1-\alpha}}, 1)$, $z_1 = (a+n, b+ns, c+ns^\alpha, 1)$, $z_2 = (a+n, e+ns+ny^{\frac{1}{1-\alpha}}, f+ns^\alpha+ny^{\frac{\alpha}{1-\alpha}}, 1)$. As before, let $x_3 = (a, b+(b+e)r, c+(c+f)r^\alpha, 1), r \neq 0, 1$. If y_3 is again chosen to be the point of $\{y_1, y_2\}^{\perp\perp}$ which is on an $\mathcal{S}-line$ with x_3 then $y_3 = (a+m, b+ms+(b+e+my^{\frac{1}{1-\alpha}})r, c+ms^\alpha(c+f+my^{\frac{\alpha}{1-\alpha}})r^\alpha, 1)$ and $< x_3, y_3 >$ intersects Π_∞ at the point w_u where $u = s + y^{\frac{1}{1-\alpha}}r$.

Thus $z_3 = (a+n, b+ns+(b+e+ny^{\frac{1}{1-\alpha}})r, c+ns^\alpha(c+f+ny^{\frac{\alpha}{1-\alpha}})r^\alpha, 1) =< x_3, y_3 > \cap \{z_1, z_2\}$. Yet again (as has been the refrain) any $\mathcal{S}-line$ meeting

$\{x_1, x_2\}^{\perp\perp}$ and $\{y_1, y_2\}^{\perp\perp}$ must also meet $\{z_1, z_2\}^{\perp\perp}$. To see that $< x_3, y_3 >$ intersects $\{P_1, P_2\}^{\perp\perp}$, first observe that the point $v_3 = (a + (c + f)y^{\frac{\alpha}{\alpha-1}}, (c + f)y^{\frac{\alpha}{\alpha-1}}s, c + (c + f)y^{\frac{\alpha}{\alpha-1}}s^{\alpha}, 1)$ is a point of $P_1 \cap P_2$, and thus $P_1 =< A, v_3, w_s >$ $, P_2 =< A, v_3, w_t >$. From this we see that $\{P_1, P_2\}^{\perp\perp}$ is the set of planes through the line $< A, v_3 >$. Finally observe that $v_3 \in P_3 =< A, x_3, w_u >$ and $P_3 \in \{P_1, P_2\}^{\perp\perp}$. Similarly, $< x_3, y_3 >$ is contained in a plane of $\{Q_1, Q_2\}^{\perp\perp}$. Hence $\mathcal{M}$ is grid-like. ∎

Now that the concept of grid-like is seen to be non-vacuous, the axiom is used to prove a theorem about the associated affine planes. Return to the notation of letting $\mathcal{S}$ be any $GQ(q + 1, q - 1)$ with a fan $\mathcal{M}$. If any ovoid of $\mathcal{M}$ is pivotal then in fact $\mathcal{M}$ must have been constructed as in section 3.1. To see this first note the following lemma.

Lemma 3.5.3 If $\mathcal{O}_\infty$ is pivotal for $\mathcal{M}$ and $\mathcal{E}_0 = \{T_1, \ldots, T_q\}$ is a parallel class of the associated affine plane $\pi(\mathcal{O}_\infty)$, then $T_1^\perp \cup \ldots \cup T_q^\perp$ is an ovoid of $\mathcal{M}$.

Proof: $\mathcal{O}_\infty$ is pivotal implies $T_1^\perp \subset \mathcal{O}_i$ for some $\mathcal{O}_i \in \mathcal{M}$. Let $x \in T_j \neq T_1$. Choose $y \in x^\perp \cap \mathcal{O}_i$ and choose $x' \in y^\perp \cap \mathcal{O}_\infty \backslash \{x\}$. Suppose $z \in \{x, x'\}^{\perp\perp} \cap T_1$. Then $z \sim y$ implies y is in $T_1^\perp$ and x is in T_1, a contradiction. Therefore $\{x, x'\}^{\perp\perp} \in \mathcal{E}_0$, i.e. $\{x, x'\}^{\perp\perp} = T_j$. As $T_j^\perp \cap \mathcal{O}_i$ is not empty, $T_j^\perp$ must be contained in $\mathcal{O}_i$. ∎

With this proved, it follows that given any parallel class of $\pi(\mathcal{O}_\infty)$ the traces of the lines in the parallel class partition the points of an ovoid of $\mathcal{M}$. Because the number of ovoids in $\mathcal{M}\backslash\{\mathcal{O}_\infty\}$ equals the number of parallel classes in $\pi(\mathcal{O}_\infty)$, each ovoid of $\mathcal{M}\backslash\{\mathcal{O}_\infty\}$ can be viewed as the union of traces of hyperbolic lines in a given parallel class of $\pi(\mathcal{O}_\infty)$. This proves the following theorem.

Theorem 3.5.4 Every regular ovoid of $\mathcal{S}$ is pivotal for exactly one fan, namely the fan constructed in the manner of Section 3.1.

We are now in a position to use the Grid-like Axiom to identify isomorphic planes.

Theorem 3.5.5 If $\mathcal{M}$ is a grid-like fan, then the affine planes associated with the pivotal ovoids of $\mathcal{M}$ are all isomorphic.

Proof: Assume that $\mathcal{M}$ is grid-like. Let $\mathcal{O}_\infty, \mathcal{O}_0, \mathcal{O}_1$ be distinct pivotal ovoids in $\mathcal{M}$. Let $\mathcal{E}_0, \mathcal{E}_1$ be parallel classes of $\pi(\mathcal{O}_\infty)$ such that each hyperbolic line of $\mathcal{E}_0$ has its trace in $\mathcal{O}_0$ and each hyperbolic line of $\mathcal{E}_1$ has its trace in $\mathcal{O}_1$. Furthermore let $\mathcal{F}_0 = \{T^\perp | T \in \mathcal{E}_0\}$ and $\mathcal{F}_1 = \{R^\perp | R \in \mathcal{E}_1\}$. Label the members of $\mathcal{E}_0$ by $\mathcal{E}_0 = \{T_1, T_2, \ldots, T_q\}$ and let R_1, R_2 be distinct members of $\mathcal{E}_1$.

Identify particular points of $\mathcal{O}_\infty$. For $i = 1, 2$ let $x_i = T_i \cap R_i$.

$\{x_1, x_2\}^{\perp\perp}$ meets each member of $\mathcal{E}_0$ and each member of $\mathcal{E}_1$. For $i = 3, \ldots, q$ let $x_i = T_i \cap \{x_1, x_2\}^{\perp\perp}$ and let R_i be the unique line of $\mathcal{E}_1$ containing x_i. For each $i = 1, \ldots, q$ let $\{\lambda_{i,1}, \ldots \lambda_{i,q}\}$ be the set of lines from $\mathcal{S}$ which are incident with x_i and let $a_{i,j} = \lambda_{i,j} \cap \mathcal{O}_0,\ b_{i,j} = \lambda_{i,j} \cap \mathcal{O}_1$.

Define the map ϕ by $\phi(a_{i,j}) = b_{i,j}$. The claim is that ϕ is an isomorphism from $\pi(\mathcal{O}_0)$ to $\pi(\mathcal{O}_1)$. Clearly ϕ maps lines of $\mathcal{F}_0$ to lines of $\mathcal{F}_1$. It remains to be shown that ϕ maps the remaining lines of $\pi(\mathcal{O}_0)$ to lines of $\pi(\mathcal{O}_1)$.

Let $y_1 \in T_1^{\perp}, y_2 \in T_2^{\perp}, z_1 = \phi(y_1), z_2 = \phi(y_2)$. Let $\mathcal{F}$ be the parallel class of $\pi(\mathcal{O}_0)$ containing the affine line $\{y_1, y_2\}^{\perp\perp}$. For each $x_i \in \{x_1, x_2\}^{\perp\perp}$, each line through x_i must meet a unique affine line of $\mathcal{F}$. Let m_i be the line of $\mathcal{S}$ through x_i and incident with a point of $\{y_1, y_2\}^{\perp\perp}$. Obviously y_1, z_1 are on m_1, and y_2, z_2 are on m_2. By the grid-like axiom, since each m_i contains x_i and some point of $\{y_1, y_2\}^{\perp\perp}$, it follows that m_i must also contain a point of $\{z_1, z_2\}^{\perp\perp}$.

By the construction if $y_i = m_i \cap \{y_1, y_2\}^{\perp\perp}$, then $\phi(y_i) = m_i \cap \{z_1, z_2\}^{\perp\perp}$. ϕ maps the affine line $\{y_1, y_2\}^{\perp\perp}$ to the affine line $\{z_1, z_2\}^{\perp\perp}$. Therefore ϕ is an isomorphism. ■

Corollary 3.5.6 If $\mathcal{M}$ is a grid-like fan, then either all or none of the affine planes associated with the ovoids of $\mathcal{M}$ are desarguesian.

Revisiting $P(T_2(\Omega), < A, B >)$, observe that the affine plane $\pi(\mathcal{O}_{AB})$ has the following structure. The affine points are q^2 planes of $\mathcal{G}$ which contain A but not B. The affine lines can be viewed as the $q^2 + q$ lines of $\mathcal{G}$ which contain A but not B with incidence inherited from $\mathcal{G}$.

For each $i \in F \cup \{\infty\}$, the pencil of lines through A in Π_i is a parallel class. $\pi(\mathcal{O}_{AB})$ completes to a projective plane $\hat{\pi}(\mathcal{O}_{AB})$ by including the line $L_\infty =< A, B >$ in the line set and all of the planes of $\mathcal{G}$ through L_∞ in the point set. Let ρ be the polarity of $\mathcal{G}$ defined by $\rho : (a, b, c, d) \mapsto [c, d, a, b]^T$. $\hat{\pi}(\mathcal{O}_{AB})$ is a plane of $\mathcal{G}^\rho$; hence $\hat{\pi}(\mathcal{O}_{AB})$ is desarguesian.

Corollary 3.5.7 All of the planes associated with the fan $\mathcal{M} = \{\mathcal{O}_{AB}, \mathcal{O}_{BA}\} \cup \mathcal{P}_{A,B}$ of $P(T_2(\Omega), < A, B >)$ are desarguesian.

4. Regularity in $\mathcal{S}$

As has been shown, $GQ(q+1, q-1)$ with regular ovoids yield $GQ(q, q)$ with regular points. In this chapter we will consider regularity of points and lines in any $GQ(q+1, q-1)$. For $q > 3$ no known such GQ has a regular point. In [TvM97] Thas and van Maldeghem give an interesting theorem greatly restricting the existence of regular points. In the first section this theorem is presented. While regular points are absent, regular pairs of points abound. In the subsequent sections, these regular pairs and the regular ovoids are examined.

4.1 A theorem of Thas and van Maldeghem

The only known example of a $GQ(q+1, q-1)$ with a regular point is $\mathcal{S}(\Omega^-)$ where Ω^- is embedded in a conic in $PG(2, 3)$. In this case all points are regular. What follows is an elaboration of a proof first given in [TvM97] showing that this is in fact the only $GQ(q+1, q-1)$ with all points regular.

Because the proof is rather lengthy, the main steps in the proof are outlined first.

(1) Construct a hyperbolic line L and its trace, $L^\perp$.

(2) Construct V_L, the set of points not collinear with L or $L^\perp$, and consider some of its subsets.

(3) Partition $V_L \cup L \cup L^\perp$ into two ovoids.

(4) Partition these ovoids into hyperbolic lines.

(5) Find a hyperbolic line and its perp in one ovoid.

(6) Observe the contradiction as such an ovoid would then contain collinear points and conclude that no such $GQ(q+1, q-1), q > 3$, exists with all points regular.

Start with a $GQ(q+1, q-1)$, say $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with all points regular. Let $y, z \in \mathcal{P}$ with y and z not collinear. Then $\{y, z\}^{\perp\perp} = L$ has size q as y is regular. Also $L^\perp = \{y, z\}^\perp$ has size q. Since every point of $\mathcal{S}$ is on q lines, each line incident with a point of L is also incident with a point of $L^\perp$. Put another way: each point collinear with a point of L is collinear with a point of $L^\perp$.

Let V_L be the set of points of $\mathcal{P}$ not collinear with any point of L. (Then observe $V_{L^\perp} = V_L$.) Each of the q points of L is on q lines, and each of these lines has $q + 1 + 1 - 2 = q$ points not in $L \cup L^\perp$. Other than points of $L \cup L^\perp$, there are $qqq = q^3$ points which are collinear with points of L. All together there are $|L| + |L^\perp| + q^3 = 2q + q^3$ points of $\mathcal{P}$ collinear with points of L. This means $|V_L| = (q + 1 + 1)\left((q + 1)(q - 1) + 1\right) - (2q + q^3) = 2q(q - 1)$.

Now pick $x \in V_L$. For each $u \in L$, $\{x, u\}^{\perp\perp}$ is a hyperbolic line. As u ranges over L this gives q hyperbolic lines. For $u' \in L \backslash \{u\}$, suppose $p \in \{x, u\}^{\perp\perp} \cap \{x, u'\}^{\perp\perp}$. Then as x is regular, $\{x, u\}^{\perp\perp} = \{x, p\}^{\perp\perp} = \{x, u'\}^{\perp\perp} = \{u, u'\}^{\perp\perp} = L$. This is a contradiction as $x \notin L$. This proves the first lemma:

Lemma 4.1.1 Every pair of hyperbolic lines through x and a point of L has only x in common.

Let W be the union of the hyperbolic lines through x and points of L, with the points of L removed: $W = \cup_{u \in L} \{x, u\}^{\perp\perp} \backslash L$. Then $|W| = q(q-2)+1 = (q-1)^2$. Similarly let $\tilde{W} = \cup_{u \in L^{\perp}} \{x, u\}^{\perp\perp} \backslash L^{\perp}$, and $|\tilde{W}| = (q-1)^2$. W and $\tilde{W}$ will later be shown to be contained in ovoids of $\mathcal{S}$.

Lemma 4.1.2 $(W \cup \tilde{W}) \subset V_L$.

Proof: Now let $v \in W, v \neq x$. For contradiction, assume there is some $u' \in L$ where v is collinear with u'. Observe that $v \in W$ implies there is some $u \in L$ such that $v \in \{x, u\}^{\perp\perp}$. Let w be the unique point of vu' collinear with u. Then the regularity of v (or u) implies that w is collinear with everything in $\{v, u\}^{\perp\perp} = \{x, u\}^{\perp\perp}$. Specifically w and x are collinear. But since w is collinear with two points of L, it is collinear with all points of L, and every line through w has a point of L. This means x is collinear with a point of L. This contradicts $x \in V_L$. The contradiction came from assuming a point of W was collinear with a point of L. Therefore $W \subset V_L$. Similarly $\tilde{W} \subset V_L$. ■

Lemma 4.1.3 $W \cap \tilde{W} = \{x\}$, and hence $|W \cup \tilde{W}| = 2(q-1)^2 - 1$.

Proof: Suppose there is some $v' \in W \cap \tilde{W}$ with $v' \neq x$. Then $\{x, v'\}^{\perp\perp}$ has a point $a \in L$ and a point $b \in L^{\perp}$. By definition of $L^{\perp}, a$ and b are collinear. But this is a contradiction as no two points of $\{x, v'\}^{\perp\perp}$ can be collinear. Therefore $|W \cup \tilde{W}| = (q-1)^2 + (q-1)^2 - 1 = 2(q-1)^2 - 1$ ∎

By construction no point of W is collinear with x. Suppose there were two collinear points, v and $v' \in W$, distinct from one another (and from x). Then $\{v, x\}^{\perp\perp} \neq \{v', x\}^{\perp\perp}$. The line vv' has a point v'' collinear with x. Let $L \cap \{v, x\}^{\perp\perp} = \{u\}$ and let $L \cap \{v', x\}^{\perp\perp} = \{u'\}$. Since v'' is collinear with v and x, it is collinear with everything in their span; particularly, v'' is collinear with u. Likewise, v'' is collinear with u'. But any point collinear with two points of L is collinear with all of L and hence is in $L^{\perp}$; i.e. $v'' \in L^{\perp}$. This is a contradiction as x is collinear with v'' and $x \in V_L = V_{L^{\perp}}$. Observe then that no two points of W are collinear. Similarly, no two points of $\tilde{W}$ are collinear; i.e. W and $\tilde{W}$ are $(q-1)^2$-caps of $\mathcal{S}$.

By construction, $W \cap L$ is empty, which means $|W \cup L| = (q-1)^2 + q = q^2 - q + 1$. Note that no two points of L can be collinear. Since $W \subset V_L$, no point of W is collinear with a point of L. As above no two points of W are collinear, in fact $W \cup L$ (and similarly $\tilde{W} \cup L$) is a $(q^2 - q + 1)$-cap of $\mathcal{S}$.

Now examine the points of V_L outside of $W \cup \tilde{W}$. Let $I = \{1, 2, \ldots q\}$ and let $\hat{M} = \{M_j | j \in I\}$ be the set of lines which contain x. Now fix some $i \in I$. Assume for contradiction that M_i has some point p collinear with two points of L. Then p is collinear with all points of L and hence $p \in L^{\perp}$. This contradicts $x \in V_L = V_{L^{\perp}}$. So different points of L are collinear with different points of M_i.

There are $q + 1 + 1 = q + 2$ points of M_i and only q points of L. This leaves exactly 2 points of M_i which are not collinear with anything in L, i.e. $|M_i \cap V_L| = 2$. One such point is x; call the other such point t_i. By construction t_i cannot be in $W \cup \tilde{W}$, as both W and $\tilde{W}$ are caps containing x. For each $j \in I$ construct t_j from M_j as above. Let $T = \{t_j |\ j \in I\}$ and let $X = V_L \backslash (W \cup \tilde{W} \cup T)$ in which case $|X| = 2q(q-1) - (2(q-1)^2 - 1) - q = q - 1$.

Again think of i as fixed in I. Let x' be a point of V_L collinear with t_i such that $x' \neq x$. Then x' cannot be on any $M_j \in \hat{M}$, else a triangle is formed. Thus for every $t_j \in T,\ x' \neq t_j$. Suppose for contradiction that $x' \in W$ and let $\{x, x'\}^{\perp\perp} \cap L = \{u\}$. As t_i is collinear with x and x', t_i must be collinear with u. This contradicts $t_i \in V_L$. Hence $x' \notin W$. Similarly $x' \notin \tilde{W}$. Therefore x' must be in X.

It has been shown that on each line through a point of V_L there is a unique other point also in V_L; this is how t_j was constructed. And observe that $x^{\perp} \cap V_L = T \cup \{x\}$. Now apply this argument to a fixed t_i. Through t_i there are q lines all together. One line, M_i, contains x. The remaining $q-1$ lines, $M_i^1, M_i^2, \ldots, M_i^{q-1}$, each have a unique point other than t_i in V_L, say $x_i^1, x_i^2, \ldots, x_i^{q-1}$, and each of these x_i^k are in X, $k = 1, \ldots, q-1$.

This argument could be applied to any of the q different from $t_j \in T$ to get $x_1^1, \ldots, x_1^{q-1}, x_2^1, \ldots, x_2^{q-1}, \ldots x_q^1, \ldots x_q^{q-1}$, all in X. But as was seen earlier, $|X| = q-1$. This means that for all $j \in I$, $\{x_j^1, \ldots, x_j^{q-1}\} = \{x_i^1, \ldots, x_i^{q-1}\} = X$. The set $X \cup \{x\}$ is a set of q points, all collinear with everything in T. In other words, $T^{\perp} = X \cup \{x\}$ and $T^{\perp\perp} = T$. T and $X \cup \{x\}$ are then hyperbolic lines. Writing $V_L = W \cup \tilde{W} \cup T \cup X$ almost gives a partition for V_L. In fact $V_L = (W \backslash \{x\}) \cup (\tilde{W} \backslash \{x\}) \cup T \cup X \cup \{x\}$ is a partition.

Lemma 4.1.4 $\mathcal{O}_{L,x} = W \cup L \cup X$ and $\mathcal{O}_{L^{\perp},x} = \tilde{W} \cup L^{\perp} \cup X$ are ovoids of $\mathcal{S}$.

Proof: $|\mathcal{O}_{L,x}| = (q-1)^2 + q + (q-1) = q^2 = ((q-1)+1)^2$. Recall that $X \cup \{x\}$ is a hyperbolic line and that $W \cup L$ is a $(q^2 - q + 1)$-cap. Since $X \subset V_L$, no point of X is collinear with a point of L. It needs only to be shown that no point of X is collinear with a point of $W \backslash \{x\}$. Let $x' \in X, v \in W \backslash \{x\}$, and suppose x' is collinear with v. Let $\{v, x\}^{\perp\perp} \cap L = \{u\}$. Every line through x' contains a point of T, say $t_j \in vx'$; then t_j is collinear with all of $\{v, x\}^{\perp\perp}$. Particularly, t_j is collinear with u. This contradicts $t_j \in V_L$. Hence $\mathcal{O}_{L,x}$ is an ovoid of $\mathcal{S}$. Likewise $\mathcal{O}_{L^\perp,x} = \tilde{W} \cup L^\perp \cup X$ is an ovoid of $\mathcal{S}$. ∎

Let $U_L = V_L \cup L \cup L^\perp$. Then $|U_L| = 2q(q-1) + q + q = 2q^2$, i.e., U_L is the size of two disjoint ovoids. In fact if $\tilde{\mathcal{O}}_{L,x} = (\tilde{W} \cup L^\perp \cup T)\backslash\{x\} = (\mathcal{O}_{L^\perp,x}\backslash X \cup \{x\}) \cup (X \cup \{x\})^\perp$, then $\tilde{\mathcal{O}}_{L,x}$ is an ovoid having no point in common with $\mathcal{O}_{L,x}$. U_L can be partitioned into two disjoint ovoids as $\mathcal{O}_{L,x} \cup \tilde{\mathcal{O}}_{L,x}$.

We now proceed to partition the ovoids $\mathcal{O}_{L,x}$ and $\tilde{\mathcal{O}}_{L,x}$ into hyperbolic lines. Notice that the construction of V_L was independent of x, so one could construct analogues of $W, \tilde{W}, T$ and X using any point of V_L. For any point $m \in V_L$, $(m^\perp \cap U_L)\backslash\{m\}$ is a hyperbolic line S. Here S is to m as T is to x. Also for any point $m \in L$, $(m^\perp \cap U_L)\backslash\{m\} = L^\perp$. Finally for any $m \in L^\perp$, $(m^\perp \cap U_L)\backslash\{m\} = L$. Hence for any point $m \in U_L$, $(m^\perp \cap U_L)\backslash\{m\}$ is a hyperbolic line. Call such a hyperbolic line the **corresponding** hyperbolic line to m.

Lemma 4.1.5 Let m, m' be a pair of distinct points of U_L contained in either $\mathcal{O}_{L,x}$ or $\tilde{\mathcal{O}}_{L,x}$. The two hyperbolic lines corresponding to m and m' are either disjoint or identical.

Proof: Without loss of generality, assume $m, m' \in \mathcal{O}_{L,x}$. Let N, N' be the hyperbolic lines corresponding respectively to m and m'; observe $N \cup N' \subset \tilde{\mathcal{O}}_{L,x}$. Let $n \in N \cap N'$. If $n \in L^\perp$, then $m, m' \in L$, which means

$N = N' = L^{\perp}$.

Now consider $n \notin L^{\perp}$ which means $n \in V_L$. Perform analogous constructions for n as were done for x above, in which case V_L consists of the following:

(1) The union of hyperbolic lines containing n and one point of $L \cup L^{\perp}$ (the analogue of $W \cup \tilde{W}$),

(2) The hyperbolic line $S = n^{\perp} \cap V_L$ (the analogue of T), and

(3) The hyperbolic line $S^{\perp} = (N^{\perp} \cap V_L)^{\perp}$ (the analogue of $X \cup \{x\}$).

Since N, N' are hyperbolic lines through n, $(N \cup N') \cap S$ is empty. If $N \cup N'$ contains a point of $L^{\perp}$, either m or m' is collinear with a point of $L^{\perp}$. But the only points of U_L collinear with points of $L^{\perp}$ are points of L; this gives a contradiction as $N \cup N' \subset \tilde{\mathcal{O}}_{L,x}$.

This says that N and N' are hyperbolic lines corresponding to points of U_L and containing no point of $L \cup L^{\perp}$. But there is only one such hyperbolic line, namely $S^{\perp} = N = N'$. Thus in any case if $n \in N \cap N'$ exists, then $N = N'$. This proves the claim. ∎

It has been shown that for any pair of points in one of the partitioning ovoids, the corresponding hyperbolic lines are contained in the other partioning ovoid and are either identical or pairwise disjoint. Since each partitioning ovoid has q^2 points and each hyperbolic line has q points, each partitioning ovoid contains $\frac{q^2}{q} = q$ disjoint hyperbolic lines corresponding to points from the other partitioning ovoid.

Let $L, L_1, \ldots, L_{q-1}$ be the q disjoint hyperbolic lines of $\mathcal{O}_{L,x}$ corresponding to the points of $\tilde{\mathcal{O}}_{L,x}$ as formed above. Likewise let $L^\perp, \tilde{L}_1, \ldots, \tilde{L}_{q-1}$ be the q disjoint hyperbolic lines of $\tilde{\mathcal{O}}_{L,x}$. Since the trace of a hyperbolic line is itself a hyperbolic line, for any $j < q$ there is some $k < q$ with $L_j^\perp = \tilde{L}_k$. Indices can be chosen such that $L_j^\perp = \tilde{L}_j$ for each $j < q$. Therefore forming the set $\mathcal{L}$ consisting of the hyperbolic lines $\{L, L_1, \ldots, L_{q-1}, L^\perp, L_1^\perp, \ldots, L_{q-1}^\perp\}$ is independent of the choice of $x \in V_L$. Choosing some $v \in \mathcal{O}_{L,x} \backslash \{L \cup \{x\}\}$ would yield the same $\mathcal{L}$. There is an ovoid $\mathcal{O}_{L,v}$ similarly determined by l and v.

Lemma 4.1.6 $\mathcal{O}_{L,v} = \mathcal{O}_{L,x}$.

Proof: Consider the hyperbolic line $\{x, v\}^{\perp\perp}$. First suppose $\{x, v\}^{\perp\perp} \notin \{L_1, \ldots, L_{q-1}\}$. In this case $\{x, v\}^{\perp\perp}$ has a point in common with each of $L, L_1, \ldots L_{q-1}$. Let $u' = \{x, v\}^{\perp\perp} \cap L$. $\mathcal{O}_{L,x}$ can be viewed as the union of the q hyperbolic lines in $\mathcal{L}$ which have exactly one point in common with $\{x, u\}^{\perp\perp}$ where u is any point of L.

$\mathcal{O}_{L,v}$ can now be viewed as the union of hyperbolic lines of $\mathcal{L}$ which have exactly one point in common with $\{v, u\}^{\perp\perp}$ where u is any point of

L. For the specific case $u = u'$ observe $\{v, u\}^{\perp\perp} = \{x, u\}^{\perp\perp}$. So $\mathcal{O}_{L,v} = L \cup L_1 \cup \ldots \cup L_{q-1} = \mathcal{O}_{L,x}$. Now suppose $\{x, v\}^{\perp\perp} \in \{L_1, \ldots, L_{q-1}\}$, say $\{x, v\}^{\perp\perp} = L_1$. Let $v' \in L_2$. Then as above $\mathcal{O}_{L,x} = \mathcal{O}_{L,v'}$, and $\mathcal{O}_{L,v} = \mathcal{O}_{L,v'}$, i.e., $\mathcal{O}_{L,v} = \mathcal{O}_{L,x}$. In either case, we have for any $v \in \mathcal{O}_{L,x} \backslash \{L \cup \{x\}\}$, $\mathcal{O}_{L,v} = \mathcal{O}_{L,x}$. ■

Choose indices such that $x \in L_1$. Note that as above, $\mathcal{O}_{L^\perp,x}$ is the union of the q hyperbolic lines from $\mathcal{L}$ which have exactly one point in common with $\{x, \tilde{u}\}^{\perp\perp}$ for any $\tilde{u} \in L^\perp$. Hence $\mathcal{O}_{L^\perp,x} = L^\perp \cup L_1 \cup L_2^\perp \cup \ldots \cup L_{q-1}^\perp$ This can be generalized.

Lemma 4.1.7 If $u \in L_i$, $\mathcal{O}_{L^\perp,u}$ can be partitioned as $L^\perp \cup L_1^\perp \cup \ldots L_{i-1}^\perp \cup \Lambda_i \cup L_{i+1}^\perp, \ldots, L_{q-1}^\perp$.

Lemma 4.1.8 $q \leq 3$.

Proof: Suppose $q > 3$ and choose a point $w \in \mathcal{O}_{L^\perp,x}$ such that $w \notin L^\perp \cup L_1 \cup L_2^\perp$, say $w \in L_3^\perp$. By lemma 4.1.6, $\mathcal{O}_{L^\perp,x} = \mathcal{O}_{L^\perp,w}$. Let $u \in L_2$; applyingLemma 4.1.7 shows that $\mathcal{O}_{L^\perp,u} = L^\perp \cup L_1^\perp \cup L_2 \cup L_3^\perp \cup \ldots L_{q-1}^\perp$ and $w \in \mathcal{O}_{L^\perp,u}$. Again by lemma 4.1.6 $\mathcal{O}_{L^\perp,u} = \mathcal{O}_{L^\perp,w}$ and transitivly, $\mathcal{O}_{L^\perp,x} = \mathcal{O}_{L^\perp,u}$. This means that $L_2 \subset \mathcal{O}_{L^\perp,x}$.

$\mathcal{O}_{L^\perp,x}$ contains a hyperbolic line, L_2, as well as its perp, $L_2^\perp$. This is a contradiction as all points of L_2 are collinear with all points of $L_2^\perp$ but no two points of an ovoid, $\mathcal{O}_{L^\perp,x}$, may be collinear. ■

The contradiction above arose from the existence of a $w \in L_3^\perp$. Of course such a w is available only if $q > 3$. In [PT84] it is shown that the only $GQ(4,2)$ is classical and has all points regular. Hence the main theorem of this section has now been proved:

Theorem 4.1.9 $\mathcal{S}$ is a thick $GQ(q+1, q-1)$ with all points regular if and only if $q = 3$.

4.2 Regular pairs

In this section, pairs of regular points are examined. The goal of the remainder of this chapter is to show that if a $GQ(q+1, q-1)$ has at least two regular ovoids pivotal for the same fan, then every regular ovoid of the GQ is pivotal for that fan.

Let I be the index set $\{0, 1, \ldots, q\}$,and $\tilde{I} = I \cup \{\infty\}$. Let $\mathcal{S}$ be a thick $GQ(q+1, q-1)$ with a fan $\mathcal{M} = \{\mathcal{O}_i | i \in \tilde{I}\}$ for which $\mathcal{O}_\infty$ is pivotal. Recall from section 3.3 that constricting about $\mathcal{O}_\infty$ yields a $GQ(q,q)$, $\mathcal{S}_\infty$, and that L_∞ is a regular line of $\mathcal{S}_\infty$.

Theorem 4.2.1 Suppose that $\{x_1, x_2\}$ is a regular pair of points in $\mathcal{S}$ with $x_1 \not\sim x_2$. Put $T = \{x_1, x_2\}^{\perp\perp} = \{x_1, \ldots, x_q\}$, and $T^\perp = \{y_1, \ldots, y_q\}$. Then exactly one of the following must occur:

(i) There are distinct $i, j \in \tilde{I}$ such that $T \subseteq \mathcal{O}_i$ and $T^\perp \subseteq \mathcal{O}_j$.

(ii) Exactly one of T, $T^\perp$ has a unique point in common with $\mathcal{O}_\infty$. Without loss of generality, assume $T^\perp \cap \mathcal{O}_\infty = \{y_1\}$ and $T \cap \mathcal{O}_\infty = \emptyset$. In this case there is a unique $\mathcal{O}_i \in \mathcal{M}$ for which $T^\perp \setminus \{y_1\} \subseteq \mathcal{O}_i$. $|T \cap \mathcal{O}_j| = 1$ for each $j \in I \backslash \{i\}$.

Proof: Since $\mathcal{O}_\infty$ is pivotal, if either T or $T^\perp$ has two points in common with $\mathcal{O}_\infty$, then it is contained in $\mathcal{O}_\infty$ and its perp is contained in some $\mathcal{O}_i$. If $T \cap \mathcal{O}_\infty = \emptyset = T^\perp \cap \mathcal{O}_\infty$, then by proposition 3.2.2 there must be distinct $i, j \in I$ for which $T \subseteq \mathcal{O}_i$ and $T^\perp \subseteq \mathcal{O}_j$. In the remaining case, exactly one of T, $T^\perp$ has a unique point in common with $\mathcal{O}_\infty$. Without loss of generality assume that $T^\perp \cap \mathcal{O}_\infty = \{y_1\}$.

There is some $i \in I$ for which $T^\perp \setminus \{y_1\} \subseteq \mathcal{O}_i$, since otherwise by proposition 3.2.2 T would have to meet $\mathcal{O}_\infty$, clearly an impossibility. Also $y_2, \ldots, y_q$ must be in traces of distinct hyperbolic lines of $\mathcal{O}_\infty$, which means they are on distinct lines ($\neq L_\infty$) through $(\mathcal{O}_i)$ in $\mathcal{S}_\infty$. By proposition 3.2.2 $|T \cap \mathcal{O}_j| = 1$ for each $j \in I \backslash \{i\}$. ■

Hence, using the notation above, in $\mathcal{S}_\infty$, $\{y_2, \ldots, y_q\} \subseteq \{(\mathcal{O}_i), x_1, \ldots, x_q\}^\perp$ and $\{y_2, \ldots, y_q\}^\perp = \{(\mathcal{O}_i), x_1, \ldots, x_q\}$. Let T_0 be the line of $\mathcal{S}_\infty$ through $(\mathcal{O}_i)$

different from L_∞, and missing $\{y_2, \ldots, y_q\}$. The lines of $\mathcal{S}$ through y_1 together with L_∞ form one ruling of a grid $\mathcal{G}$. For ease of notation, suppose that $\mathcal{O}_i = \mathcal{O}_0$, and $x_j \in \mathcal{O}_j$, $1 \le j \le q$. Then the other ruling of $\mathcal{G}$ consists of $y_1^\perp \cap \mathcal{O}_k = T_k$, $0 \le k \le q$, and $x_j \in \mathcal{O}_j \cap y_1^\perp$.

Observation 4.2.2 If $y_1 \in \mathcal{O}_\infty$ and $y_2 \in \mathcal{O}_i$ and if $\{y_1, y_2\}$ is regular in $\mathcal{S}$, then $\{y_1, y_2\}^{\perp\perp} \setminus \{y_1\} \subseteq \mathcal{O}_i$.

Corollary 4.2.3 With the notation adopted above, if for some $i \in I$, $\mathcal{O}_i$ is regular, then it is pivotal for $\mathcal{M}$.

Proof: If $\mathcal{O}_i$ is regular for some $i \in I$, and x, y are distinct points of $\mathcal{O}_i$, then $\{x, y\}$ is regular and $\{x, y\}^{\perp\perp} \subseteq \mathcal{O}_i$. This does not permit the second case of the above theorem to hold, hence the first case must hold. ■

Now suppose that $\mathcal{O}$ is a regular ovoid of $\mathcal{S}$ possibly different from $\mathcal{O}_\infty$. Recall that $\pi(\mathcal{O}_\infty)$ and $\pi(\mathcal{O})$ are affine planes whose "lines" are the hyperbolic lines contained in them. Clearly $\pi(\mathcal{O}_\infty) \cap \pi(\mathcal{O})$ is a subspace of $\pi(\mathcal{O}_\infty)$ and of $\pi(\mathcal{O})$. This proves the following lemma.

Lemma 4.2.4 Any regular ovoid of $\mathcal{S}$ different from $\mathcal{O}_\infty$ meets $\mathcal{O}_\infty$ in exactly $0,\ 1$ or q points.

The three possibilities for a regular ovoid meeting $\mathcal{O}_\infty$ are considered separately in the next three lemmas.

Lemma 4.2.5 If $\mathcal{O}$ is a regular ovoid with $|\mathcal{O} \cap \mathcal{O}_\infty| = q$, then $\mathcal{O} = T_1 \cup T_2^\perp \cup \cdots \cup T_q^\perp$, where $\mathcal{O} \cap \mathcal{O}_\infty = T_1$, and $\mathcal{O}_\infty = T_1 \cup T_2 \cup \cdots \cup T_q$ is a parallel class partition of the points of $\pi(\mathcal{O}_\infty)$. Hence $\mathcal{O}_k = T_1^\perp \cup T_2^\perp \cup T_q^\perp$ for some $k \in I$.

Proof: Suppose that $\mathcal{O}_\infty = \{z_1, \ldots, z_q, x_{q+1}, \ldots, x_{2q}, \ldots, x_{q^2}\}$, where $T_1 = \{z_1, \ldots, z_q\}$, $T_2 = \{x_{q+1}, \ldots, x_{2q}\}$, $\ldots$, $T_q = \{x_{(q-1)q+1}, \ldots, x_{q^2}\}$ are the disjoint hyperbolic lines of one parallel class of $\pi(\mathcal{O}_\infty)$, and $\mathcal{O} = \{z_1, \ldots, z_q, y_{q+1}, \ldots, y_{q^2}\}$, with $\mathcal{O} \cap \mathcal{O}_\infty = T_1$. Let $y \in \mathcal{O} \cap \mathcal{O}_k$ for some $k \in I$. Then for each $z_i \in \mathcal{O}$, $\{z_i, y\}^{\perp\perp} \setminus \{z_i\} \subseteq \mathcal{O}_k$ by theorem 4.2.1, so $\cup_{i=1}^q \left(\{z_i, y\}^{\perp\perp} \setminus \{z_i\}\right)$ gives $1 + q(q-2) = q^2 - 2q + 1$ points of $\mathcal{O} \cap \mathcal{O}_k$.

Since $2(q^2 - 2q + 1) > q^2 - q$ for $q \geq 3$, $\mathcal{O} \setminus \mathcal{O}_\infty$ must be contained in just one $\mathcal{O}_k$, i.e., $|\mathcal{O} \cap \mathcal{O}_k| = q^2 - q$, and hence $\mathcal{O} \cap \mathcal{O}_k = \{y_{q+1}, \ldots, y_{q^2}\}$. Clearly no y_j is collinear with any z_i. From this, $y_j^\perp \cap \mathcal{O}_\infty = \{x_{n(q+1)}, \ldots, x_{(n+1)q}\}$ is disjoint from T_1. thus the parallel class of $\mathcal{O}_\infty$ containing T_1 must contain $q-1$ hyperbolic lines $T_2, \ldots, T_q$ whose traces cover $\mathcal{O} \cap \mathcal{O}_k$. So $\mathcal{O} \cap \mathcal{O}_k = T_2^\perp \cup \cdots T_q^\perp$; $\mathcal{O}$ was constructed to contain T_1. Finally $\mathcal{O}_k \in \mathcal{M}$ implies $\mathcal{O}_k$ consists of the traces of the hyperbolic lines contained in some specific parallel class of $\pi(\mathcal{O}_\infty)$. Hence:

$$\mathcal{O}_\infty = T_1 \cup \cdots \cup T_q,$$

$$\mathcal{O}_k = T_1^\perp \cup \cdots \cup T_q^\perp,$$

$$\mathcal{O} = T_1 \cup T_2^\perp \cup \cdots \cup T_q^\perp.$$

This completes the proof of the theorem. ■

Corollary 4.2.6 Continuing with the notation of the above proof, let $x_1 \in T_1$, $x_2 \in T_2^\perp$, $\{x_1, x_2\}^{\perp\perp} = T$. Then for $j \in I \backslash \{k\}$, $|T^\perp \cap \mathcal{O}_j| = 1$.

Proof: By hypothesis $\{x_1, x_2\}^{\perp\perp} \subseteq \mathcal{O}$. If two points of $\{x_1, x_2\}^{\perp\perp}$ belong to the same $T_j^\perp$, then of course $\{x_1, x_2\}^{\perp\perp}$ would have to equal $T_j^\perp$. Hence $T = \{x_1, x_2\}^{\perp\perp}$ has one point in each of $T_1, T_2^\perp, \ldots, T_q^\perp$. Now let $\{x_1, x_2\}^\perp = \{w_1, \ldots, w_q\} = T^\perp$. Clearly $T^\perp \cap \mathcal{O}_\infty = T^\perp \cap \mathcal{O}_k = \emptyset$, since both $\mathcal{O}_\infty$ and $\mathcal{O}_k$ are ovoids. If two points of $T^\perp$ belong to the same $\mathcal{O}_j$, then since $T \cap \mathcal{O}_\infty \neq \emptyset$, it would have to be that $T \subseteq \mathcal{O}_\infty$. Hence $|T^\perp \cap \mathcal{O}_j| = 1$ for each $j \in I \backslash \{k\}$. ■

Theorem 4.2.7 If $\mathcal{O}$ is a regular ovoid with $|\mathcal{O} \cap \mathcal{O}_\infty| = 1$, then $|\mathcal{O} \cap \mathcal{O}_i| = q-1$ for $0 \leq i \leq q$. If $\mathcal{O} \cap \mathcal{O}_\infty = \{z\}$ and $x \in \mathcal{O} \cap \mathcal{O}_i$, then $\{z, x\}^{\perp\perp} \setminus \{z\} = \mathcal{O} \cap \mathcal{O}_i$.

Proof: Suppose $\mathcal{O} \cap \mathcal{O}_\infty = \{z\}$. From the observation following theorem 4.2.1 we see that if $x \in \mathcal{O} \setminus \mathcal{O}_\infty$, say $x \in \mathcal{O}_i$, then $\{z,x\}^{\perp\perp} \setminus \{z\} \subseteq \mathcal{O}_i$. First suppose that $\{z,x\}^{\perp\perp} \setminus \{z\} \subseteq \mathcal{O} \cap \mathcal{O}_i$, $\{z,y\}^{\perp\perp} \setminus \{z\} \subseteq \mathcal{O} \cap \mathcal{O}_i$, and $z \notin \{x,y\}^{\perp\perp}$, hence $\{x,y\}^{\perp\perp} \cap \mathcal{O}_\infty = \emptyset$. Say:

$$\{z,x\}^{\perp\perp} = \{z, x = x_2, \ldots, x_q\}; \{z,y\}^{\perp\perp} = \{z, y = y_2, \ldots, y_q\}.$$

Since $\{x,y\}^{\perp\perp} \cap \mathcal{O}_\infty = \emptyset$, if $\{x,y\}^{\perp} \cap \mathcal{O}_\infty = \emptyset$ also, then $\{x,y\}^{\perp} \subseteq \mathcal{O}_j$ for some $j \in I \setminus \{i\}$, and $\{x,y\}^{\perp\perp} \subseteq \mathcal{O}_i$. On the other hand, if $\{x,y\}^{\perp} \cap \mathcal{O}_\infty \neq \emptyset$, then $\{x,y\} \subseteq \mathcal{O}_i$ implies $\{x,y\}^{\perp} \subseteq \mathcal{O}_\infty$ and $\{x,y\}^{\perp\perp} \subseteq \mathcal{O}_i$. In any case the following holds : For $x, y \in \mathcal{O} \cap \mathcal{O}_i$, $\{x,y\}^{\perp\perp} \subseteq (\mathcal{O} \cap \mathcal{O}_i) \cup \{z\}$. Thus $(\mathcal{O} \cap \mathcal{O}_i) \cup \{z\}$ is a subspace of $\pi(\mathcal{O})$. Hence if $|\mathcal{O} \cap \mathcal{O}_i| \geq q$, then $\mathcal{O} = (\mathcal{O} \cap \mathcal{O}_i) \cup \{z\}$.

Suppose $\mathcal{O}_i = T_1^{\perp} \cup \cdots \cup T_q^{\perp}$, where $\mathcal{O}_\infty = T_1 \cup \cdots \cup T_q$. Say $T_1 = \{z_1, \ldots, z_q\}$, and z_1 is the point of $\mathcal{O}_i$ for which $(\mathcal{O}_i \setminus \{z_1\}) \cup \{z\} = \mathcal{O}$. Then the lines of $\mathcal{S}$ covered by z_1 are the same lines as those covered by z, an obvious impossibility as $\{z, z_1\}$ determines at most one line. Hence it must be that for each $i \in I$, $|(\mathcal{O} \cap \mathcal{O}_i)| = q-1$, and if $x \in \mathcal{O} \cap \mathcal{O}_i$, then $\{z,x\}^{\perp\perp} \setminus \{z\} = \mathcal{O} \cap \mathcal{O}_i$. ■

Theorem 4.2.8 Let $\mathcal{O}$ be a regular ovoid with $|\mathcal{O} \cap \mathcal{O}_\infty| = 0$. Then exactly one of the following two possibilities must hold:

(i) $\mathcal{O}$ is some $\mathcal{O}_i \in \mathcal{M},$ in which case

$\mathcal{O}_i$ is pivotal for the fan $\mathcal{M}$;

(ii) There is a unique $\mathcal{O}_i, i \in I,$ for which

$|\mathcal{O} \cap \mathcal{O}_\infty| = |\mathcal{O} \cap \mathcal{O}_i| = 0,$ and $|\mathcal{O} \cap \mathcal{O}_j| = q$

for all $j \in I \backslash \{i\}$. In this case $\mathcal{O} \cup \{(\mathcal{O}_i)\}$

is an ovoid of $\mathcal{S}_\infty$.

Proof: Suppose $\mathcal{O} \cap \mathcal{O}_\infty = \emptyset$. Let x, y be distinct points of $\mathcal{O}$, thus $T = \{x, y\}^{\perp\perp} \subseteq \mathcal{O}$, and $T \cap \mathcal{O}_\infty = \emptyset$. If $T^\perp \cap \mathcal{O}_\infty = \emptyset$, then $T \subseteq \mathcal{O}_i$ for some $i \in I$, and $T^\perp \subseteq \mathcal{O}_k$ for some $k \in I \backslash \{i\}$. If $T^\perp \cap \mathcal{O}_\infty \neq \emptyset$, then either $T^\perp \subseteq \mathcal{O}_\infty$ and $T \subseteq \mathcal{O}_i$ for some i, or $|T^\perp \cap \mathcal{O}_\infty| = 1$ and $T^\perp \setminus \mathcal{O}_\infty \subseteq \mathcal{O}_i$ for some $i \in I$ while $|T \cap \mathcal{O}_j| = 1$ for each $j \in I \backslash \{i\}$. In all cases, if x, y are distinct points of $\mathcal{O} \cap \mathcal{O}_i$ for some i, then $\{x, y\}^{\perp\perp} \subseteq \mathcal{O} \cap \mathcal{O}_i$. Hence $\mathcal{O} \cap \mathcal{O}_i$ is a subspace of $\pi(\mathcal{O})$, and must have size $0, 1, q$ or q^2. One possibility is that $\mathcal{O} = \mathcal{O}_i \in \mathcal{M}$.

Suppose that $\mathcal{O} \notin \mathcal{M}$; $|\mathcal{O} \cap \mathcal{O}_i| = 0, 1$ or q, for each $i \in I$. Let a_n be the number of $\mathcal{O}_j$ with $|\mathcal{O} \cap \mathcal{O}_j| = n \in \{0, 1, q\}$, for $j \in I$. Clearly each $a_n \geq 0$, and

(i) $a_0 + a_1 + a_q = q + 1,$

(ii) $0 \cdot a_0 + 1 \cdot a_1 + q \cdot a_q = q^2,$

from which $a_1 = q(q - a_q)$. Put $m = q - a_q$, so $a_1 = mq$. It follows that $q - m = a_q = q + 1 - a_0 - a_1$, implying $a_1 + a_0 = m + 1 = a_0 + mq$, or $k(q-1) = 1 - a_0$, which must be nonnegative. This allows only two possibilities: $a_0 = 1$ or 0.

If $a_0 = 1$, then $m = 0 = a_1$ and $a_q = q$. In this case $\mathcal{O}$ is disjoint from $\mathcal{O}_\infty$ and from one other member of $\mathcal{M}$, say $\mathcal{O}_i$, and meets each of the others $\mathcal{O}_1, \ldots, \mathcal{O}_q$ in q points, i.e. in an affine line of $\pi(\mathcal{O})$. It is straightforward to check that $\mathcal{O} \cup \{(\mathcal{O}_i)\}$ is an ovoid of $\mathcal{S}_\infty$. For the other case, suppose $a_0 = 0$. Here $m = 1$ and $q = 2$. But as only thick GQ are under consideration, $q > 2$. So in fact this case not occur. ■

Suppose now that in addition to having $\mathcal{O}_\infty$ pivotal for the fan $\mathcal{M}$, the ovoid $\mathcal{O}_0$ is regular and hence also pivotal for $\mathcal{M}$.

Theorem 4.2.9 Suppose $\{x, y\}$ is a regular pair of points with $x \not\sim y$. Put $T = \{x, y\}^{\perp\perp}$, $T^\perp = \{x, y\}^\perp$. Then there must exist distinct $i, j \in \tilde{I}$ for which $T \subseteq \mathcal{O}_i$, $T^\perp \in \mathcal{O}_j$.

Proof: Suppose not. Then one of $T^\perp$, T has a unique point in common with $\mathcal{O}_\infty$, say $|T^\perp \cap \mathcal{O}_\infty| = 1$. Then there is some $i \in I$ for which $|T^\perp \cap \mathcal{O}_i| = q - 1$. Clearly $\mathcal{O}_i$ is not regular, so $i \neq 0$. Then $|T \cap \mathcal{O}_j| = 1$ for each $j \in I \backslash \{i\}$, including $j = 0$. But $|T \cap \mathcal{O}_0| = 1$ says there must be some

$k \in I\backslash\{0\}$ with $|T \cap \mathcal{O}_k| = q-1$, an impossibility for $q > 2$. ■

This leads to the main theorem of this section.

Theorem 4.2.10 Let $\mathcal{O}_\infty$ and $\mathcal{O}_0$ be distinct pivotal ovoids of $\mathcal{M}$; if $\mathcal{O}$ is a regular ovoid of $\mathcal{S}$, then in fact $\mathcal{O}$ is a pivotal member of $\mathcal{M}$.

Proof: Suppose $\mathcal{O}$ is a regular ovoid with $\mathcal{O} \notin \mathcal{M}$. Then $|\mathcal{O} \cap \mathcal{O}_\infty| = 0, 1$ or q, and $|\mathcal{O} \cap \mathcal{O}_0| = 0, 1$, or q.

Case (i) $|\mathcal{O} \cap \mathcal{O}_\infty| = q$. In this case there is a $k \in I$ with $|\mathcal{O} \cap \mathcal{O}_k| = q^2 - q$; $k \neq 0$ and $|\mathcal{O} \cap \mathcal{O}_0| = 0$. But theorem 4.2.8 applied to $\mathcal{O}_0$ in place of $\mathcal{O}_\infty$ says there is some $j \in \tilde{I}\backslash\{0\}$ with $|\mathcal{O} \cap \mathcal{O}_j| = 0$ and $|\mathcal{O} \cap \mathcal{O}_m| = q$ for $m \in \tilde{I}\backslash\{0, j\}$. Hence this case cannot arise.

Case (ii) $|\mathcal{O} \cap \mathcal{O}_\infty| = 1$. Here $|\mathcal{O} \cap \mathcal{O}_i| = q-1$ for $0 \leq i \leq q$. But as $|\mathcal{O} \cap \mathcal{O}_0| \neq q-1$ if $q > 2$, this case cannot arise.

Case (iii) $|\mathcal{O} \cap \mathcal{O}_\infty| = 0$. Then also it follows that $|\mathcal{O} \cap \mathcal{O}_0| = 0$. If $\mathcal{O} \notin \mathcal{M}$ then $|\mathcal{O} \cap \mathcal{O}_j| = q$ for all $j \in I\backslash\{0\}$. But suppose $x \in \mathcal{O} \cap \mathcal{O}_j$, $y \in \mathcal{O} \cap \mathcal{O}_k$, with j and k distinct members of $I\backslash\{0\}$. Then $\{x, y\}^{\perp\perp}$ does not belong to a single member of $\mathcal{M}$, contradicting theorem 4.2.9. Hence this case does not arise, completing the proof. ■

5. Characterizations

In [ST84], De Soete and Thas gave characterization axioms for $GQ(q-1, q+1)$ (with no restriction on q). This inspired Payne to formulate axioms $A_1 - A_7$ characterizing the known $GQ(q+1, q-1)$. Namely the GQ satisfying $A_1 - A_7$ are exactly the known $GQ(q+1, q-1)$ arising from a q-arc as described in section 2.4. De Soete and Thas then revised their approach [ST86] and consequently Payne's $A7$ was made redundant. In [PM98] a relationship was discovered between Payne's axioms and conditions on admissible permutations giving plane amalgamations. In this chapter, these axiom systems are reviewed.

5.1 $(0, 2)$-sets

The motivation for this and the next section is the desire to embed an abstract GQ into affine or projective space. One of the main hurdles in doing this is determining how to represent sets of non-collinear points of the GQ as affine/projective lines. If the points in question form regular pairs, then the hyperbolic line they determine may suffice. However as seen in the previous chapter, not all pairs will be regular; hence a new notion is needed: an abstraction of a hyperbolic line. De Soete and Thas give one such notion with their interesting concept of a $(0, 2)$-set. What follows here is a compilation of some of their ideas from [ST84] and [ST86].

If $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a $GQ(s, t)$, then a nonempty subset K of $\mathcal{P}$ is said to be a $(0, 2)$**-set** of $\mathcal{S}$ provided the points of K are pairwise noncollinear and $|x^\perp \cap K| \in \{0, 2\}$ for all $x \in \mathcal{P} \backslash K$. Observe that in the GQ $T_2^*(\Omega^+)$, if L is a line of $\mathcal{G}$ which meets Π_∞ in a point outside of Ω^+ then the points of $L \backslash \Pi_\infty$ form a $(0, 2)$-set.

Lemma 5.1.1 If K is a (0,2)-set of $\mathcal{S}$ then $|K| = s + 1$.

Proof: Let $x \in K$, and let L be a line through x. For $x \neq k \in K$, k is collinear with a unique point p of L. As K is a $(0, 2)$-set, p is collinear with no point of $K \backslash \{x, k\}$. As there is a bijection between points of K and points of L, $|K| = |L| = s + 1$ ∎

Observe that if L is a line which completely misses K then each point of K is collinear with some point p of L. Since K is a $(0, 2)$-set, p is collinear with exactly one other point of K. Thus as the points of K arise in disjoint pairs, $s + 1$ must be even. This gives the following lemma.

Lemma 5.1.2 If K is a $(0, 2)$-set of $\mathcal{S}$ with $t > 1$ then s is odd. (Having $t > 1$ allows the existence of such an L mentioned above).

The first axiom used ensures a collection of $(0, 2)$-sets partitioning the noncollinear point pairs in $\mathcal{S}$:

Basic Axiom: $\mathcal{S}$ contains a collection $\mathcal{B}_1$ of $(0,2)$-sets such that each pair of noncollinear points of $\mathcal{S}$ is in a unique member of $\mathcal{B}_1$.

Observation 5.1.3 Let $\mathcal{P}' = \mathcal{P}, \mathcal{B}' = \mathcal{B} \cup \mathcal{B}_1$. $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \in)$ is a $2-((s+1)(st+1), s+1, 1)$ design with s odd. In the case where $s = q-1$ and $t = q+1$ observe that such a design has the parameters of affine 3-space (and that q is even). Members of $\mathcal{B}_1$ will be referred to as blocks.

The goal of the remaining part of this section is to see that $\mathcal{S}'$ really is affine 3-space. By the Basic Axiom if $x \in \mathcal{P}$ and $L \in \mathcal{B}$ with $x \notin L$ then there exists a subset of s $(0,2)$-sets in $\mathcal{B}_1$, say $\mathcal{L}_{x,L} = L_1, \dots L_s$, containing x and a point of L. The next axiom relates lines skew to L with these $(0,2)$-sets (Observe that if L' is a line skew to L and meeting each member of $\mathcal{L}$, then $\mathcal{L}_{x,L} = \mathcal{L}_{x,L'}$).

Axiom T: Let $x \notin M \in \mathcal{B}$ with $M \cap L = \emptyset$. If there are two distinct $(0,2)$-sets in $\mathcal{L}_{x,L}$ which are disjoint from M then for every $L_i \in \mathcal{L}_{x,L}$, L_i is disjoint from M.

Let i, j be distinct indices in $\{1, \dots s\}$ with $y_i = L_i \cap L$, $y_j = L_j \cap L$. As L_j is a $(0,2)$-set, $y_i^{\perp}$ contains y_j and some other point (not on L) y_j' of L_j. Likewise as L_i is a $(0,2)$-set, $y_j'^{\perp}$ contains y_i and some other point y_i' of L_i.

Let M be the line of $\mathcal{S}$ determined by y_i' and y_j' and observe that $L \cap M = \emptyset$, otherwise a triangle is formed. This proves the following:

Lemma 5.1.4 For $x \in \mathcal{P}, L \in \mathcal{B}$ with $x \notin L$ and for each pair of distinct indices $i, j \in \{1, \ldots, s\}$ there is a line $M \in \mathcal{B}$ which misses L and which meets both L_i and L_j.

The next lemma can be used to weaken the hypothesis of Axiom T. Unless stated otherwise, for $1 \leq i \leq s$, y_i is defined as the intersection of L with L_i.

Lemma 5.1.5 If $M \in \mathcal{B}$ and if at least two different i and j give $M \cap L_i \neq \emptyset \neq M \cap L_j$ then for each $h \in \{1, \ldots, s\}$, $M \cap L_h \neq \emptyset$.

Proof: If M misses L, then the conclusion follows from Axiom T. Consider the case where M and L have a common point, say $y_j \in L_j$. Suppose M misses some $(0,2)$-set $L_r \in \mathcal{L}$. Let $y_r = L_r \cap L$. Since y_j is collinear with y_r, L_r must have some other point y_r' in $y_j^{\perp}$. Likewise y_r' must be collinear with some other point y_j' of L_j. Let L' be the line of $\mathcal{S}$ determined by y_j' and y_r'. By Axiom T, L' must meet every member of $\mathcal{L}$. Observing that $L' \cap M = \emptyset$ and applying Axiom T to L' forces M to meet every member of $\mathcal{L}$. This contradicts $M \cap L_r = \emptyset$. Thus M must in fact meet all of the members of $\mathcal{L}$. ■

With the above lemma proved, it makes sense to replace Axiom T

with the above lemma taken as an axiom referred to as **Axiom** T^*. Now observe that this axiom forces regularity of lines.

Lemma 5.1.6 If M and L are as in Axiom T^*, then they form a regular pair.

Proof: If M meets L then by definition they form a regular pair. Assume M misses L. For distinct $k, r \in \{1, \ldots, s\}$ there is a unique point $y_{kr} = y_k^\perp \cap L_r,\ y_{kr} \neq y_r$. By Axiom T^* for a fixed k, $\{y_{kr} | r \neq k\} \cup \{y_k\}$ is a set of s mutually collinear points. Call the line they determine N_k. In this way there arise s lines, $N_1, \ldots, N_s$ each of which have a point in common with each L_i. The lines $N_1, \ldots N_s$ then completely cover the points of the various L_i, and the points of M are likewise covered by the lines N_i.

Let $z = L \cap x^\perp$; for $1 \leq i \leq s+1$ let $z_i = N_i \cap M$ where N_{s+1} is the line of $\mathcal{S}$ determined by x and z. To see that N_{s+1} really does meet M, observe that for $1 \leq i \leq s,\ z_i$ cannot be collinear with z (as z_i is already collinear with the point y_i on L). x and z must both be collinear with the unique point of M which is not a z_i for $1 \leq i \leq s$. This unique point is exactly the point z_{s+1}.

For $N_k \in \{L, M\}^\perp$ different from N_{s+1} observe that $y_{kr} = N_k \cap L_r,\ k \neq r$, and $y_k = N_k \cap L_k \cap L$. As each L_k is a $(0,2)$-set, there exists some $y'_{kr} \in L_k$ different from y_k such that y_{kr} is collinear with y'_{kr}. By Axiom T, the line determined by y_{kr} and y'_{kr} has one point in common with each of the $L_i \backslash \{x\}$. For a fixed k as r varies over $\{1, \ldots, s\} \backslash \{k\}$, this gives $s-1$ disjoint lines K_r.

Let $\mathcal{K}$ denote this set of lines.

Note that as these lines together with L form a set of s disjoint lines which cover the s^2 points of the various L_i, M must be in $\mathcal{K}$. It follows that $\mathcal{K} \cup \{L\}$ form one ruling of an $(s \times s)$ grid G with the other ruling given by $N_1, \ldots, N_s$. As L and M both meet N_{s+1}, each line of $\mathcal{K}$ must also meet N_{s+1}.

Let w be the unique point of N_1 collinear with x (i.e. W is the only point of N_1 not on some L_i. Let K be the line of $\mathcal{S}$ determined by x and w, then in fact K contains the unique point on each N_k which is not on some L_i. Observe now that N_{s+1} and K extend G to an $(s+1 \quad \times \quad s+1)$ grid $G(x, L) = \{L, M\}^{\perp} \cup \{L, M\}^{\perp\perp}$ with $\{L, M\}^{\perp\perp} = \mathcal{K} \cup \{L, K\}$ (where M is in fact one of the $s-1$ lines of $\mathcal{K}$). Hence the pair $\{L, M\}$ is regular. ■

With this proved, now an affine plane can be constructed.

Lemma 5.1.7 In the notation of the previous lemma, let $\mathcal{P}^* = \{y \in \mathcal{P} | y$ is on a line of $\{L, M\}^{\perp}\}$ and let $\mathcal{B}^* = G(x, L) \cup \mathcal{B}_1^*$ where $\mathcal{B}_1^*$ consists of all $(0,2)$-sets of $\mathcal{B}_1$ which contain at least two points of $\mathcal{P}^*$. The incidence structure given by $(\mathcal{P}^*, \mathcal{B}^*, \in)$ is a $2-((s+1)^2, (s+1), 1)$ design, i.e. an affine plane of order $s+1$ with s odd.

Proof: It is straightforward to see that $|\mathcal{P}^*| = (s+1)^2$ and that each block of $\mathcal{B}^*$ contains $s+1$ points. It remains to be seen that any two points $\mathcal{P}^*$ are contained in a unique block of $\mathcal{B}^*$.

Let $u, v \in \mathcal{P}^*$. If u is collinear with v in $\mathcal{S}$, then the line they determine in $\mathcal{S}$ is precisely the block of $\mathcal{B}^*$ that contains them. Suppose now that u and v are not collinear in $\mathcal{S}$, by the Basic Axiom, there is a unique $(0,2)$-set in $\mathcal{B}_1$ which contains u and v. This $(0,2)$-set is then necessarily in $\mathcal{B}_1^*$. What follows demonstrates that this $(0,2)$-set has all of its points contained in $\mathcal{P}^*$.

Without loss of generality say $u \notin L$. If either u or v is x, the block containing u and v is obvious. Assume $u \neq x \neq v$. In what follows, we see that the grids $G(x, L), G(u, L)$, and $G(v, L)$ are all the same.

Case 1: u and x are not collinear in $\mathcal{S}$. Let L_k be the $(0,2)$-set of $\mathcal{B}_1$ containing both u and x; observe that the points of L_k are in both grids $G(x, L)$ and $G(u, L)$. Let N_r be the line through u meeting L and M; then N_r is a member of both grids $G(x, L)$ and $G(u, L)$. Let y_r' be a point of N_r distinct from u and y_r. As y_r' is collinear with the point u of L_k, it is also collinear with some other point u' of L_k. This puts L, N_r, y_r', and u' in both grids. Hence the two grids $G(x, L)$ and $G(u, L)$ are actually identical.

Case 2: u and x lie on the line K. Pick $w \in G(x, L) \backslash \{L\}$ such that w is not collinear with x nor with u. As seen above $G(x, L) = G(w, L)$; hence $u \in G(w, L)$. ■

Observation 5.1.8 The affine plane above is a substructure of the design $\mathcal{S}'$; for any non-incident pair $(x, L) \in \mathcal{P} \times \mathcal{B}$ such an affine plane can be constructed.

Furthermore, the following corollary arises.

Corollary 5.1.9 If $x \in \mathcal{P}, L \in \mathcal{B}_1, x \notin L$ with $|x^\perp \cap L| = 2$ then the substructure of $\mathcal{S}'$ generated by x and L is an affine plane of order $s + 1$.

Proof: Let $x^\perp \cap L = \{y_1, y_2\}$ and let L' be the line of $\mathcal{S}$ through x and y_2. The substructure of $\mathcal{S}'$ generated by y_1 and L' is contained in the substructure of $\mathcal{S}'$ generated by x and L; and vice-versa. But the first of these is an affine plane of order $s + 1$ as indicated above. ■

The term **Type 1** will be used to describe the affine planes of order $s+1$ as constructed in the preceding lemma, observation, and corollary. Each point in such a plane is incident with two affine lines coming from $\mathcal{P}$ and s affine lines coming from $\mathcal{B}_1$. Moreover for a non-incident point-line pair (x, L') in such a plane, if L' is a block, then x is collinear in $\mathcal{S}$ with two points of L'.

Lemma 5.1.10 Let α be a Type 1 affine plane in $\mathcal{S}'$ whose point set is denoted $\mathcal{P}^\alpha$ with $\bar{\alpha}$ the projective completion of α. If $u \in \mathcal{P}' \backslash \mathcal{P}^\alpha$ and $u^\perp \cap \mathcal{P}^\alpha = \{x_1, \ldots x_{s+1}\}$ and if x' and x'' are the two projective points of $\bar{\alpha}$ defined by the two parallel classes of lines from $\mathcal{B}$ in α, then the set $\mathcal{O} = \{x_1, \ldots x_{s+1}, x', x''\}$ is a hyperoval in $\bar{\alpha}$

Proof: First note that as x' and x'' are both off of α, no line of $\bar{\alpha}$ through x' and x'' can also contain any of the x_i. Let R be a line of α. If $R \in \mathcal{B}$ then there is a unique x_i on R collinear with u; furthermore exactly one of x' and x'' is incident with R in $\bar{\alpha}$. On the other hand if $R \in \mathcal{B}_1$ then as R is a $(0,2)$-set, $|u^\perp \cap R| \in \{0, 2\}$, i.e. R meets $\mathcal{O}\backslash\{x', x''\}$ in 0 or 2 points; furthermore as R is in not a line of B, its projective completion contains neither x' nor x''. Thus every line of $\bar{\alpha}$ meets $\mathcal{O}$ in 0 or 2 points. ■

Up to this point it has been assumed that $\mathcal{S}$ is a $GQ(s,t)$ with the only restriction being s is odd. For the remainder of this section, further assume that $t = s+2$. For $L, M \in \mathcal{B}$, L and M will be said to be **Type 1 parallel**, written $L \parallel M$ provided there is some Type 1 affine plane α in which L and M are parallel, written $L \parallel_\alpha M$.

Lemma 5.1.11 For any point x and any $L \in \mathcal{B}$ there is a unique $M \in \mathcal{P}$ such that $x \in M$ and $L \parallel M$.

Proof: Recall that $|\mathcal{P}| = (1+s)(1+s(s+2)) = (1+s)^3$; this gives $(1+s)(s(s+2))$ points of $\mathcal{P}$ not on L. Each Type 1 affine plane containing L has $s(s+1)$ points not on L. Any two of these planes meet exactly at L. This gives $\frac{(1+s)s(s+2)}{s(s+1)} = s+2$ Type 1 affine planes through L. In each of these planes, there are s lines other than L in a parallel class with L. Including L

itself, there are $1+s(s+2)=(1+s)^2$ lines M in $\mathcal{P}$ such that $M \parallel_\alpha L$ for some α. As these lines are disjoint, they cover all $(1+s)^3$ points of $\mathcal{P}$ exactly once. ■

Lemma 5.1.12 The parallelism given by $\parallel$ is in fact an equivalence relation on the lines of $\mathcal{B}$.

Proof: That $\parallel$ is both reflexive and symmetric is straightforward. To see that $\parallel$ is transitive requires effort. Let α, β be two affine planes of Type 1 and let $L, M, N \in \mathcal{B}$ such that $L \parallel_\alpha M$ and $M \parallel_\beta N$. If L, M, N, α, β are not all distinct, transitivity follows immediately. Assume instead they are in fact all distinct. What is now needed is the demonstration of a Type 1 affine plane γ in which $L \parallel_\gamma M$.

Let $x \in N$, and let γ be the Type 1 affine plane generated by x and L. Let N' be the unique line γ through x which is parallel (in γ) to L as ensured by the previous lemma. If $N = N'$, transitivity is demonstrated.

Assume for contradiction that $N \neq N'$, and let K be a line of α in $\{L, M\}^\perp$. As was demonstrated above, $x^\perp \cap \alpha$ is an $(s+1)$-arc. Let R be a line through x meeting α at a point y. In α, y is incident with just one line parallel to K, say K'. If $R \parallel K$, this would give two lines of $\mathcal{B}$ through y parallel to K, a contradiction. Hence none of the $s+1$ lines joining x to a point of α are parallel to K. As K contains a point of L and as L is the unique line through that point and parallel to N', N' cannot be parallel to K. Similarly as K contains a point of M, N cannot be parallel to K. This gives $s+3$ lines through x, none of which are parallel to K, contradicting the previous lemma. Hence $N = N'$; i.e. $L \parallel N$. ∎

As $\mathcal{S}$ is a $GQ(s, s+2)$, there are $s+3$ parallel classes with $(s+1)^2$ lines in each parallel class. Let $\mathcal{M} = \{\mathcal{R}, \mathcal{R}_1, \ldots, \mathcal{R}_{s+2}\}$ be the collection of parallel classes. Observe that $\mathcal{M}$ is a packing of spreads for $\mathcal{S}$. To see that in fact $\mathcal{R}$ is pivotal for this packing note the following. If $L, M \in \mathcal{R}$ then by the definition of parallel class, $\{L, M\}$ is a regular pair and $\{L, M\}^{\perp\perp} \subset \mathcal{R}$. Furthermore $\{L, M\}^{\perp}$ is a set of mutually parallel lines; i.e. $\{L, M\}^{\perp} \subset \mathcal{R}_i$ for some $i \in \{1, \ldots s+2\}$. Now constrict $\mathcal{S}$ about $\mathcal{R}$ to get a new $GQ(s+1, s+1)$, say $\mathcal{S}^*$ with a regular point (∞) such that $\mathcal{S} = P(\ \mathcal{S}^*, (\infty)\ )$. Let x be a point of $\mathcal{S}$ not in a given Type 1 affine plane α; then $|x^{\perp} \cap \alpha| = s+1$. Note that x is on some line L of $\mathcal{R}_1$; likewise x is on some line M of $\mathcal{R}$. In $\mathcal{S}$ the lines L and M generate a Type 1 affine plane β.

With this parallelism of lines shown to be an equivalence relation, a similar relation can be defined on Type 1 affine planes. Two Type 1 affine planes α, β are said to be **parallel** provided there exist distinct concurrent affine lines L, M in α and distinct concurrent affine lines L', M' in β with

$L \parallel L'$ and $M \parallel M'$.

The symbol $\parallel$ is again used to denote this parallelism as well as parallelism of lines. The context should make clear which sense is intended. Let α, β be parallel Type 1 affine planes with L, L', M, M' as in the above definition.

α and β have no common points. If they shared some point x, then L would be parallel to some N in α through x and L' would be parallel to some N' through x. Hence $N \parallel N'$ with $x \in N \cap N'$, a contradiction.

If J, K are distinct concurrent lines in α then there exist a pair of distinct concurrent lines J', K' in β such that $J \parallel J', K \parallel K'$. To see this let $y \in \beta$ and consider the Type 1 affine plane generated by y and J and that generated by y and K; let J', K' be the respective intersections of these planes with β.

The relation $\parallel$ is an equivalence relation on Type 1 affine planes in $\mathcal{S}$. Reflexivity and symmetry are immediate. Transitivity follows from the previous paragraph.

The Type 1 affine planes in a parallel class partition the points of $\mathcal{S}$. Hence each parallel class consists of $\frac{(s+1)^3}{(s+1)^2} = s + 1$ planes.

Let α, β be distinct parallel planes with x a point of β. Of the $s + 3$ lines (in $\mathcal{S}$) through x, $s+1$ meet lines of α. This gives $(1+s)^2 - (1+s) = s(s+1)$ points of α which are coincident with x from blocks of $\mathcal{B}_1$, i.e. $(0, 2)$-sets, no two of which have points other than x in common.

If N is a line of β which misses x, there are s points of N not collinear in $\mathcal{S}$ with x. These s points determine s $(0,2)$-sets with x. This accounts for $s(s+1)+s = s^2+2s$ $(0,2)$-sets through x, i.e. this accounts for all $(0,2)$-sets through x. To see that each point is on s^2+2s $(0,2)$-sets, recall that $\mathcal{S}'$ is a $2-((s+1)^3, s+1, 1)$ design, and hence each point is on s^2+3s+3 blocks. Of these, $s+3$ are lines of $\mathcal{B}$, and the remaining s^2+2s are $(0,2)$-sets.

Now let α, β be non-parallel affine planes of Type 1 with $\mathcal{T} = \{\alpha = \alpha_1, \alpha_2, \ldots, \alpha_{s+1}\}$ the parallel class containing α. For $\alpha_i \in \mathcal{T}$, any points of $\beta \cap \alpha_i$ must all lie in a common block of $\mathcal{B}'$: For suppose x, y, z are three points of $\beta \cap \alpha_i$ not contained in a block of $\mathcal{B}'$. The blocks $[xy]$ and $[xz]$ can be thought of as affine lines in both affine planes having a common point, forcing $\alpha_i \parallel \beta$, a contradiction. This means that $\beta \cap \alpha_i$ has at most $s+1$ points. On the other hand as $\mathcal{T}$ partitions $\mathcal{P}'$ and as β has $(s+1)^2$ points. Each α_i must have exactly $s+1$ points of β. Thus $\beta \cap \alpha_i \in \mathcal{B}'$, and in fact α and β must meet in a common block of $\mathcal{B}'$.

The contrapositive of this is stated as the following observation.

Observation 5.1.13 If two Type 1 affine planes in $\mathcal{S}$ do not have a common block then they are parallel.

Let $\alpha' \in \mathcal{T} \backslash \{\alpha\}$ with $L = \beta \cap \alpha$ and $L^* = \beta \cap \alpha'$. As L and L^* are disjoint lines in β, they are parallel and hence are either both in $\mathcal{B}$ or both in $\mathcal{B}_1$. Further let β' be a Type 1 affine plane parallel to β with $L' = \alpha' \cap \beta'$. By the previous argument L^* and L' are parallel lines from either $\mathcal{B}$ or $\mathcal{B}_1$. This forces L and L' to be parallel lines from either $\mathcal{B}$ or $\mathcal{B}_1$.

Lemma 5.1.14 Let γ be a Type 1 affine plane distinct from α and β which contains L. L' is in some plane γ' parallel to γ.

Proof: Let $x \in L'$ and let γ' be the unique plane through x parallel to γ. Put $M = \gamma' \cap \alpha$ and $N = \gamma' \cap \beta$. This forces $L \parallel_\beta N$ and $L \parallel_\alpha M$. If $\gamma = \gamma'$ then $M = L = N$. If $\gamma \neq \gamma'$ suppose $M \cap N = z$. Because $z \in \alpha$ and $z \in \beta$, z must be a point of L, but $L \parallel_\alpha M$ (and $L \parallel_\beta N$), a contradiction. $M \cap N$ is then empty, and hence $M \parallel_{\gamma'} N$. Let $M' = \gamma' \cap \alpha', N' = \gamma' \cap \beta'$, in which case $M \parallel_{\gamma'} M', N \parallel_{\gamma'} N'$. Thus $M' \parallel_{\gamma'} N', x \in M', x \in N'$ forces $M' = N' \subset \alpha' \cap \beta' = L'$. Hence L' is in γ'. ■

At this point define a parallelism on $\mathcal{B}_1$ as follows. For $L, M \in \mathcal{B}_1$, L and M are said to be parallel (again written $L \parallel M$) provided there exist Type 1 affine planes $\alpha, \alpha', \beta, \beta'$ with $\alpha \parallel \alpha', \beta \parallel \beta'$ such that $\alpha \cap \beta = L, \alpha' \cap \beta' = M$.

Lemma 5.1.15 The parallelism given by $\parallel$ is an equivalence relation on $\mathcal{B}_1$.

Proof: Reflexivity follows by letting $\alpha = \alpha', \beta = \beta'$; symmetry is immediate. For transitivity let $L, M, N \in \mathcal{B}_1$ with $L \parallel M, M \parallel N$ in which case there are Type 1 affine planes $\alpha, \alpha', \beta, \beta', \pi, \pi', \nu, \nu'$ with $\alpha \parallel \alpha', \beta \parallel \beta', \pi \parallel \pi', \nu \parallel \nu'$ such that $\alpha \cap \beta = L, \alpha' \cap \beta' = M = \pi \cap \nu, N = \pi' \cap \nu'$. Letting π, ν, α' play the respective roles of α, β, γ in the previous lemma forces the existence of some Type 1 affine plane α'' which is parallel to α' and which contains N. Likewise, letting β' play the role of γ forces the existence of some Type 1 affine plane β'' which is parallel to β' and which contains N. In fact $N = \alpha'' \cap \beta''$ and by the transitivity of $\parallel$ on Type 1 affine planes $\alpha \parallel \alpha'', \beta \parallel \beta''$. Hence $L \parallel N$. ■

Corollary 5.1.16 Each parallel class of $\mathcal{B}_1$ partitions the points of $\mathcal{P}'$.

Proof: Let $L \in \mathcal{B}_1, x \in \mathcal{P}'$, and let α, β be distinct Type 1 affine planes which meet at L. Furthermore let α', β' be distinct Type 1 affine planes containing x such that $\alpha \parallel \alpha', \beta \parallel \beta'$. Put $L' = \alpha' \cap \beta'$; so $x \in L'$ and $L \parallel L'$. Suppose there is some $L'' \in \mathcal{B}_1$ which contains x and which is parallel to L. This forces the existence of planes γ and δ through L'' with $\gamma \parallel \alpha, \delta \parallel \beta$. But as classes of parallel planes partition P', $\gamma = \alpha', \delta = \beta'$. Hence $L'' = L'$; i.e. there is exactly one member of $\mathcal{B}_1$ parallel to L which contains x (namely L'). Observe that if $x \in L$ then $L = L'$. ■

Corollary 5.1.17 If two members of $\mathcal{B}_1$ are parallel as lines in a Type 1 affine plane then they are parallel under $\parallel$ as defined above.

Proof: Let $L, M \in \mathcal{B}_1$ with $L \parallel_\alpha M$ for some Type 1 affine plane α. Let β be some other Type 1 affine plane through L, let y be a point of M, let β' be the Type 1 affine plane through y and parallel to β, and let M' be the common block on α and β'. Observe then that $M' \parallel L$, and hence $M' \parallel_\alpha L$. This forces $M' = M$, and therefore $L \parallel M$. ■

The following lemmas and corollaries relate parallel blocks and planes.

Lemma 5.1.18 Let α be a Type 1 affine plane which does not contain the block L. α and L have no common point if and only if L belongs to one of the parallel classes defined by the blocks of α.

Proof: First assume that L is parallel to some block $M \in \alpha$. If $L, M \in \mathcal{B}$ then they are parallel in an affine plane which meets α exactly at M and hence L has no point of α. If $L, M \in \mathcal{B}_1$ then there are type 1 affine planes $\gamma, \gamma', \delta\delta'$ with $\gamma \parallel \gamma', \delta \parallel \delta', L = \gamma \cap \delta, M = \gamma' \cap \delta'$. Suppose L met α in a point x. This would force γ to meet α in a block N through x If N met M then γ could not be parallel to γ', but $N \parallel_\alpha M$ implies $N \parallel M$ which implies $N \parallel L$, a contradiction. Hence L must miss α.

On the other hand, assume now that α and L have no common points. Then L is in a parallel class containing a block of α as follows. If $L \in \mathcal{B}$ choose x to be any point of α. If $L \in \mathcal{B}_1$, choose x to be a point of α collinear in $\mathcal{S}$

with two points of L. Let β be the Type 1 affine plane generated by L and x; let $M = \alpha \cap \beta$. Because L misses α, $L \parallel_\beta M$, and hence $L \parallel M$. ■

Corollary 5.1.19 Parallel affine planes of Type 1 define the same $s+2$ parallel classes of blocks.

Proof: This is immediate from the lemma. ■

Corollary 5.1.20 If α and β are Type 1 affine planes such that two distinct intersecting blocks, L, M, of α are parallel respectively to two distinct intersecting blocks, L', M', of β then $\alpha \parallel \beta$.

Proof: Suppose α and β are not parallel and hence they have a common block N. If $|L \cap N| = 1$, then $|L' \cap N| = 1$. As $L \parallel L'$, there is a Type 1 affine plane γ containing L, L', and N. But this is a contradiction as L and N determine the Type 1 affine plane α. Hence $L \parallel_\alpha N$. This forces M to meet N in exactly one point. A similar contradiction arises. Thus α and β cannot have a common block; i.e. $\alpha \parallel \beta$. ■

For the remainder of this section assume $s > 3$.

Lemma 5.1.21 Let $y_1z_1u_1$ and $y_2z_2u_2$ be two triangles of a Type 1 affine plane α which are perspective from the point w with $w, y_1, z_1, u_1, y_2, z_2$, and u_2

distinct points and $y_1z_1, y_1u_1, z_1u_1, y_2z_2, y_2u_2, z_2u_2, wy_1, wz_1$, and wu_1 distinct lines. If $y_1z_1 \parallel_\alpha y_2z_2 \in \mathcal{B}$ and $z_1u_1 \parallel_\alpha z_2u_2 \in \mathcal{B}$ then $u_1y_1 \parallel_\alpha u_2y_2$.

Proof: Note that since members of $\mathcal{B}$ are lines of $\mathcal{S}$ no three of them can form a triangle; i.e. $u_1y_1, u_2y_2 \in \mathcal{B}_1$. Let $L \in \mathcal{B}$ be a line meeting α at the point w and construct a Type 1 affine plane β_1 from u_1y_1 which meets L in some point w_1 different from w.

Count the number of different planes which could be constructed in this manner: For $p_1 \in u_1y_1$ there is a unique point w_1 of L which is collinear in $\mathcal{S}$ with p_1. As u_1y_1 is a $(0,2)$-set, there is a unique other point $p_2 \in u_1y_1$ which is also collinear with w_1. If w is collinear in $\mathcal{S}$ with both u_1 and y_1, this leaves $\frac{s-1}{2}$ pairs of points in u_1y_1 which are collinear with a point of $L\backslash w$; if w is not collinear in $\mathcal{S}$ with both u_1 and y_1, this leaves $\frac{s+1}{2}$ pairs of points in u_1y_1 which are collinear with a point of $L\backslash w$. This means there are at least $\frac{s-1}{2}$ different β_1 which can be constructed in this manner. Similarly there are at least $\frac{s-3}{2}$ ways to construct a Type 1 affine plane β_2 through u_2y_2 which meets $L\backslash\{w, w_1\}$ in a point w_2.

If $\beta_1 \parallel \beta_2$ then $u_1y_1 \parallel u_2y_2$. Assume β_1 and β_2 are not parallel. Suppose for contradiction that $w_2z_2 \parallel w_1z_1$; this would force $w_2z_2y_2 \parallel w_1z_1y_1$ and $w_2z_2u_2 \parallel w_1z_1u_1$. It has been shown that when two parallel planes are intersected with a third plane, the resulting lines of intersection are parallel.

Specifically, $(Ly_1 \cap w_1z_1y_1) \parallel (Ly_1 \cap w_2z_2y_2)$, i.e. $y_1w_1 \parallel y_2w_2$. Similarly, $u_1w_1 \parallel u_2w_2$. By the previous corollary, $\beta_1 \parallel \beta_2$, a contradiction. Hence w_2z_2 and w_1z_1 cannot be parallel. As w, z_1, z_2 are all collinear in the plane α, this argument shows also that $w_1y_1 \not\parallel w_2y_2$ and $w_1u_1 \not\parallel w_2u_2$ because either of these parallelism would imply $w_1z_1 \parallel w_2z_2$.

Label these points of intersection: $z_3 = w_1z_1 \cap w_2z_2, u_3 = w_1u_1 \cap w_2u_2, y_3 = w_1y_1 \cap w_2y_2$. Recall that by hypothesis $u_1z_1 \parallel u_2z_2$ and observe that u_3z_3 is the line of intersection of the planes $u_1z_1w_1$ and $u_2z_2w_2$. From this $u_3z_3 \cap u_1z_1 = \emptyset = u_3z_3 \cap u_2z_2$; i.e. $u_3z_3 \parallel_{u_1z_1w_1} u_1z_1$ and $u_3z_3 \parallel_{u_2z_2w_2} u_2z_2$. Thus $u_1z_1 \parallel u_3z_3$ and similarly $y_1z_1 \parallel y_3z_3$.

This forces u_3z_3 and y_3z_3 to be members of $\mathcal{B}$, and hence by the previous corollary $\alpha \parallel u_3z_3y_3$. Finally observe that the plane $u_1y_1w_1$ hits the parallel planes α and $u_3z_3y_3$ in the respective parallel lines y_1u_1 and y_3u_3, and likewise $u_2y_2w_2$ hits the parallel planes α and $u_3z_3y_3$ in the respective parallel lines y_2u_2 and y_3u_3. By transitivity $y_1u_1 \parallel y_2u_2$. ■

One final lemma is needed before the main theorem of this section will emerge.

Lemma 5.1.22 Let α and β be distinct parallel Type 1 affine planes. Let u be a point not in α or β, and let L_1, L_2 be distinct blocks through u which do not belong to to the parallel classes of blocks defined by α (or β). If y_i is the common point of α and L_i, and if z_i be the common point of α and $L_i,\ i=1,2$, then $y_1y_2 \parallel z_1z_2$.

Proof: Since $\alpha \parallel \beta,\ y_1y_2 \cap z_1z_2 = \emptyset$. If $y_1y_2 \in \mathcal{B}$ then the points u, y_1, y_2, z_1, z_2 all belong to a Type 1 affine plane, in which case $y_1y_2 \parallel z_1z_2$. Similarly $z_1z_2 \in \mathcal{B}$ implies $y_1y_2 \parallel z_1z_2$. Assume that neither y_1y_2 nor z_1z_2 are in $\mathcal{B}$. For $i=1,2$ let $M_i \in \mathcal{B}$ be a line through y_1 such that M_1 meets M_2 at some point w. Let w' be the point of β on uw, and observe that $M_1u \cap \beta = z_1w',\ M_2u \cap \beta = z_2w'$. As $\alpha \parallel \beta,\ M_1 \parallel_{M_1u} z_1w'$ and $M_2 \parallel_{M_2u} z_2w'$. Since $M_1, M_2 \in \mathcal{B}$ it follows that $z_1w', z_2w' \in \mathcal{B}$.

Now choose a line $L \in \mathcal{B}$ through u which meets both α and β but misses both z_1w' and z_2w'. If L meets z_1z_2 then the Type 1 affine plane containing L and z_1z_2 must meet α at exactly the line y_1y_2 (as $\alpha \parallel \beta$); in this case $y_1y_2 \parallel z_1z_2$ and the proof is completed. So now assume that L misses z_1z_2 and let $u' = L \cap \beta$. As shown with β_2 in the previous lemma, there are $\frac{s-3}{2}$ planes different from α which contain y_1y_2 and a point of $L\backslash\{u'\}$. Let γ be such a plane.

If $u \in \gamma$ then as above u, y_1, y_2, z_1, z_2 are all coplanar (here in γ) and hence $y_1y_2 \parallel z_1z_2$; so further assume u is not in γ and let $u'' = L \cap \gamma$. Let $v_1 = y_1u'' \cap \beta,\ v_2 = y_2u'' \cap \beta,\ w'' = wu'' \cap \beta$. To see that such points actually exists use lemma 5.1.18 applied to β and observe that the parallel classes of β are the same as those of α. The plane $u''wy_1$ contains the lines M_1 of α and v_1w'' of β; hence $M_1 \parallel v_1w''$ and likewise $M_2 \parallel v_2w''$, forcing $v_1w'', v_2w'' \in \mathcal{B}$. By transitivity observe that $z_1w' \parallel v_1w''$ and $z_2w' \parallel v_2w''$.

$y_1 \in z_1u$ implies $y_1 \in Lz_1$ which implies $y_1u'' \subset Lz_1$ and hence $v_1 \in Lz_1$. Clearly u' and z_1 are in Lz_1. But as these three points (v_1, z_1, and u') are also in β, they must lie in a common block. Similarly v_2, z_2, u', are on a common block, as are w'', w', u'.

The two triangles formed by v_1, v_2, w'' and z_1, z_2, w' are perspective from the point u'. Since $z_1w' \parallel v_1w''$ and $z_2w' \parallel v_2w''$, lemma 5.1.21 indicates that $z_1z_2 \parallel v_1v_2$. As γ meets α and β in parallel lines, $y_1y_2 \parallel v_1v_2$. Therefore $y_1y_2 \parallel z_1z_2$. ∎

This leads to the main theorem of this section:

Theorem 5.1.23 Let $\mathcal{S}$ be a $GQ(s, s+2), s \neq 1$ which contains a set $\mathcal{B}_1$ of $(0,2)$-sets such that every pair of non-collinear points is contained in a unique member of $\mathcal{B}_1$ and such that Axiom (T) holds. It follows that $\mathcal{S}$ is isomorphic to $T_2^*(\mathcal{O})$ (the GQ obtained by expanding $T_2(\mathcal{O})$ about a regular point) for some oval $\mathcal{O}$.

Proof: As was shown early on, under these hypotheses, s must be odd. If $s = 3$ then by [PT84] the result follows. Assume that $s > 3$. $\mathcal{S}'$ as derived above is shown to be affine three-space as follows (let $\mathcal{P} = \mathcal{P}', \mathcal{B}, \mathcal{B}_1, \mathcal{B}'$ be as defined above):

From the construction, two points are either in a unique member of $\mathcal{B}$ or a unique member of $\mathcal{B}_1$.

The two block parallelisms have been shown to be equivalence relations on $\mathcal{B}$ and $\mathcal{B}_1$ respectively; jointly they form an equivalence relation on $\mathcal{P}'$. Let $x \in \mathcal{P}', L \in \mathcal{B}'$. From lemma 5.1.11 and corollary 5.1.16 there exists a unique $M \in \mathcal{B}'$ through x which is parallel to L.

Let L and M be parallel blocks with $x_1 \in L, x_2 \in M$. Let $w \in x_1x_2\backslash\{x_1, x_2\}, x_1' \in L\backslash\{x_1\}$. Let α be a Type 1 affine plane through x_1x_1' but not through w; let β be the Type 1 affine plane through x_2 and parallel to α. From lemma 5.1.18 $M \subset \beta$. Let $x_2' = wx' \cap \beta$. Such a point is also ensured by lemma 5.1.18. Since M is the unique line of β which is parallel to L and which contains x_2 it follows that $M = x_2x_2'$; i.e. it is shown that for any $x' \in L\backslash\{x_1\}$ and $w \in x_1x_2\backslash\{x_1, x_2\}$ the block wx' meets M (the Axiom of Veblen).

By construction each block contains $s + 1$ (i.e. more than 2) points.

By [Len54] $\mathcal{S}'$ is the design of an affine space. As the point set has size $(s + 1)^3$, in fact this is $AG(3, s + 1)$; thus $\mathcal{S}$ is embedded in $AG(3, s + 1)$ with s odd. By [Tha78] it follows that $\mathcal{S}$ is in fact isomorphic to $T_2^*(\mathcal{O})$ as $s+1$

is even. ■

5.2 Axioms of Payne

Whereas De Soete and Thas concerned themselves with characterizing $T_2^*(\mathcal{O})$, i.e. the quadrangles obtained by expanding $T_2(\mathcal{O})$ about a regular point, Payne gave seven axioms characterizing the quadrangles obtained by expanding $T_2(\mathcal{O})$ about a regular line. The last of these was latter shown to be redundant. In this section results from [Pay85a] concerning the remaining six axioms are presented.

Axioms A_1, A_2, and A_4 imply the existence of a $GQ(q+1, q-1)$, say $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with q even and having a fan of ovoids $\mathcal{M} = \{\mathcal{O}_\infty, \mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_q\}$ for which $\mathcal{O}_\infty$ is pivotal. Furthermore propositions 3.2.1 and 3.2.2 hold in this context and $\mathcal{S}$ can be constricted about the pivotal ovoid $\mathcal{O}_\infty$ to form $\mathcal{S}_\infty$ a $GQ(q, q)$ as in section 3.3 with L_∞ regular.

Let $\mathcal{O}_\infty = T_1 \cup T_2 \cup \cdots \cup T_q$ and $\mathcal{O}_0 = T_1^\perp \cup T_2^\perp \cup \cdots \cup T_q^\perp$. For $a_1, a_2 \in \mathcal{O}_\infty$ we write $a_1 \equiv a_2$ provided $a_1^\perp \cap \mathcal{O}_0 \cap a_2^\perp \neq \emptyset$; in fact this is true if and only if $a_1^\perp \cap \mathcal{O}_0 = a_2^\perp \cap \mathcal{O}_0$. It is straightforward to check that $\equiv$ is an equivalence relation. Let $[a]$ be the equivalence class of a. Thus for $a \in \mathcal{O}_\infty$, $[a]$ is the hyperbolic line whose perp is in $\mathcal{O}_0$. If $[b] = \{b_1, \ldots b_q\} = a^\perp \cap \mathcal{O}_0$ then the q^2 lines joining points of the class of $[a]$ with points of $[b]$ form a set of lines called a **quiver**. Note that a quiver together with its endpoints forms a regular bipartite graph of degree q.

Two lines of a quiver are concurrent if and only if they meet at a point of $\mathcal{O}_0$ or $\mathcal{O}_\infty$. Two nonconcurrent lines L, M are in the same quiver if and only if the point of $\mathcal{O}_\infty$ on L is collinear with the point of $\mathcal{O}_0$ on M if and only if the point of $\mathcal{O}_0$ on L is collinear with the point of $\mathcal{O}_\infty$ on M.

Each point of $\mathcal{O}_\infty \cup \mathcal{O}_0$ determines a unique quiver. There are q quivers, each containing q^2 lines. This accounts for all q^3 lines of $\mathcal{S}$. Also, each line is in a unique quiver determined by its 2 points of $\mathcal{O}_0 \cup \mathcal{O}_\infty$.

There are q^3 points of of $P \setminus (\mathcal{O}_\infty \cup \mathcal{O}_0)$ covered by the lines of a quiver. Thus each point of $P \setminus (\mathcal{O}_\infty \cup \mathcal{O}_0)$ is on a unique line of each quiver. Each line not in a given quiver Q is concurrent with exactly q pairwise non-concurrent lines of Q.

Now assume Payne's axiom A_3, which is given below.

A_3. Let L_1, M_1 be nonconcurrent lines of $\mathcal{S}$ meeting lines L_2, M_2 at four distinct points of $P \setminus (\mathcal{O}_\infty \cup \mathcal{O}_0)$. Let a_j be the point of L_j in $\mathcal{O}_\infty$, $j = 1, 2$. Let b_j be the point of M_j in $a_j^\perp$, $j = 1, 2$. Then $b_1 \in \mathcal{O}_0$ if only if $b_2 \in \mathcal{O}_0$.

Let L, M be non-concurrent lines of some quiver Q, i.e. L, M join a span $T \subseteq \mathcal{O}_\infty$ with $T^\perp \subseteq \mathcal{O}_0$. Consider the following three results:

Observation 5.2.1 A_3 implies that lines of $\{L, M\}^\perp$ not in Q (i.e. not incident with points of $\mathcal{O}_\infty \cup \mathcal{O}_0$) are all in a common quiver Q'.

Proposition 5.2.2 L, M belong to a $q \times q$ grid G having q lines in Q' (the lines of $\{L, M\}^\perp$) and q lines in Q. The points of G are precisely those points on the lines of $\{L, M\}^\perp$ not in $\mathcal{O}_\infty \cup \mathcal{O}_0$.

Proof:

Let K_1, K_2, K_3 be lines in $\{L, M\}^\perp$. Let Let $x \in K_2$, $x \notin L$, $x \notin M$, $x \notin \mathcal{O}_\infty \cup \mathcal{O}_0$. For $j = 1, 2, 3$, N_j is the line through x meeting K_j. From the last observation (applied to K_1, K_2), L and N_1 must be in the same quiver, and L and N_3 must be in the same quiver. Because L is in exactly one quiver (namely Q) $N_1, N_3 \in Q$. But as x cannot be on 2 lines of Q, $N_1 = N_3$. ■

A **triad** of lines is a set of three pairwise disjoint lines. A triad is said to be **centric** provided there is some fourth line which intersects each of the lines of the triad.

Proposition 5.2.3 If L, M, K is a triad of lines in Q then K belongs to the $q \times q$ grid G containing L, M if and only if (L, M, K) is centric.

Proof: If K is in G then each line in the other ruling is a center for L, M, K. If K is not in G, then let $a = K \cap \mathcal{O}_\infty, b = \cap \mathcal{O}_0$. Observe that in G there must be lines L_1 and L_2 through a and b respectively. Any center of

K, L, M would form a triangle with two of L, M, L_1, L_2. ■

Recall that L_∞ is a regular line of $\mathcal{S}_\infty$. Let K be some other line through $(\mathcal{O}_0)$, say $K = \{a_1, a_2\}^\perp \subseteq \mathcal{O}_0$ where $a_1, a_2 \in \mathcal{O}_\infty$. Let $\mathcal{O}_\infty$ have as some of its points $a_1, a_2, \ldots, a_q$, where $\{a_i, a_j\}^\perp = \{b_1, \ldots, b_q\} = K \subseteq \mathcal{O}_0$. K then determines a quiver Q for which a line L of $\mathcal{S}$ is in Q ifand only if L is not concurrent with K in $\mathcal{S}_\infty$.

Let M be a line of $\mathcal{S}_\infty$ not concurrent with K. The regularity of L_∞ implies the regularity of $\{K, M\}$ provided M is concurrent with L_∞.

Now consider the case M is not concurrent with L_∞. In this case M is a line of $\mathcal{S}$ not concurrent with K. M belongs to some quiver Q', as the quivers partition the lines of $\mathcal{S}$. For each $i \geq 0$, let m_i be the point of M on $\mathcal{O}_i$. Through each $m_i,\ i \geq 1$ there is a unique line L_i from Q. Note that every L_i meets every line of Q'. This gives a $q \times q$ grid , G, of lines in $\mathcal{S}$.

Each line of Q' is on a unique point of the ovoid $\mathcal{O}_0$. These q points together with the point $(\mathcal{O}_0)$ form a line $T^\perp$ in $\mathcal{S}_\infty$. Now note that G can be extended to a $(q+1) \times (q+1)$ grid , $G_\infty = G \cup \{K, T^\perp\}$, of lines in $\mathcal{S}_\infty$. This means that $\{K, M\}$ is a regular pair of lines in $\mathcal{S}_\infty$. Hence K is a regular line of $\mathcal{S}_\infty$. This proves the following:

Proposition 5.2.4 $(\mathcal{O}_0)$ is a coregular point of $\mathcal{S}_\infty$

From [PT84], having q even is enough to force all coregular points to be regular. Specifically applied to the case at hand, this means that $(\mathcal{O}_0)$ is a regular point of $\mathcal{S}_\infty$. proposition 3.4.3 then forces the ovoid $\mathcal{O}_0$ to be a pivotal member of $\mathcal{M}$. Hence the two ovoids $\mathcal{O}_\infty$ and $\mathcal{O}_0$ play interchangeable roles.

Since $(\mathcal{O}_0)$ is regular, $\mathcal{S}_\infty$ can be expanded about $(\mathcal{O}_0)$ to form a new GQ, $\mathcal{S}_\infty^0 = (\mathcal{P}_\infty^0, \mathcal{B}_\infty^0, \mathcal{I}_\infty^0)$, of order $(q-1, q+1)$ in the following way.

$\mathcal{P}_\infty^0 = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_q$.

$\mathcal{B}_\infty^0 = \mathcal{B} \cup \{T^\perp \mid T$ is a hyperbolic line of $\mathcal{O}_\infty$ or $\mathcal{O}_0$ for which $T^\perp \cap (\mathcal{O}_\infty \cup \mathcal{O}_0) = \emptyset\}$.

Incidence $\mathcal{I}_\infty^0$ is that of $\mathcal{S}_\infty$ restricted to $\mathcal{S}_\infty^0$ with the additional statement that traces of hyperbolic lines from $\mathcal{O}_0$ are incident with the points they contain in $\mathcal{S}$. Note that if the roles of $\mathcal{O}_\infty$ and $\mathcal{O}_0$ are interchanged one could first construct the $GQ(q,q)$, $\mathcal{S}_0 = (\mathcal{P}_0, \mathcal{B}_0, \mathcal{I}_0)$, by constricting about the pivotal ovoid $\mathcal{O}_0$ and then expand about the regular point $(\mathcal{O}_\infty)$ of $\mathcal{S}_0$ to get $\mathcal{S}_0^\infty = (\mathcal{P}_0^\infty, \mathcal{B}_0^\infty, \mathcal{I}_0^\infty)$, a $GQ(q-1, q+1)$. After having done so, the resulting $\mathcal{S}_0^\infty$ is *identical* (not just isomorphic) to $\mathcal{S}_\infty^0$. With so many incidence structures, confusion can arise with resect to the symbols $\perp$ and $\sim$. Unless otherwise indicated these symbols will be used with regard to $\mathcal{S}$. To indicate collinearity or concurrence in $\mathcal{S}_\infty^0$ the symbol $\approx$ will be used.

We now assume the fifth and sixth axioms of Payne:

A_5. Let (L_1, L_2, L_3) be a centric triad of lines with the property that the points of $\mathcal{O}_\infty$ on L_1, L_2, L_3 are all collinear with all three points of $\mathcal{O}_0$ on L_1, L_2, L_3. Then, if for some $(a, b) \in \mathcal{O}_\infty \times \mathcal{O}_0$ with a and b noncollinear, both L_1 and L_2 are incident with points of $\{a, b\}^\perp$, then L_3 is also incident with a point of $\{a, b\}^\perp$.

A_6. Let x_1, x_2, x_3 be distinct points of some $\mathcal{O}_j$, $1 \leq j \leq q$. Let (L_1, L_2, L_3) and (M_1, M_2, M_3) be two triads of lines such that L_i meets M_i at x_i, $1 \leq i \leq 3$. Suppose that the three points of $\mathcal{O}_\infty$ on L_1, L_2, L_3 (resp., M_1, M_2, M_3) are each collinear with the three points of $\mathcal{O}_0$ on L_1, L_2, L_3 (resp., M_1, M_2, M_3). Then (L_1, L_2, L_3) is centric if and only if (M_1, M_2, M_3) is centric.

Whereas De Soete and Thas constructed $(0, 2)$-sets, Payne constructs pseudolines. Let x and y be noncollinear with $x \in \mathcal{O}_i, y \in \mathcal{O}_j, 1 \leq i < j \leq q$ (so $x, y \in \mathcal{P}^0_\infty$). From part ii) of proposition 3.2.2 $\{x, y\}^\perp$ contains a unique point of $\mathcal{O}_\infty$, say a and a unique point of $\mathcal{O}_0$, say b. Define the **pseudoline** through x and y, written $[[xy]]$ to be $\{a, b\}^\perp$. From part i) of proposition 3.2.2 $[[xy]]$ contains a unique point of each $\mathcal{O}_k, 1 \leq k \leq q$; observe that any two such points determine the same $[[xy]]$. Each $x \in \mathcal{P}^0_\infty$ is collinear with q points of $\mathcal{O}_\infty$ and with q points of $\mathcal{O}_0$

Observation 5.2.5 For x and y noncollinear with $x \in \mathcal{O}_i, y \in \mathcal{O}_j, 1 \leq i < j \leq q$, if G is any grid containing x and y then G contains all q points of $[[xy]]$.

It follows that if Q is a quiver containing half of the lines of G, each line of Q contains a unique member of $[[xy]]$.(This is just axiom A_5 restated.)

Observation 5.2.6 For $i = 1, 2$ let G_i be a $q \times q$ grid with lines from quivers Q_i and Q_i' such that both G_i have lines incident with distinct points $x, y \in \mathcal{O}_j, 1 \leq j \leq q$. The set of q points of $\mathcal{O}_j$ incident with lines of G_1 is the same set of q points of $\mathcal{O}_j$ incident with lines of G_2. (This is a restatement of axiom A_6).

The term pseudoline was defined for pairs of points in different ovoids of $\{\mathcal{O}_1, \ldots, \mathcal{O}_q\}$. Here the definition is extended to certain point pairs within the same ovoid $\mathcal{O}_j \in \{\mathcal{O}_1, \ldots, \mathcal{O}_q\}$: Let $x, y \in \mathcal{O}_j, 1 \leq j \leq q$ such that $\{x, y\}^\perp$ contains no point of $\mathcal{O}_\infty \cup \mathcal{O}_0$ and let L_x, and L_y be the lines through x and y respectively in some quiver Q. Hence L_x, L_y are in some grid G. Define the **pseudoline** $[[x, y]]$ to be be the set of q points in $G \cap \mathcal{O}_j$. $[[xy]]$ contains a unique point of each line in Q.

From the previous observation, choosing different quivers for Q still yields the same pseudoline $[[x, y]]$. Clearly $[[x, y]]$ is determined by any two of its points. Observe now that point pairs $(x, y) \in \mathcal{P}_\infty^0$ can be classified as follows:

(1) $x \sim y$ and hence $x \approx y$.

(2) $x \not\sim y$ and x and y are in different ovoids of $\mathcal{M}$; $[[xy]]$ is a pseudoline.

(3) x, y are in the same ovoid with $\{x,y\}^\perp \cap (\mathcal{O}_\infty \cup \mathcal{O}_0) = \emptyset$; $[[xy]]$ is a pseudoline.

(4) x, y are in the same ovoid $\mathcal{O}_j$ with $\{x,y\}^\perp \subset \mathcal{O}_\infty$ (resp. $\mathcal{O}_0$); $\{x,y\}^{\perp\perp}$ is line of $\mathcal{B}_\infty$ (resp. $\mathcal{B}_0$) completely contained in $\mathcal{O}_j$ in $\mathcal{S}$. (This last case corrects errata from VII.2(i),(ii) in [Pay85a].)

Now consider the two cases where $[[xy]]$ is defined. Define the **slope** of the pseudoline $[[xy]]$ as follows. If $[[xy]]$ is as in case 2 above with $[[xy]] = \{a,b\}^\perp$ where $a \in \mathcal{O}_\infty, b \in \mathcal{O}_0$ then $[[xy]]$ is said to have **slope** $s[[xy]] = [a]$. If $[[xy]]$ is as in case 3 above then $[[xy]]$ is said to have **slope** $s[[xy]] = \infty$. This gives $q+1$ different slopes.

Proposition 5.2.7 Let $x, y \in \mathcal{P}^0_\infty$ be in distinct ovoids of $\mathcal{M}$ with $x \not\approx y$; so the pseudoline $[[x,y]] = \{a,b\}^\perp = \{x_1, \ldots x_q\}$ where $a \in \mathcal{O}_\infty, b \in \mathcal{O}_0, x_i \in \mathcal{O}_i$. For each $a' \in \mathcal{O}_\infty - [a]$ (respectively, $b' \in \mathcal{O}_0 - [b]$) there is a unique $i \in \{1, \ldots q\}$ with $a' \in x_i^\perp$ (respectively, $b' \in x_j^\perp$).

Proof: For $1 \leq i < k \leq$, $(x_i^\perp \cap \mathcal{O}_\infty) \cap (x_k^\perp \cap \mathcal{O}_\infty) = \{a\}$; thus $\bigcup_{i=1}^q (x_i^\perp \cap \mathcal{O}_\infty)$ is a set of $q(q-1)$ points of $\mathcal{O}_\infty - \{a\}$. If any of the $x_i^\perp$ contained a point a' (different from a) of $[a]$ then $\{a, a'\}^\perp \subset \mathcal{O}_i$; this contradicts $[a]^\perp \in \mathcal{O}_0$. So $\bigcup_{i=1}^q (x_i^\perp \cap \mathcal{O}_\infty) = \mathcal{O}_\infty - [a]$. Hence for each $a' \in \mathcal{O}_\infty - [a]$ there is a unique i with $a' \in x_i^\perp$. Because the roles of $\mathcal{O}_\infty$ and $\mathcal{O}_0$ are interchangeable, this completes the proof. ∎

Proposition 5.2.8 Let $x, y \in \mathcal{P}_\infty^0$ be in the same ovoid $\mathcal{O}_j \in \mathcal{M}$ and hence $[[xy]] = \{x_1, \ldots x_q\} \in \mathcal{O}_j$. For each $a' \in \mathcal{O}_\infty$ (respectively, $b' \in \mathcal{O}_0$) there is a unique x_i (respectively x_j) $\in [[xy]]$ such that $a' \in x_i \perp$ (respectively $b' \in x_j^\perp$).

Proof: For x_1, x_2 distinct members of $[[xy]]$ observe that $(x_1^\perp \cap \mathcal{O}_\infty) \cap (x_2 \cap \mathcal{O}_\infty) = \emptyset$ since otherwise $[[x_1x_2]]$ would not be defined (but $[[x_1x_2]] = [xy]$. This forces $|\bigcup_{i=1}^q (x_i^\perp \cap \mathcal{O}_\infty)| = q^2$ in which case $\bigcup_{i=1}^q (x_i^\perp \cap \mathcal{O}_\infty) = \mathcal{O}_\infty$. Similar arguments hold replacing $\mathcal{O}_\infty$ with $\mathcal{O}_0$. ∎

Corollary 5.2.9 Let $z \in \mathcal{P}_\infty^0 - [[xy]]$ where $[[xy]]$ is a pseudoline with slope m, and suppose that $z \not\approx w$ for each $w \in [[xy]]$. For each slope $m' \neq m$ there is a unique $w \in [[xy]]$ for which $s[[zw]] = m'$.

The following lemma demonstrates the connection between pseudolines of $\mathcal{S}$ and $(0, 2)$-sets of $\mathcal{S}_\infty^0$. This is the first step in proving the main theorem of this section which relates Payne's axioms to those of De Soete and Thas.

Lemma 5.2.10 Let $x, y \in \mathcal{P}_\infty^0$ with $x \not\approx y$; $[[xy]]$ is a $(0, 2)$-set in $\mathcal{S}_\infty^0$.

Proof: First consider the case where $[[xy]]$ is as in (iii) above, i.e., $x, y \in \mathcal{O}_i \in \mathcal{M}$. Let z be a point of $\mathcal{P}_\infty^0 - [[xy]]$ and collinear in $\mathcal{S}_\infty^0$ with some point of $[[xy]]$. As $[[xy]]$ is determined by any two of its points, assume this point to be x. Here there are two subcases.

Begin by assuming $z \in \mathcal{O}_i$ and hence $\{x, z\}^\perp \subset \mathcal{O}_\infty$ or $\{x, z\}^\perp \subset \mathcal{O}_0$. Without loss of generality suppose $\{x, z\}^\perp \subset \mathcal{O}_\infty$. As w ranges over $[[xy]]$, the various $w^\perp \cap \mathcal{O}_0$ partition $\mathcal{O}_0$. There is a particular w which is collinear with at least one point of $z^\perp \cap \mathcal{O}_0$. But as z and w are both in $\mathcal{O}_i$ it follows that $\{z, w\}^\perp \subset \mathcal{O}_0$ and hence $z \approx w$ (and also $w \neq x$). Furthermore w is uniquely determined this way. If there were some $u \in [[x, y]]$ with $z \approx u$ this would force $z \sim u$, a contradiction as $z, u \in \mathcal{O}_i$.

Next assume $z \in \mathcal{O}_j \neq \mathcal{O}_i$. In this case $z \approx x$ implies $z \sim x$ in which case there is a line $L \in \mathcal{B}$ through x and z. Let Q be the quiver containing L, and let M be the line of Q through y. If G is the grid containing Q, there is another quiver Q' in G; let K be the unique line of Q' through z in G. Observe that K contains a unique point of w of $[[xy]]$. On the other hand if $x \; \mathcal{I} \; L \; \mathcal{I} \; z \; \mathcal{I} \; K \; \mathcal{I} \; w$ with $w \in [[xy]], L \neq K$ then L and K are in the unique grid determined by L and y. Hence $[[xy]]$ is a $(0, 2)$-set of $\mathcal{S}^0_\infty$.

Now consider the case where $[[xy]]$ is as in (ii) above, i.e. there are distinct $\mathcal{O}_j, \mathcal{O}_k \in \mathcal{M}$ with $x \in \mathcal{O}_j, y \in \mathcal{O}_k$ where $x \not\sim y$. Write $[[xy]] = \{a, b\}^\perp$ with $a \in \mathcal{O}_\infty, b \in \mathcal{O}_0$. Choose $z \in \mathcal{P}^0_\infty - [[xy]]$ with $z \approx x$. Begin by assuming $z \notin \mathcal{O}_j$; which would mean that $x \sim z$ by some line $L \in \mathcal{B}$ with L in the quiver Q. Let M be the line through y in Q; let Q' be the other quiver in the grid determined by L and M. Let K be the unique line of Q' through z. By observation 5.2.5 K has a unique point w of $[[xy]]$. Observe that w is the unique other point of $[[xy]]$ collinear in $\mathcal{S}_\infty^0$ with z.

Finally assume $z \approx x$ with $z \in \mathcal{O}_j$. Once again assume $\{x, z\}^\perp \subset \mathcal{O}_\infty$; thus $a \in \{x, z\}^\perp$. Since x is the unique point of $\mathcal{O}_j$ on $[[xy]]$, is w is a point different from x with $z \approx w$ it follows that $z \sim w$. Let K be the line of $\mathcal{B}$ through a and z. The point b is collinear with a unique point w on K. This forces $w \in \{a, b\}^\perp$, i.e. $w \in [[xy]]$. Observe that this makes w the unique point of $[[xy]]$ other than x with which z is collinear in $\mathcal{S}_\infty^0$. Hence $[[xy]]$ is a $(0, 2)$-set. ■

If β is a set of q^2 points of $\mathcal{P}_\infty^0$, β will be called a **special plane** provided for $x, y \in \beta$, $x \approx y$ implies $\{x, y\}^{\perp\perp} \cap \mathcal{P}_\infty^0 \subseteq \beta$ and $x \not\approx y$ implies $[[xy]] \subseteq \beta$. Observe that a special plane β together with the lines and pseudolines it contains is actually an affine plane of order q. Moreover two distinct special planes intersecting in at least three distinct points must have a line of $\mathcal{S}_\infty^0$ or a pseudoline of $\mathcal{S}$ as their intersection. Three types of special planes immediately arise.

Observation 5.2.11 Each ovoid of $\mathcal{M}\backslash\{\mathcal{O}_\infty, \mathcal{O}_0\}$ is a special plane.

Observation 5.2.12 The q^2 points on a $q \times q$ grid G of $\mathcal{S}$ (restricted to points of $\mathcal{P}_\infty^0$) is a special plane denoted β_G. This follows from observations 5.2.5 and

5.2.6.

Lemma 5.2.13 For $a \in \mathcal{O}_\infty \cup \mathcal{O}_0$, $\beta = a^\perp \cap \mathcal{P}_\infty^0$ is a special plane.

Proof: Let $a \in \mathcal{O}_\infty$ with x, y distinct points in β. Consider first the case $x \approx y$: If $x \sim y$ by some line $L \in \mathcal{B}$ then $\{x,y\}^{\perp\perp} = L$. So $\{x,y\}^{\perp\perp} \cap \mathcal{P}_\infty^0 = L \backslash \{a\} \subset a^\perp \cap \mathcal{P}_\infty^0$. If $x \approx y$ but $x \not\sim y$, then x and y belong to the same ovoid $\mathcal{O}_i$ of $\mathcal{M}$. As $a \in \{x,y\}^\perp$ it follows that $\{x,y\}^\perp \subset \mathcal{O}_\infty$ and hence $\{x,y\}^{\perp\perp} \subset \mathcal{O}_i$. In fact $\{x,y\}^{\perp\perp} \subset \mathcal{O}_i \cap a^\perp \subset \beta$.

Finally consider the case $x \not\approx y$. x and y are in distinct ovoids of $\mathcal{M}$, and there is a unique $b \in \mathcal{O}_0$ with $\{x,y\}^\perp \cap \mathcal{O}_0 = \{b\}$. In this case $[[xy]] = \{a,b\}^\perp$ which is clearly contained in β. ■

Let Π^* be the set of special planes formed in one of these three ways. Let (x, L) be a nonincident point-line pair from $\mathcal{S}_\infty^0$. First consider the case where $L \in \mathcal{B}$. There is a unique point on L which is collinear with x in $\mathcal{S}$. If this point is in $\mathcal{O}_\infty \cup \mathcal{O}_0$ call it a and observe that $a^\perp \cap \mathcal{P}_\infty^0$ is the unique plane of Π^* containing x and L. If this point is in $\mathcal{P}_\infty^0$ call it y and let $K \in \mathcal{B}$ be the line through x and y. Since L and K determine a unique grid G, β_G is the unique plane of Π^* through x and L.

Now consider $L \in \mathcal{B}_\infty^0 \backslash \mathcal{B}$, in which case there is some $i \in \{1, \ldots q\}$ with $L \subset \mathcal{O}_i$. Suppose $L^\perp \in \mathcal{O}_\infty$. If $x \in \mathcal{O}_i$ then $\mathcal{O}_i$ is the unique plane of

Π^* containing x and L. If $x \in \mathcal{O}_j \neq \mathcal{O}_i$ then there is a unique $y \ \mathcal{I}_\infty^0 \ L$ such that $x \approx y$ by some $M \in \mathcal{B}_\infty^0$. As x and y are in different ovoids of $\mathcal{M}$, M is actually in $\mathcal{B}$. Let $M \cap \mathcal{O}_\infty = a$; then $a^\perp \cap \mathcal{P}_\infty^0$ is the unique plane of Π^* containing x and L. This last case corrects the proof of VII.6 in [Pay85a]. The following lemma has now been proved.

Lemma 5.2.14 If (x, L) is a nonincident point-line pair from $\mathcal{S}_\infty^0$ then x and L lie in a unique plane of Π^*.

If β is the plane of Π^* determined by the nonincident point-line pair (x, L) from $\mathcal{S}_\infty^0$, then by this lemma there exist $q-1$ pseudolines in β which contain x and a point of L. Let $\mathcal{L} = \{L_1, \ldots, L_{q-1}\}$ be the set of these pseudolines. This leads to the following lemma which is the penultimate (and perhaps most significant) claim of this section.

Lemma 5.2.15 Let x, L, β and $\mathcal{L}$ be as described above with $x \notin M \in \mathcal{B}_\infty^0$ and $L \cap M = \emptyset$. If at least two distinct members, say L_i and L_j of $\mathcal{L}$ meet M then every member of $\mathcal{L}$ meets M.

Proof: Let $y_i = L_i \cap M, y_j = L_j \cap M$. Obviously x, y_1, and y_2 are in β. By the previous lemma (x, M) determine a unique plane $\beta' \in \Pi^*$ which also contains x, y_1, and y_2, i.e. $\{x, y_1, y_2\} \subset \beta \cap \beta'$. But the intersection of two distinct planes of Π^* must be a line. Since x, y_1, and y_2 are not mutually collinear (by an affine line), it follows that $\beta' = \beta$; hence M is parallel to L

in β. Therefore each affine line which meets L in β must also meet M in β. Namely, each of the pseudolines of $\mathcal{L}$ meets M. ■

Observe that this lemma is exactly Axiom T from the previous section. As an immediate consequence we have the following theorem.

Theorem 5.2.16 $\mathcal{S}_\infty^0$ is $T_2^*(\Omega)$ for some oval Ω

Which leads to the following corollary.

Corollary 5.2.17 $\mathcal{S}$ is the result of expanding about a regular line in some $T_2(\Omega)$, i.e. $\mathcal{S} = gq(\Omega^-)$ for some planar q-arc Ω^-.

5.3 Coordinatizing the GQ

To see another characterization using axioms $A_1 - A_6$, a coordinatization scheme is needed. A short diversion is made here to see how the finite field $GF(q)$ can be used to coordinatize $\mathcal{S}_\infty$ and $\mathcal{S}$. (For more on coordinatizations consult [Pay74, Pay77, ST88].) In all that follows q is assumed to be even.

A pair (α, β) of permutations of F is **admissible** provided that whenever u_1, u_2, u_3 are distinct elements of F and z_1, z_2, z_3 are distinct elements of F with subscripts taken modulo 3, then

$$0 = \sum_{i=1}^{3} u_i(z_{i+1} - z_{i-1})$$

implies that

$$0 \neq \sum_{i=1}^{3} u_i^{\alpha}(z_{i+1}^{\beta} - z_{i-1}^{\beta}).$$

In [Pay77] it was shown that if a GQ, $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathcal{I}')$, of order (q, q) has a regular point (∞) incident with a regular line $[\infty]$, for which both the associated projective planes are desarguesian, then there this an admissible pair (α, β) fixing 0 and 1 which may be used to coordinatize the GQ $\mathcal{S}'$ as follows.

The points of $\mathcal{S}'$ have labels (∞), (m), (a, b), (m, v, w) for arbitrary $a, b, m, v, w \in F$. The lines have labels $[\infty]$, $[a]$, $[m, v]$, $[a, b, c]$ for arbitrary $m, v, a, b, c \in F$. Incidence $\mathcal{I}'$ is defined as follows:

$$[a, b, c]\mathcal{I}'(a, b)\mathcal{I}'[a]\mathcal{I}'(\infty)\mathcal{I}'[\infty]\mathcal{I}'(m)\mathcal{I}'[m, v]\mathcal{I}'(m, v, w),$$

and

$$(m, v, w)\mathcal{I}'[a, b, c] \text{ if and only if } w = ma + b \text{ and } v = a^{\alpha}m^{\beta} + c.$$

Now expand $\mathcal{S}'$ about the regular line $[\infty]$ to obtain a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ of order $(q + 1, q - 1)$. The lines of $\mathcal{S}$ are given by

$$\mathcal{B} = \{[a, b, c] : a, b, c \in F\};$$

The points of $\mathcal{S}$ are given by $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$, where

(i) $\mathcal{P}_1 = \{m, v, w) : m, v, w \in F\}$;

(ii) $\mathcal{P}_2 = \{(a, b) : a, b \in F\}$;

(iii) $\mathcal{P}_3 = \{((a, c)) : a, c \in F\}$.

Then incidence $\mathcal{I}$ is defined as follows:

$(m, v, w)\ \mathcal{I}\ [a, am + w, a^{\alpha} m^{\beta} + v]\ \forall\ m, v, w, a \in F$;

$(a, b)\ \mathcal{I}\ [a, b, c]\ \forall\ a, b, c \in F$;

$((a, c))\ \mathcal{I}\ [a, b, c]\ \forall\ a, b, c \in F$.

Consider the following collinearities and concurrences which are listed as a proposition. The proof is a straightforward exercise.

Proposition 5.3.1 The following collinearities and concurrencies hold in $\mathcal{S}$.

(i) $(a,b) \sim ((x,y))$ iff $a = x$, in which case they are on $[a,b,y]$.

(ii) $(m,v,w) \sim (a,b)$ iff $w = am + b$, in which case

$(m,v,w) = (m,v,am+b)\mathcal{I}[a,b,a^{\alpha}m^{\beta}+v]I(a,b)$.

(iii) $(m,v,w) \sim ((a,c))$ iff $v = a^{\alpha}m^{\beta}+c$, in which case

$(m,v,w) = (m,a^{\alpha}m^{\beta}+c,w)\mathcal{I}[a,am+w,c]I((a,c))$.

(iv) $(m,v,w) \sim (m',v',w')$ if $m \neq m'$ and

$\left(\frac{w+w'}{m+m'}\right)^{\alpha} = \frac{v+v'}{m^{\beta}+m'^{\beta}}$,

in which case they lie on the line $\left[\frac{w+w'}{m+m'}, am+w, a^{\alpha}m^{\beta}+v\right]$.

(v) $[a,b,c] \sim [a,b,c']$ at (a,b).

(vi) $[a,b,c] \sim [a,b',c]$ at $((a,c))$.

(vii) $[a,b,c] \sim [a',b',c']$ with $a \neq a'$ iff

$\left(\frac{b+b'}{a+a'}\right)^{\beta} = \left(\frac{c+c'}{a^{\alpha}+a'^{\alpha}}\right)$, in which case they meet at

$\left(\frac{b+b'}{a+a'}, \frac{a^{\alpha}c'+a'^{\alpha}c}{a^{\alpha}+a'^{\alpha}}, \frac{ab'+a'b}{a+a'}\right)$.

Let the elements of F be labeled in any way as $F = \{m_1, m_2, \ldots, m_q\}$.

For each i, $1 \leq i \leq q$, let $\mathcal{O}_i = \{(m_i, v, w) : v, w \in F\}$. Put $\mathcal{O}_0 = \{(a, b) : a, b \in F\}$, and $\mathcal{O}_\infty = \{((a, c)) : a, c \in F\}$. It follows that $\mathcal{M} = \{\mathcal{O}_i : i \in \tilde{I}\}$ is a fan of ovoids for which $\mathcal{O}_\infty$ and $\mathcal{O}_0$ are pivotal.

Proposition 5.3.2 The two affine planes $\pi(\mathcal{O}_\infty)$ and $\pi(\mathcal{O}_0)$ are desarguesian.

Proof: Let (a, b) and (c, d) be distinct points of $\pi(\mathcal{O}_0)$. If $a = c$ then $\{(a, b), (c, d)\}^\perp = \{((a, y))|y \in F\}$, and $\{(a, b), (c, d)\}^{\perp\perp} = \{(a, y)|y \in F\}$. If $b = d$ then $\{(a, b), (c, d)\}^\perp = \{((x, b))|x \in F\}$, and $\{(a, b), (c, d)\}^{\perp\perp} = \{((x, b))|x \in F\}$. If $a \neq c$ and $b \neq d$, then $\{(a, b), (c, d)\}^\perp = \{(\frac{b+d}{a+c}, v, a(\frac{b+d}{a+c}) + b|v \in F\}$ and $\{(a, b), (c, d)\}^{\perp\perp} = \{(x, y)|y = (\frac{b+d}{a+c})x + a(\frac{b+d}{a+c}) + b,\ x \in F\}$. Thus the lines of $\pi(\mathcal{O}_0)$ either look like $y = mx + B$ or like $x = B$ with $m, B \in F$, i.e. the points and lines of $\pi(\mathcal{O}_0)$ can be coordinatized by F and hence $\pi(\mathcal{O}_0)$ is desarguesian. Similarly, $\pi(\mathcal{O}_\infty)$ can be shown to be desarguesian. ■

As a kind of converse, now suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a $GQ(q+1, q-1)$ with a fan $\mathcal{M} = \{\mathcal{O}_i : i \in \tilde{I}\}$ for which both $\mathcal{O}_\infty$ and $\mathcal{O}_0$ are pivotal. Suppose also that the affine planes $\pi(\mathcal{O}_\infty)$ and $\pi(\mathcal{O}_0)$ are desarguesian.

Lemma 5.3.3 $\mathcal{S}$ is coordinatized as above.

Proof: Constrict about $\mathcal{O}_\infty$ to obtain a GQ $\mathcal{S}_\infty$ that is an amalgamation of planes. Then by theorem 3.4.6 $\mathcal{S}_\infty$ is an amalgamation of desarguesian planes and hence $\mathcal{S}_\infty$ is coordinatized by an admissible pair (α, β) as above.

Since expanding $\mathcal{S}_\infty$ about the regular line L_∞ is the inverse process of constricting $\mathcal{S}$ about $\mathcal{O}_\infty$, one may assume that $\mathcal{S}$ is coordinatized as above. ■

The following definitions hold for any $GQ(s,t)$. A **whorl about a point** x is a collineation fixing everything incident with x. A whorl is called a **homology** provided it is either the identity or it fixes a unique point y outside of $x^\perp$. In this case y is called the **center of the homology**. A whorl is called a **symmetry** provided it fixes each point of $x^\perp$. There can be at most s symmetries about a point x. If x does have s symmetries about it, x is said to be a **center of symmetry**; in this case x is also regular. Homologies and symmetries are defined dually for lines with the term **axis** used in place of center. We now wish to examine homologies and symmetries in $\mathcal{S}_\infty$. The next results come from [Pay77] and are reviewed in Chapter 12 of [PT84].

Proposition 5.3.4 β is additive if and only if for some $[a_0]$, the pair $\{[a_0], M\}$ is regular for some line M not meeting $[\infty]$, if and only if $[a]$ is an axis of symmetry for all $a \in F$. In this case, the symmetries about $[a_0]$ are given as follows (where it suffices to give the action on points not collinear with (∞) and lines not concurrent with $[\infty]$).

For $\sigma \in F$,

$(m, v, w) \mapsto (m + \sigma, v + a_0^\alpha \sigma, w + a_0 \sigma);$

$[a, b, c] \mapsto [a, b + \sigma(a + a_0), c + \sigma(a^\alpha + a_0^\alpha)].$

Proposition 5.3.5 β is multiplicative if and only if there is a full group of $q - 1$ homologies fixing $(0, 0, 0)$ and (∞) and all points in their perp. In this case the homologies are determines as follows:

For $0 \neq t \in F$,

$(m, v, w) \mapsto (tm, t^\beta v, tw);$

$[a, b, c] \mapsto [a, tb, t^\beta c].$

Proposition 5.3.6 α is additive if and only if for some (m_0), the pair $\{(m_0), p\}$ is regular for some point p not collinear with (∞), if and only if (m) is a center of symmetry for all $m \in F$. In this case the symmetries about (m_0) are:

For $\sigma \in F$:

$(m, v, w) \mapsto (m, v + \sigma^\alpha(m^\beta + m_0^\beta), w + \sigma(m + m_0));$

$[a, b, c] \mapsto [a + \sigma, b + \sigma m_0, c + \sigma^\alpha m_0^\beta].$

Proposition 5.3.7 α is multiplicative if and only if there is a full group of $q - 1$ homologies fixing $[0, 0, 0]$ and $[\infty]$ and all lines in their perp. In this case the homologies are determined as follows:

For $0 \neq t \in F$,

$(m, v, w) \mapsto (m, t^\alpha v, tw);$

$[a, b, c] \mapsto [ta, tb, t^\alpha c].$

5.4 Relating Payne's Axioms to α and β

In this section the coordinatization developed above will be used to obtain a characterization of $GQ(q+1, q-1)$. This characterization is related to that of Payne discussed earlier. Here Payne's axioms will be examined vis-a-vis the admissible pair (α, β).

Suppose β is additive. In $\mathcal{S}$, let $L_1 = [a_1, b_1, c_1]$, $L_2 = [a_2, b_2, c_2], M_1 = [a_3, b_3, c_3], M_2 = [a_4, b_4, c_4]$ be such that $L_1 \not\sim M_1$ and such that L_1, M_1 meet L_2, M_2 in four distinct points outside of $\mathcal{O}_\infty \cup \mathcal{O}_0$. Let $x_i = L_i \cap \mathcal{O}_\infty$, and let y_i be the point of M_i collinear with x_i, $i = 1, 2$. Suppose $y_1 \in \mathcal{O}_0$; then $y_1 = (a_3, b_3) \sim ((a_1, b_1)) = x_1$. This forces $a_1 = a_3$. As lines in $\mathcal{S}_\infty$, we see that L_1, M_1 both meet the line $[a_1]$.

Since β is additive each line of the form $[a]$ in $\mathcal{S}_\infty$ is a regular line. Specifically $[a_1]$ is regular, and hence $\{L_1, M_1\}$ is a regular pair of lines in $\mathcal{S}_\infty$. Observe that $[a_1], L_2, M_2 \in \{L_1, M_1\}^\perp$. In $\mathcal{S}_\infty$, $[a_2]$ meets $[a_1]$ at the point (∞); $[a_2]$ meets L_2 at the point (a_2, b_2). Then $[a_2]$ must meet M_2 at the point (a_2, b_4). But note that $(a_2, b_4) \sim ((a_2, c_2)) = x_2$. This means that $y_2 = (a_2, b_4)$, thus $y_2 \in \mathcal{O}_0$. Thus the following is proved.

Proposition 5.4.1 β is additive implies A_3.

While the above is rather geometric in nature, the same proposition can be proved using coordinates.

Consider first the following lemma.

Lemma 5.4.2 If β is additive, then $a_1^\alpha(\ (\frac{b_1+b_3}{a_1+a_2})^\beta \ + \ (\frac{b_1+b_3}{a_1+a_4})^\beta \ ) + a_2^\alpha(\frac{b_1+b_3}{a_1+a_2})^\beta + a_4^\alpha(\frac{b_1+b_3}{a_1+a_4})^\beta \neq 0.$

Proof: Put $X = z_1 + z_2, \ Y = z_2 + z_3, \ Z = z_3 + z_1 = X + Y$. Then the admissibility condition says $u_0X + u_1Y + u_2(X+Y) = 0 \Rightarrow u_0^\alpha X^\beta + u_1^\alpha Y^\beta + u_2^\alpha(X+Y)^\beta \neq 0.$ ∎

Now consider a proof of proposition 5.4.1 which uses coordinates. Again, let β be additive and let L_i, M_i, x_i, and $y_i, i = 1, 2$ be as hypothesized above. Observe that as L_1 meets L_2 and M_2 outside of $\mathcal{O}_\infty \cup \mathcal{O}_0$, $a_1 \neq a_2, a_4$. It remains to be shown that $y_2 \in \mathcal{O}_0$. But this happens if and only if $a_2 = a_4$. Suppose that $a_2 \neq a_4$ and develop a contradiction. Referring back to the coordinatization section recall that $L_1 \sim L_2, \ M_1 \sim L_2, \ L_1 \sim M_2$, and $M_1 \sim M_2$ if and only if the following equations hold (respectively):

$$(\frac{b_1 + b_2}{a_1 + a_2})^\beta = \frac{c_1 + c_2}{a_1^\alpha + a_2^\alpha} \tag{5.1}$$

$$(\frac{b_3 + b_2}{a_1 + a_2})^\beta = \frac{c_3 + c_2}{a_1^\alpha + a_2^\alpha} \tag{5.2}$$

$$(\frac{b_1 + b_4}{a_1 + a_4})^\beta = \frac{c_1 + c_4}{a_1^\alpha + a_4^\alpha} \tag{5.3}$$

$$(\frac{b_3 + b_4}{a_1 + a_4})^\beta = \frac{c_3 + c_4}{a_1^\alpha + a_4^\alpha} \tag{5.4}$$

Adding equation (5.1) to (5.2) and adding (5.3) to (5.4) give the following:

$$(\frac{b_1 + b_3}{a_1 + a_2})^\beta = \frac{c_1 + c_3}{a_1^\alpha + a_2^\alpha} \tag{5.5}$$

$$(\frac{b_1 + b_3}{a_1 + a_4})^\beta = \frac{c_1 + c_3}{a_1^\alpha + a_4^\alpha} \tag{5.6}$$

Solving for c_1+c_3 in (5.5) and (5.6) gives $c_1+c_3 = (\frac{b_1+b_3}{a_1+a_2})^\beta(a_1^\alpha+a_2^\alpha) = (\frac{b_1+b_3}{a_1+a_4})^\beta(a_1^\alpha + a_4^\alpha)$. But this gives $a_1^\alpha(\ (\frac{b_1+b_3}{a_1+a_2})^\beta \ + \ (\frac{b_1+b_3}{a_1+a_4})^\beta \ ) + a_2^\alpha(\frac{b_1+b_3}{a_1+a_2})^\beta + a_4^\alpha(\frac{b_1+b_3}{a_1+a_4})^\beta = 0$. Observe that this contradicts the above lemma. This provides another proof of proposition 5.4.1.

Recall that A_3 forces $(\mathcal{O}_0)$ to be a corregular point of $\mathcal{S}_\infty$, and from page 225 of [Pay77] it can be seen that this forces β to be additive. This gives this following theorem.

Theorem 5.4.3 If $\mathcal{S}$ and $\mathcal{S}_\infty$ are coordinatized as above, then A_3 holds if and only if β is additive.

At this point the relationship between β and A_5 is examined. Start with the following lemmas.

Lemma 5.4.4 Let a, a_1, a_2 be distinct elements of F. Then $\{(a_1, b_1), ((a_2, c_2))\}^\perp$ has a point on $[a, b, c]$ if and only if $\left(\frac{b_1+b}{a_1+a}\right)^\beta = \frac{c_2+c}{a_2^\alpha+a^\alpha}$, in which case that point is $\left(\frac{b_1+b}{a_1+a}, \frac{a_2^\alpha c+a^\alpha c_2}{a_2^\alpha+a^\alpha}, \frac{ab_1+a_1b}{a_1+a}\right)$.

Proof: This follows from the collinearities and concurrences given previously. ■

Lemma 5.4.5 For $0 \neq a \in F$, $\left(\frac{1}{a}\right)^\beta \neq \frac{1}{a^\alpha}$.

Proof: In the definition of admissibility, put $u_1 = 0$, $u_2 = 1$, $u_3 = a \neq 0,\ 1$; $z_1 = 0$, $z_2 = a^{-1}$, $z_3 = 1$. It follows that $0 = \sum u_i(z_{i+1} - z_{i-1})$, and the condition that $0 \neq \sum u_i^\alpha(z_{i+1}^\beta - z_{i-1}^\beta)$ is exactly the inequality of the lemma. ■

Observe now the relationship between A_5 and α (in the presence of A_3).

Theorem 5.4.6 Let $\mathcal{S}$ and $\mathcal{S}_\infty$ be coordinatized as above. Also, suppose that $\mathcal{S}$ satisfies property A_3. Then $\mathcal{S}$ satisfies property A_5 if and only if β is multiplicative, in which case β is an automorphism of F. In this case, by III.3 of [Pay77] β may be assumed to be the identity permutation. Moreover, $\mathcal{S}$ can be viewed as some $\mathcal{S}(\Omega^-)$.

Proof: First suppose that A_5 holds. Choose $a_1 \neq 0, 1$, and then pick a_2 such that $\left(\frac{1}{a_1}\right)^\beta = \frac{1}{a_2^\alpha}$, in which case by the above lemma $0, 1, a_1, a_2$ are all distinct. Next consider a very special setup for A_5. Consider the centric triad $\{L_1, L_2, L_3\}$, where

$$L_1 = [0,0,0] \text{ with points } ((0,0)), (0,0), (y,0,0),$$

$$L_2 = [0,1,1] \text{ with points } ((0,1)), (0,1), (y+1,1,1),$$

$$L_3 = [0,b,b^\beta] \text{ with points } ((0,b^\beta)), (0,b), (y+b,b^b,b),$$

$$\text{where } b \neq 0,1, \text{ and } [1,y,y^\beta] \text{ is a transversal of } \{L_1,L_2,L_3\}.$$

This latter holds for all $y \in F$. Now using the a_1 and a_2 chosen above, for any $b_1 \in F$ choose $c_2 \in F$ so that $\left(\frac{b_1}{a_1}\right)^\beta = \frac{c_2}{a_2^\alpha}$, from which it follows that $\{(a_1,b_1),((a_2,c_2))\}^\perp$ has a point on $[0,0,0]$. Similarly, $\{(a_1,b_1),((a_2,c_2))\}^\perp$ has a point on $[0,1,1]$ since $\left(\frac{b_1+1}{a_1}\right)^\beta = \frac{c_2+1}{a_2^\alpha}$ by the additivity of β. By the hypothesis that A_5 holds, $\{(a_1,b_1),((a_2,c_2))\}^\perp$ must have a point on $[0,b,b^\beta]$, implying that $\left(\frac{b_1+b}{a_1}\right)^\beta = \frac{c_2+b^\beta}{a_2^\alpha}$. Again using the additivity of β, we see that $\left(\frac{b}{a_1}\right)^\beta = \frac{b^\beta}{a_2^\alpha} = b^\beta \cdot \left(\frac{1}{a_1}\right)^\beta$. This must hold for all $b \neq 0,1$. Thus $\left(b \cdot \frac{1}{a_1}\right)^\beta = b^\beta \cdot \left(\frac{1}{a_1}\right)^\beta$ for all $b, \frac{1}{a_1} \in F \setminus \{0\}$. It follows that β is multiplicative.

Conversely, suppose that β is multiplicative so that β is actually an automorphism of F. In this case by III.3 of [Pay77], α may be replaced by $\alpha\beta^{-1}$ and β may be replaced by the identity map. It is then straightforward to show that the condition that $(\alpha\beta^{-1}, id)$ be admissible is the same as the condition that $\alpha\beta^{-1}$ be a permutation that gives an oval. (Put $z_i = u_i$, $i = 0, 1, 2$, modulo 3.) Hence it suffices to show that if $\beta = id$ and if we write α in place of $\alpha\beta^{-1}$, then the GQ $\mathcal{S}'$ coordinatized by the admissible pair (α, id) is isomorphic to $T_2(\Omega)$ for some oval Ω. We sketch this step.

Let the oval Ω consist of the points $(1, a, a^\alpha, 0)$, $a \in F$, together with the point $(0, 1, 0, 0)$, and having nucleus equal to $(0, 0, 1, 0)$. Coordinatize the GQ $T_2(\Omega)$ as follows. The point $(m, w, v, 1) \in PG(3, q)$ will be coordinatized as (m, v, w) as a point of $\mathcal{S}'$. The plane $[a, 1, 0, b]^T$ will be coordinatized as (a, b) as a point of $\mathcal{S}'$. The plane $[1, 0, 0, m]^T$ will be coordinatized as (m) as a point of $\mathcal{S}'$. The plane $[0, 0, 0, 1]$ will be coordinatized as the point (∞) of $\mathcal{S}'$.

Let $< (m, w, v, 1), (1, a, a^\alpha, 0) >=< (0, am+w, a^\alpha m+v, 1), (1, a, a^\alpha, 0) >$ be coordinatized as $[a, am + w, a^\alpha m + v]$ as a line of $\mathcal{S}'$, and let the line $< (m, w, v, 1), (0, 1, 0, 0) >=< (m, 0, v, 1), (0, 1, 0, 0) >$ will be coordinatized as $[m, v]$ as a line of $\mathcal{S}'$. Finally, the line $< (1, a, a^\alpha, 0), (0, 0, 1, 0) >$ is coordinatized as $[a]$ and $< (0, 1, 0, 0), (0, 0, 1, 0) >$ is coordinatized as $[\infty]$. It now is straightforward to check that (m, v, w) is incident with $[a, b, c]$ if and only if $b + w = am$ and $c + v = a^\alpha m$, and all the other incidences are also identical to those of $\mathcal{S}'$. ∎

Recalling that the coordinatization depended upon $\mathcal{S}_\infty$ having been an amalgamation of desarguesian planes and further recalling that these planes were pairwise isomorphic to the planes associated with the pivotal ovoids of $\mathcal{S}$, the above theorem leads to the following characterization.

Theorem 5.4.7 Let $\mathcal{S}$ be a GQ of order $(q+1, q-1)$ with a family of ovoids $\mathcal{M} = \{\mathcal{O}_\infty, \mathcal{O}_0, \mathcal{O}_1, \ldots \mathcal{O}_q\}$ for which $\mathcal{O}_\infty, \mathcal{O}_0$ are both pivotal. Then $\mathcal{S}$ is isomorphic to a $\mathcal{S}(\Omega^-)$ if and only if the affine planes associated with $\mathcal{O}_\infty$ and $\mathcal{O}_0$ are both desarguesian and Payne's axioms A3, A5 hold.

6. Collineations

A **collineation** is a permutation of points which preserves collinearity. In the first section of this chapter, collineations of the known $GQ(q+1, q-1)$ with q even are discussed. In most cases the collineations of the GQ are simply projective collineations. However in certain instances additional collineations are found. In the second section results of Grundhöfer, Joswig and Stroppel [GJS94] concerning $P(W(q), x)$ are discussed.

6.1 Collineations of $\mathcal{S}(\Omega^-)$

Let $F = GF(q)$ with q even, and let $\mathcal{S}$ be a $GQ(q+1, q-1)$ derived from a q-arc Ω^- in $\pi = PG(2, q)$. Let A and B be the points of π for which $\Omega^+ = \Omega^- \cup \{A, B\}$ is a hyperoval. Let $\mathcal{O}_{AB}$ (resp., $\mathcal{O}_{BA}$) be the ovoid of $\mathcal{S}$ consisting of the planes of $PG(3, q)$ through A but not B (resp., through B but not A). If $\pi_1, \ldots, \pi_q$, are the planes of $PG(3, q)$ different from π but containing the line AB, let $\mathcal{O}_i$ be the set of points of π_i not in π. Then $\mathcal{M} = \{\mathcal{O}_{AB}, \mathcal{O}_{BA}\} \cup \{\mathcal{O}_i : 1 \leq i \leq q\}$ is a fan of ovoids of $\mathcal{S}$ for which both $\mathcal{O}_{AB}$ and $\mathcal{O}_{BA}$ are pivotal. Constrict $\mathcal{S}$ about the ovoid $\mathcal{O}_{AB}$ to obtain a GQ of order q with regular point $(\mathcal{O}_{BA})$.

Observe that the points of $AG(3,q) = PG(3,q)\backslash\pi$ are all accounted for as points of $\mathcal{S}$. Some of the lines of $AG(3,q)$ are lines of $\mathcal{S}$, and those that are not must be intersections of grids of $\mathcal{S}$ (as observed earlier in observation 5.2.6). Hence it is not at all surprising that some of the collineations of $\mathcal{S}$ would also be projective collineations.

However because some of the points of $\mathcal{S}$ are in fact planes of $AG(3,q)$, it is possible that there are other collineations of $\mathcal{S}$ which are not collineations of $PG(3,q)$. It is clear that any collineation of $\mathcal{S}$ must preserve the fan $\mathcal{M}$. But in some cases ovoids consisting of planes in $AG(3,q)$ could get mapped to ovoids consisting of points of $AG(3,q)$.

If the points of $PG(3,q)$ are viewed as row vectors in F^4, any collineation of $PG(3,q)$ can be viewed as multiplication by an invertible 4×4 matrix over F followed by an automorphism of F. This follows from what is often called the Fundamental Theorem of Projective Geometry. The group formed by these collineations is called the **projective semilinear group of** F^4 and is written $P\Gamma L(4,q)$.

If $\mathcal{O}_{AB}$ and $\mathcal{O}_{BA}$ are the only pivotal members of $\mathcal{M}$, they must be left invariant or interchanged. Every other ovoid of $\mathcal{M}$ consists of the union of points on projective lines which mutually intersect in a point of the q-arc Ω^-. Because $\mathcal{O}_{AB}$ and $\mathcal{O}_{BA}$ are sets of planes in 3-space it follows that each collineation of $\mathcal{S}$ must be induced by a collineation of $PG(3, q)$ that leaves Ω^- invariant.

If either $\mathcal{O}_{AB}$ or $\mathcal{O}_{BA}$ is moved to some ovoid, $\mathcal{O}_i$, other than $\mathcal{O}_{AB}$ or $\mathcal{O}_{BA}$, then $\mathcal{O}_i$ is a regular ovoid and thus must be a member of $\mathcal{M}$ pivotal for $\mathcal{M}$. Consider the related $GQ(q, q)$, $\mathcal{S}_\infty$. In this case, in addition to $(\mathcal{O}_{BA})$, the point $(\mathcal{O}_i)$ of the regular line L_∞ is regular. By proposition 5.3.6, all points of L_∞ are regular, implying that every member of $\mathcal{M}$ is pivotal.

Also in this case if (α, id) is the admissible pair used to coordinatize the GQ of order $(q + 1, q - 1)$, α is additive, and the oval is a translation oval. Moreover, the original GQ $\mathcal{S}$ is the point-line dual of the $GQ(q - 1, q + 1)$ constructed from the hyperoval Ω^+. The details are worked out in [Pay90] for a slightly different coordinatization, but are reviewed below for the coordinatization being used here.

The point is that there are conceivably more collineations of $\mathcal{S}$ than just those induced by the collineations of $PG(3, q)$ leaving the q-arc invariant. However, even in this case the collineations of the point-line dual $GQ(q-1, q+1)$ are exactly those induced by the collineations of $PG(3, q)$ leaving invariant the hyperoval. This is used below to conclude that in fact the only case in which S actually admits collineations other than those induced by the collineations of $PG(3, q)$ leaving the q-arc invariant is the case in which the original hyperoval is regular and one of the two points deleted from the conic to give the associated q-arc is the nucleus of the conic, which is always uniquely determined.

Theorem 6.1.1 Let $\beta \in Aut(F)$ be an automorphism of maximal order and put $\alpha = \beta^{-1}$. Then let Ω^+ be the hyperoval defined by

$$\Omega^+ = \{(1, a, a^\alpha, 0) : a \in F\} \cup \{(0, 1, 0, 0), (0, 0, 1, 0)\},$$

i.e.,

$$\Omega^+ = \mathcal{C} \cup N \text{where} \mathcal{C} = \{(x, y, z, 0) \in PG(3, q) : z^\beta = x^{\beta-1}y\},\ N = \{(0, 0, 1, 0)\}$$

Define the q-arc Ω^- by

$$\Omega^- = \{(1, a, a^\alpha, 0) : a \in F\}.$$

Now define a map ρ from the GQ $\mathcal{S}(\Omega^+)$ of order $(q-1, q+1)$ to the GQ $\mathcal{S}(\Omega^-)$ of order $(q+1, q-1)$ that maps points (resp., lines) of $\mathcal{S}(\Omega^+)$ (given as objects of $PG(3, q)$) to lines (resp., points) of $\mathcal{S}(\Omega^-)$ (using the coordinatization given in the section 5.3) as follows:

$\rho : (m, w, v, 1) \ \mapsto \ [m, v, w^\alpha],$

$\rho :< (0, b, c, 1), (1, a, a^\alpha, 0) > \mapsto \ (a^\alpha, b^\alpha, c),$

$\rho :< (m, 0, v, 1), (0, 1, 0, 0) > \mapsto \ (m, v),$

$\rho :< (m, w, 0, 1), (0, 0, 1, 0) > \mapsto \ ((m, w^\alpha)).$

Then ρ is a duality.

The proof is a straightforward matter of checking that incidences are preserved.

First consider the case that Ω^+ is a regular hyperoval. In this case every collineation which stabilizes Ω^+ acts 3-transitively on the points of the conic $\mathcal{C}$ while fixing the nucleus N. Here choose $\beta = 2$. In this case there is a collineation θ of $PG(3,q)$ leaving Ω^+ invariant and defined on points by

$$\theta : (x, y, z, w) \ \mapsto \ (x + y, y, z + y, w).$$

Clearly θ induces a collineation (also called θ) of $\mathcal{S}(\Omega^+)$. Using ρ to transform θ into a collineation $\bar{\theta}$ of $\mathcal{S}(\Omega^-)$ provides the following theorem.

Theorem 6.1.2 Let Ω^- be the q-arc containing the points $(1, a, a^{1/2}, 0)$, and let $\mathcal{S}(\Omega^-)$ be the corresponding $GQ(q+1, q-1)$. Then there is a collineation of $\mathcal{S}(\Omega^-)$ defined as follows:

$\bar{\theta} : \ [m, v, w] \ \mapsto \ [m + w^2, v + w^2, w];$

$\bar{\theta} : \ (a, b, c) \ \mapsto \ (\frac{q}{a+1}, \frac{b}{a+1}, c + \frac{b^2}{a+1}), \quad a \neq 1;$

$\bar{\theta} : \ (1, b, c) \ \mapsto \ (b^2, c + b^2);$

$\bar{\theta} : \ (a, b) \ \mapsto \ (1, a^{1/2}, b + a);$

$\bar{\theta} : \ ((a, c)) \ \mapsto \ ((a + c^2, c)).$

Again the matter of checking that incidences are preserved is straightforward.

The important thing to notice here is that the ovoid of points $((a, c))$ (corresponding to $\mathcal{O}_\infty$ in the general treatment) is left invariant, but the ovoid of points (a, b) (corresponding to $\mathcal{O}_0$ in the general treatment) is interchanged with the ovoid of points of the form $(1, b, c)$. The following result includes an explicit collineation in the regular hyperoval case that makes it clear that all ovoids of the fan other than the one consisting of points of the form $((a, c))$ are in the same orbit of the collineation group of $\mathcal{S}(\Omega^-)$. Moreover, since the nucleus of the regular hyperoval is fixed by all collineations of $PG(3, q)$ that stabilize the hyperoval, in this case the ovoid of points $((a, c))$ is stabilized by all collineations of $\mathcal{S}(\Omega^-)$.

Using the same technique as above to find a collineation of $\mathcal{S}(\Omega^-)$ when Ω^- completes to a translation hyperoval, one finds the following collineation.

Theorem 6.1.3 Let $\alpha \in Aut(F)$ be an automorphism of maximal order so that (α, id) is an admissible pair. For $b, d \in F$ with $d \neq 0$, the map $\theta(b, d)$ given below is a collineation of $\mathcal{S}(\Omega^-)$.

$$\theta(b,d) : [m, v, w] \;\mapsto\; [m, mb + vd, m^\alpha b + wd];$$

$$\theta(b,d) : (a, b, c) \;\mapsto\; (ad + b, bd, cd);$$

$$\theta(b,d) : (m, v) \;\mapsto\; (m, mb + vd);$$

$$\theta(b,d) : ((m, w)) \;\mapsto\; ((m, m^\alpha b + wd)).$$

When $2 \neq \alpha \neq 2^{-1}$, i.e. when the hyperoval does not contain a conic, the full collineation group of $\mathcal{S}(\Omega^-)$ leaves invariant each of the two special ovoids with points of the forms (a, b) and $((a, c))$, respectively, even though all the ovoids in the fan are pivotal for the fan, and the other q members of the fan are all in the same orbit, on which the group of collineations acts doubly transitively. Thus the only case where $\mathcal{S}$ has collineations outside of $P\Gamma L(4, q)$ is when $\Omega^- \cup \{A, B\}$ contains a conic whose nucleus is A or B.

6.2 Collineations of $P(W(q), x)$

While most of this thesis has been concerned only with q even, for this section the parity of q is not restricted. Let M be the matrix associated with the alternating form g of Section 2.1. Let $GL(n, q)$ be the $n \times n$ invertible matrices over $F = GF(q)$.

The full collineation group of the classical symplectic quadrangle $W(q)$ is given by the semidirect product $\Delta = PGSp_4F \rtimes Aut(F)$ where $PGSp_4F$ is the factor group $\{B \in GL(4, q) | BMB^T = \lambda M, \lambda \in F\}/\{cI | c \in F\}$. Theorem 2.2 of [GJS94] shows that the full collineation group of $P(W(q), x)$ is Δ_x, the stabilizer of x in Δ. If x is chosen to be the point with represented by $(1, 0, 0, 0)$ then Δ_x is the subgroup of Δ generated by the matrices of the form

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ y & a_1 & a_2 & 0 \\ z & a_3 & a_4 & 0 \\ b & c & d & |A| \end{bmatrix}$$

where $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ is any matrix of $GL(2, F)$, $|A|$ is its determinant,

$$\begin{bmatrix} y \\ z \end{bmatrix} = |A|^{-1} A \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix},$$

and b, c, d are any elements of F.

[PM98] gave what were thought to be corrections to errata of [GJS94]. However due to [Gru99] it is now clear that one of the "errors" was in fact a misinterpretation of notation on our part. In order to reconstruct $P(W(q), x)$ using methods of [Str92], the authors of [GJS94] refer to a subgroup of Δ_x which they write as $\Gamma = (PSp_4F)_x$. PSp_4F is the subgroup of $PGSp_4F$ where the λ referred to above is a square in F. In [PM98] PSp_4F was mistakenly interpreted as $PGSp_4F$ and hence the confusion. When q is even then of course $PSp_4F = PGSp_4F$, but when q is odd $PSp_4F < PGSp_4F$ as then only half of the elements of F are squares.

7. Other constructions

Attempts have been made to construct truly new GQ whose parameters differ by 2. For example Mason [Mas97] provides a recipe of sorts which may produce new GQ of order q with q an odd power of 2 which would in turn give new $GQ(q-1, q+1)$. However to date no such new quadrangles have been found.

Alternative constructions for the known GQ have also been developed. Such constructions give different insight into the nature of the GQ and perhaps may lead to new developments. Two such alternatives are presented here. While these constructions do not yield new GQ, the hope is that viewing them from different points of view may help us to understand their properties. It is also conceivable that a method for developing new GQ may be found by examing these alternative methods.

7.1 Constructions using groups and cosets

This first alternative construction is due to Payne [Pay92]. It is similar in nature to Kantor's construction in [Kan80]. Instead of beginning in projective 3-space, the geometry is developed algebraically using groups and cosets.

Let G be a group of order s^3 for some $s > 1$ with a family $\mathcal{F}^+ = \{A_*, A_\infty, A_1, \ldots, A_s\}$ of $s+2$ subgroups of order s. $\mathcal{F}^+$ is called a **fourgonal partition** of G provided $A_iA_j \cap A_k = \{e = \text{identity element of } G\}$ for all distinct triple $i, j, k \in I^* = \{*, \infty, 1, \ldots, s\}$. Suppose that $\mathcal{F}^+$ is in fact a fourgonal partition of G. Three related GQ can be constructed.

Proposition 7.1.1 If $\mathcal{P}^+$ consists of the members of G and if $\mathcal{B}^+ = \{A_ig | A_i \in \mathcal{F}^+, g \in G\}$ then $\mathcal{S}^+ = (\mathcal{P}^+, \mathcal{B}^+, \in)$ is a $GQ(s-1, s+1)$ ($\mathcal{S}^+$ is often written $\mathcal{S}(G, \mathcal{F}^+)$).

Proof: It is straightforward to see that each line contains s points. Let $g \in G$ and observe that for each $i \in I^*$, $g \in A_ih$ if and only if $A_ih = A_ig$; i.e. g is on at most $s+2$ lines. $A_ig = A_jg$ provided for each $a_i \in A_i$ there is some $a_j \in A_j$ such that $a_ig = a_jg$ if and only if $a_i = a_j$ if and only if $A_i = A_j$. In fact each point is on exactly $s+2$ lines. Two points g, h are on two lines say A_ig, A_jg provided there exist $a_i \in A_i, a_j \in A_j$ with $h = a_ig = a_jg$ if and only if $a_i = a_j = e$ (as $A_i \cap A_j = \{e\}$) if and only if $g = h$. Thus two distinct points are on at most one line.

Consider the line given by $L = A_jg$ and let L_i, L_k meet L at a_ig and a_kg respectively (so $a_i, a_k \in A_j$). Then $L_i = A_ia_ig, L_k = A_ka_kg$ for some $A_i, A_k \in \mathcal{F}^+ \backslash \{A_j\}$. Now $x \in L_i \cap L_k$ provided there exist $b \in A_i, c \in A_k$ with

$x = ba_ig = ca_kg$ if and only if $ba_i = ca_k$ if and only if $c = b(a_ia_k^{-1})$ in which case $c \in A_iA_j$. Hence $c \in A_iA_j \cap A_k$, forcing $c = e$ in which case $b = a_ka_i^{-1} \in A_j$. Now $b \in A_i \cap A_j$ implies $b = e$, and hence $a_i = a_k$. As L_i, L_j now have at least two common points (x and a_ig), this implies $L_i = L_k$, hence no triangles can exist. Finally observe that there are $|G| = s^3 = [1+(s-1)][1+(s-1)(s+1)]$ points and there are $\sum_{i\in I^*}\frac{|G|}{|A_i|} = (s+2)s^2 = [1+(s+1)][1+(s-1)(s+1)]$ lines. (Observe that $A_ig = A_jh$ if and only if $A_i = A_jhg^{-1} \leq G$ if and only if $A_jhg^{-1} = A_j$ if and only if $A_i = A_j$.) Thus $\mathcal{S}^+$ is a $GQ(s-1, s+1)$. ■

For each $i \in I^*$ let $\mathcal{A}_i$ be the factor group $G/A_i = \{A_ig|g \in G\}$ and let $\mathcal{M}^+ = \{\mathcal{A}_i|i \in I^*\}$.

Proposition 7.1.2 $\mathcal{M}^+$ is a packing of spreads of $\mathcal{S}^+$; moreover if $A_* \lhd G$ then $\mathcal{A}_*$ is a normal spread of $\mathcal{M}^+$.

Proof: As any two cosets of the same subgroup must be disjoint and as each A_i has index s^2 in G, it is clear that $\mathcal{M}^+$ is a packing of spreads. Assume $A_* \lhd G$. Let $L_1, L_2 \in \mathcal{A}_*$ and choose $g \in L_1$ (i.e. $L_1 = A_*g = gA_*$). Let h be the unique point of L_2 collinear with g, that is $L_2 = A_*h = hA_*$ and $h \in A_ig$ for some (unique) $A_i \in \mathcal{F}^+\backslash\{A_*\}$. Specifically there is some $b \in A_i$ such that $h = bg$. Choose some $a \in A_*$ which implies $ha \in L_2$. If L is the line $A_i(ha)$ then also $L = A_i(bga) = A_i(ga)$. Hence L meets L_2 at ha, and L meets

L_1 at ga. Therefore $\{L_1, L_2\}^\perp = \{A_i(ga)|a \in A_*\} = \{A_i(x)|x \in L_1\} \subset \mathcal{A}_i$.

Let $A_i = \{b_1 = e, b_2, \ldots, b_s\}$ and hence for $a \in A_*$ the line $A_i(ga)$ is $\{ga, b_2ga, \ldots, b_sga\}$. For $k = 1, \ldots, s$ let $L_k = A_*(b_kg) = (b_kg)A_* = \{b_kga|a \in A_*$. Thus for any $a \in A_*, k \in \{1, \ldots, s\}$, $L_k \cap A_i(ga) = b_kga$. Conclude that $\{L_1, L_2\}^{\perp\perp} = \{L_k|1 \leq k \leq s\} \subset \mathcal{A}_*$. ■

Suppose now $A_* \lhd G$; let $\mathcal{F} = \mathcal{F}^+ \backslash \{A_*\}$, $I^\infty = \{\infty, 1, 2, \ldots, s\}$, and for each $i \in I^\infty$ let $A_i^* = A_*A_i$. Constrict about the normal spread $\mathcal{A}_*$ in $\mathcal{S}(G, \mathcal{F}^+)$ to get a new $GQ(s,s)$ called $\mathcal{S}(G, \mathcal{F}) = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ where $\mathcal{P} = \mathcal{P}^+ \cup \{\{L, L'\}^\perp | L, L' \in \mathcal{A}_*, L \neq L'\} \cup \{(\infty)\}$ and $\mathcal{B} = (\mathcal{B}^+ \backslash \mathcal{A}_*\} \cup \{[\mathcal{A}_i] | i \in I^\infty\}$. In $\mathcal{S}(G, \mathcal{F})$, points of G are incident with the cosets which contain them. $\{L, L'\}^\perp$ is incident with $[\mathcal{A}_i]$ provided $\{L, L'\}^\perp \subset \mathcal{A}_i$. $\{L, L'\}^\perp$ is incident with A_ig provided $A_ig \subset \{L, L'\}^\perp$, and the point (∞) is incident with each $[\mathcal{A}_i]$.

Proposition 7.1.3 The point (∞) is a regular point of $\mathcal{S}(G, \mathcal{F})$.

Proof: It has been shown that $\mathcal{O}_\infty$ is a normal ovoid of a $GQ(s, s-2)$ if and only if constricting about $\mathcal{O}_\infty$ yields a $GQ(s,s)$ with the new line at infinity L_∞ regular. Interpreting the point-line dual of this statement for the current setting proves the proposition. ■

At this point it is natural to ask what further conditions are need to get a $GQ(s+1, s-1)$ which is somehow related to $\mathcal{S}(G, \mathcal{F})$. Assume that G is elementary abelian of order s^3, that is $s = p^r$ for some r and some prime p, furthermore each non-identity element of G has order p. Also assume the existence of the family $\mathcal{F}$ but not necessarily the family $\mathcal{F}^+$ assumed above (the existence of A_* is not assumed). $\mathcal{F}$ is said to be a **Kantor family** provided the following hold:

i) For each $A_i \in \mathcal{F}$ there is a subgroup A_i^* of G having order s^2 with $A_i \leq A_i^*$.

ii) $A_i A_j \cap A_k = \{e\}$ for distinct $i, j, k \in I^\infty$.

iii) $A_i^* \cap A_j = \{e\}$ for distinct $i, j \in I^\infty$.

The A_i^* are called the tangent spaces of $\mathcal{F}$ at A_i. Property ii) (which is the fourgonal partition property restricted to $\mathcal{F}$) is often called K1, and property iii) is often called K2. Observe that without assuming the existence of A_* one may still construct $\mathcal{S}(G, \mathcal{F})$ be replacing the old "points" of the form $\{L, L'\}^\perp$ (where L, L' were distinct members of A_*) with cosets $A_i^* g, i \in I^\infty, g \in G$ Each $A_i^* g$ is incident with $[\mathcal{A}_i]$ and with each coset $A_i h$ contained in $A_i^* g$.

Proposition 7.1.4 Each $[\mathcal{A}_i], i \in I^\infty$, is a regular line of $\mathcal{S}(G, \mathcal{F})$.

Proof: Consider a specific line of the form $[\mathcal{A}_i]$. Let L be a line of $\mathcal{S}(G, \mathcal{F})$ not meeting $[\mathcal{A}_i]$; this makes L a coset $A_j g$ for some $g \in G$ with $i \neq j$.

For each $a \in A_j$, ag is a point of A_jg collinear with the point $A_i^*(ag)$ on $[\mathcal{A}_i]$ by the line $A_i(ag)$. The point A_j^*g of A_jg is collinear with the point (∞) on $[\mathcal{A}_i]$ by the line $[\mathcal{A}_j]$. Thus $\{[\mathcal{A}_i], A_jg\}^\perp = \{A_i(ag)|a \in A_j\} \cup \{[\mathcal{A}_j]\}$. Note that for each $b \in A_i$ the line $A_j(bg)$ meets each $A_i(ag)$ at the point $a(bg) = b(ag)$ (as G is abelian). Hence $\{[\mathcal{A}_i], A_jg\}^{\perp\perp} = \{A_j(bg)|b \in A_i\} \cup \{[\mathcal{A}_i]\}$ which implies $|\{[\mathcal{A}_i], A_jg\}^{\perp\perp}| = s + 1$. ■

Observe that this means that the point (∞) is a coregular point of $\mathcal{S}(G,\mathcal{F})$. If $\mathcal{S}(G,\mathcal{F})$ were in fact the result of constricting about a normal spread in $\mathcal{S}(G,\mathcal{F}^+)$ then also (∞) would be a regular point of $\mathcal{S}(G,\mathcal{F})$. By 1.5.2.iv of [PT84] this holds if and only if s is even and particularly s is a power of 2.

Since $\mathcal{S}(G,\mathcal{F})$ is a $GQ(s,s)$ with a regular line (choose say $[\mathcal{A}_\infty]$), a ($GQ(s+1, s-1)$ can be formed by expanding $\mathcal{S}(G,\mathcal{F})$ about $[\mathcal{A}_\infty]$. Let $\mathcal{F}^- = \mathcal{F}\backslash\{A_\infty\}$, $I = \{1, \ldots, s\}$, and call the resulting GQ $\mathcal{S}(G,\mathcal{F}^-)$. Observe that $\mathcal{S}(G,\mathcal{F}^-)$ can be viewed as follows. The lines of $\mathcal{S}(G,\mathcal{F}^-)$ are just the cosets A_ig where $A_i \in \mathcal{F}^-, g \in G$. There are three types of points: the elements of G, the cosets A_i^*g where $i \in I, g \in G$, and the cosets $A_\infty A_ig$ where $i \in I, g \in G$. In fact the point set has a "natural" partitioning into ovoids.

Observation 7.1.5 Let $\mathcal{O}_*$ be the points of the form A_i^*g where $i \in I, g \in G$

and let $\mathcal{O}_\infty$ be the points of the form $A_\infty A_i g$ where $i \in I, g \in G$. Finally let $G/A_\infty^* = \{\mathcal{O}_1, \ldots, \mathcal{O}_s\}$. The set $\mathcal{M} = \{\mathcal{O}_i | i \in I^*\}$ is a fan of ovoids of $\mathcal{S}(G, \mathcal{F}^-)$ for which $\mathcal{O}_\infty$ is pivotal. (This follows by straightforward examination of traces and spans of pairs in $\mathcal{O}_\infty$.)

[Pay92] provides a reinterpretation of the ideas discussed in sections 5.1 and 5.2 in terms of the coset geometric constructions above. Specifically the following characterizations are noted:

i) When $s = q$ is odd, $\mathcal{S}(G, \mathcal{F}^-) \cong P(Q(4, q), L)$ if and only if for any distinct members $i, j, k, u, v \in \{1, \ldots, q\}$, $|A_\infty A_i \cap A_j A_k \cap A_u A_v| = 1$ or q, and for any distinct members $i, j, u, v \in \{1, \ldots q\}$, $|A_i^* \cap A_j^* \cap A_u A_v| = 1$ or q.

ii) When $s = q$ is even, $\mathcal{S}(G, \mathcal{F}^-) \cong \mathcal{S}(\Omega^-)$ if and only if for any six distinct members $i, j, k, l, m, n \in \{*, \infty, 1, \ldots, q\}$ $|A_i A_j \cap A_k A_l \cap A_m A_n| = 1$ or q.

7.2 Constructions using groups and designs

While others have used design and graph theoretic approaches to study generalized quadrangles (e.g., [CL91, EP72, HP93]), De Bruyn's work [Bru99] is of particular interest and his alternative construction of $GQ(s-1, s+1)$ is given here.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a **Steiner system** with parameters $(2, k, v)$

with $1 < k < v$. This means $|\mathcal{P}| = v$, each block has size k and every pair of points is on a unique block. For a fixed point x there are $v-1$ other points which determine a unique block with x. Each of these blocks has $k-1$ points other than x. Hence each point is on $\frac{v-1}{k-1}$ blocks. Let $k = s+1,\ v = st+1$. Since there are v points, each of which lies on $\frac{v-1}{k-1}$ blocks, where each block has k points, it follows that $|\mathcal{B}| = v(\frac{v-1}{k-1})(\frac{1}{k}) = \frac{(st+1)st}{s(s+1)}$.

Now let G be some multiplicative group of order $s+1$ with a map $\Delta : \mathcal{P} \times \mathcal{P} \to G$ defined by $(x, y) \to \delta_{xy}$ where x, y, z are in a common block of $\mathcal{B}$ if and only if $\delta_{xy}\delta_{yz} = \delta_{xz}$. In such a case $(\mathcal{D}, G, \Delta)$ is called an admissible triple. Observe that because every point in $\mathcal{P}$ is collinear with itself, it follows that for any $x \in \mathcal{P},\ \delta_{xx}\delta_{xx} = \delta_{xx}$. Hence δ_{xx} is the identity element e of G. Hence also $(\delta_{xy})^{-1} = \delta_{yx}$.

Lemma 7.2.1 For B a block of $\mathcal{D}$ and $x \in B,\ \{\delta_{xy} | y \in B\} = G$.

Proof: Let y, z be distinct members of B and suppose $\delta_{xy} = \delta_{xz}$. $e = (\delta_{xz})^{-1}\delta_{xy} = \delta_{zx}\delta_{xy} = \delta_{zy}$ implies $z = y$, a contradiction. Since $|B| = s+1 = |G|$, the lemma is proved. ■

Lemma 7.2.2 For B a block of $\mathcal{D}$ and $x \in B, w \notin B$ $\{\delta_{xu}\delta_{uw} | u \in B\} = G$.

Proof: Let u, v be distinct members of B and suppose $\delta_{xu}\delta uw = \delta_{xv}\delta_{vw}$. Then $\delta_{uw} = (\delta_{xu}^{-1})\delta_{xv}\delta_{vw} = \delta_{ux}\delta_{xv}\delta_{vw} = \delta_{uv}\delta_{vw}$, a contradiction as

$w \notin B$. Since $|B| = s+1 = |G|$, the lemma is proved. ■

For any such admissible triple, let Γ be the loopless graph whose vertex set consists of ordered pairs from $G \times \mathcal{P}$ where two distinct vertices (g_1, x), (g_2, y) are adjacent provided either $x = y$ or $g_2 = g_1\delta_{xy}$. The following is at the heart of [Bru99].

Proposition 7.2.3 Γ is the collinearity graph of a $GQ(s,t)$.

Proof: Clearly Γ contains $|G| \times |\mathcal{P}| = (s+1)(st+1)$ vertices. Consider some vertex (g, x) of Γ. For each of the s elements $h \in G \backslash \{g\}, (g, x)$ is adjacent to (h, x). Also since x is on t blocks of $\mathcal{D}$, each one of which has s points different from x, there are st vertices of the form $(g\delta_{xy}, y)$ adjacent to (g, x). Hence each vertex of Γ has degree $s(t+1)$.

Suppose that $p_1 = (g_1, x)$ is adjacent to $p_2 = (g_2, x)$ in Γ; let $p_3 = (g_3, z)$ be a common neighbor. If $x = y \neq z$ then $g_3 = g_1\delta_{xz}$ and $g_3 = g_2\delta_{yz} = g_2\delta_{xz} \Rightarrow g_1 = g_2 \Rightarrow p_1 = p_2$, a contradiction; hence if $x = y$ then the neighborhood of $\{p_1, p_2\}$ is $C_1 = \{(g, x) | g \in G\}$, i.e. p_1, p_2 are in a unique maximal clique of size $s+1$. If $p = (h, w)$ is a vertex not in C_1 then observe that p is adjacent to exactly one vertex of C_1, namely (h, x).

On the other hand if x, y and z are all distinct then $g_3 = g_1\delta_{xz} = g_2\delta_{yz}$. But $g_2 = g_1\delta_{xy}$, so in fact $g_3 = g_1\delta_{xy}\delta_{yz}$. From this it follows that $\delta_{xy}\delta_{yz} = \delta_{xz}$, and thus x, y, and z are all contained in a common block B of $\mathcal{D}$. Hence the neighborhood of $\{p_1, p_2\}$ is $C_2 = \{(g_1\delta_{xu}, u) | u \in B\}$. Again p_1, p_2 are in a unique maximal clique of size $s+1$. If $p = (h, w)$ is a vertex not in C_2 then p is adjacent to exactly one vertex of C_2, namely $(g_1\delta_{xw}, w)$ if $w \in b$ or $(g_1\delta_{xu}, u)$ where u is the unique member of B such that $\delta_{xu}\delta_{uw} = hg_1^{-1}$. The unique existence of such a u is given by the preceding lemma.

Let $\mathcal{Q}$ be the geometry whose points are the vertices of Γ and whose lines are the maximal cliques of size $s+1$ in Γ. Based on the counts above, every point of $\mathcal{Q}$ is on $t+1$ lines. Since each point outside of a given maximal clique has been shown to be adjacent with a unique member of that maximal clique, $\mathcal{D}$ is a $GQ(s, t)$. ■

With this theorem proved, it is also shown that if an admissible triple has $\mathcal{D} = AG(2, q)$ with direction vectors given by $E = \{e_1, \ldots e_{q+1}\}$ satisfying $e_k + \lambda e_i + (1-\lambda)e_j \neq 0$ for all $\lambda \in GF(q)$ and for all triples $e_i, e_j, e_k \in E$, then the $GQ(q-1, q+1)$ associated with this admissible triple is one arising from a hyperoval in the standard way.

A. Projective planes and 3-space

Volumes have been written on finite projective geometry. While [Dem68] will likely remain the standard reference for finite geometers, [Cox74] and [BR98] provide a good introduction to projective geometry for the uninitiated reader.

The material below is meant to serve as a cursory review of some of the basic ideas from finite projective geometry. Because many of the examples of GQ in this thesis involve hyperovals, a list of the known hyperovals in Desarguesian planes is given. As most of the projective geometry used in this thesis comes from projective 3-space, this appendix is likewise restricted. Since the axioms of a projective plane serve as a foundation for projective geometry, this seems to be a natural place to begin.

A **projective plane** is a set of points and a set of lines together with an incidence relation satisfying the following three axioms:

P1) Any two points are incident with a unique common line.

P2) Any two lines are incident with a unique common point.

P3) There are at least four points, no three of which are incident with a common line.

It can be shown that any finite projective plane must satisfy the following counts: The total number of points can be written as $q^2 + q + 1$ for some $q > 1$; $q^2 + q + 1$ is also the total number of lines. Each line has $q + 1$ points incident with it; similarly each point is incident with $q + 1$ lines.

From a combinatorial point of view, these counts are often taken as the definition of a finite projective plane; that is a finite projective plane can be defined as a set of q^2+q+1 points and a set of lines which each are incident with $q + 1$ points in such a way that any two points are incident with a unique line (see for example [CL91]). These two definitions can be shown to be equivalent.

The number q is called the **order of the projective plane**. In all known examples q is a power of a prime number. In fact for any given prime power, at least one projective plane of that order exists. It is not known whether planes of non-prime power order exist, however the following theorem due to Bruck and Ryser [BR49] does give some significant restrictions.

Theorem If q is congruent to 1 or 2 mod 4 there cannot be a projective plane of order q unless $q = a^2 + b^2$ for some integers a and b.

The only other restriction on the existence of finite projective planes was given by Lam, Swiercz, and Thiel. Using coding theoretic techniques they demonstrated via a massive computer search that no projective plane of order 10 could exist [LST89]. The case $q = 6$ is ruled out by the Bruck-Ryser Theorem. Hence 12 is the smallest order for which the existence of a projective plane is unknown.

A method for constructing planes is provided later; first consider a classification of planes related to triangles. Two triangles ABC and XYZ are said to be **perspective from a point** P provided the lines AX, BY, and CZ meet at P. The triangles are said to be **perspective from a line** l provided $AB \cap XY$, $BC \cap YZ$, and $AC \cap XZ$ all lie on the line l. The following condition helps to classify projective planes:

P4) (Desargues' Axiom) Any two triangles which are perspective from a point must also be perspective from a line.

Not all planes satisfy P4. Those that do are called Desarguesian planes; those that do not are called non-Desarguesian. Because all Desarguesian planes of a given order are isomorphic to one another, much of the research related to projective planes is dedicated to constructing/examining planes which are non-Desarguesian. An **affine plane** is a projective plane with one line (and all of the points on that line) deleted. Deleting such a line introduces the notion of parallelism. In an affine plane parallelism is an equivalence relation. As with projective planes, finite affine planes can be defined combinatorially: A finite affine plane can be defined as a set of q^2 points and a set of lines which each are incident with q points in such a way that any pair of points is incident with a unique line.

An affine plane is Desarguesian iff its associated projective plane, called the **projective completion**, is Desarguesian. There are various equivalent statements of Desargues' Axiom; notably a finite plane is Desarguesian precisely if it can be coordinatized by a finite field.

A straightforward example of a projective plane can be constructed as follows. Let V be a rank 4 vector space over a finite field $F = GF(q)$. Let W be a rank 3 subspace of V. If P_W is the set of all rank 1 subspaces of W and L_W is the set of all rank 2 subspaces of W, then P_W and L_W form the points and lines, respectively, of a projective plane where incidence is defined by subspace containment. Such a plane, written $PG(2, q)$ and called **projective 2-space**, is necessarily Desarguesian. In fact any Desarguesian plane can be viewed this way.

Let P_V be the set of rank 1 subspaces of V and L_V be the set of rank 2 subspaces of V; then P_V and L_V are the points and lines of what is called **projective 3-space** written $PG(3, q)$. Vector space counting arguments show that the number of points in $PG(3, q)$ equals the number of planes in $PG(3, q)$ which is given by $q^3 + q^2 + q + 1$; the number of lines in $PG(3, q)$ is $q^4 + q^3 + 2q^2 + q + 1$.

Just as an affine plane can be constructed from a projective plane, so too **affine 3-space** can be constucted from projective 3-space by removing a projective plane with all of its points and lines. The affine 3-space associated with $PG(3, q)$ is written $AG(3, q)$.

A point of $PG(3, q)$ looks like $x =< (x_0, x_1, x_2, x_3) >$ where the x_i are field elements, not all zero. It is common to write the vector (x_0, x_1, x_2, x_3) to represent the point x generated by it. In the setting of vector spaces, the symbol $\perp$ is used to indicate orthogonality. Recall that if $S \leq V$ then $rank(S^\perp) = rank(V) - rank(S)$. Hence every plane Π of $PG(3, q)$ can be viewed as the set of points orthogonal to some given point x; i.e. $\Pi = x^\perp$. If $x = (x_0, x_1, x_2, x_3)$ it is common to write the plane $x^\perp$ as the column vector $[x_0, x_1, x_2, x_3]^T$.

The points of $PG(2, q)$ are isomorphic to the rank 1 subspaces of F^3 and the lines of $PG(2, q)$ are isomorphic to the rank 2 subspaces of F^3. For this reason it is common to write points of $PG(2, q)$ as vectors in F^3.

B. Arcs and ovals

In a projective plane, a **k-arc** is a set of k points, no three of which are collinear. An **oval** is a $q+1$-arc in a plane of order q. A permutation α of $F = GF(q)$ is called an **oval permutation** provided $\frac{x^\alpha - y^\alpha}{x-y} \neq \frac{x^\alpha - z^\alpha}{x-z}$ whenver x, y, z are distinct elements of F. If q is odd, an oval is a largest possible arc in a plane; but if q is even, an oval can be extended (uniquely) by including one additional point called the **nucleus**. Such an oval together with its nucleus is called a hyperoval.

To see a simple example of an oval, consider the points of $PG(2, q)$ which satisfy $x_1^2 = x_0 x_2$ (i.e. the conic $\{(1, t, t^2) | t \in F\} \cup \{(0, 0, 1)\}$). The famed Italian geometer Beniamino Segre [Seg55] demonstrated that if q is odd, these are the only ovals up to isomorphism. However if q is even and sufficiently large, non-conical ovals and hyperovals do exist. For any map α on F let $\mathcal{H}_\alpha = \{(1, t, t^\alpha) | t \in F\} \cup \{(0, 1, 0), (0, 0, 1)\}$. $\mathcal{H}_\alpha$ is a hyperoval if and only if α is an oval permutation. It is straightforward to see that given any α which generates $Aut(F)$, $\mathcal{H}_\alpha$ is a hyperoval; when α is the squaring map or its inverse (i.e. when $\mathcal{H}_\alpha$ contains a conic) the hyperoval is said to be **regular**.

To see that in fact $\mathcal{H}_\alpha$ is a hyperoval when $<\alpha>= Aut(F)$ consider the 3×3 matrix M whose rows are distinct points of $\mathcal{H}_\alpha$. If any of the rows come from $\{(0,1,0),(0,0,1)\}$ then clearly M is invertible. Consider now the remaining cases which have the form

$$M = \begin{bmatrix} 1 & t & t^\alpha \\ 1 & u & u^\alpha \\ 1 & v & v^\alpha \end{bmatrix}$$

where t, u, v are distinct members of F. Then M reduces to

$$\begin{bmatrix} 1 & t & t^\alpha \\ 0 & 1 & (t+u)^{\alpha-1} \\ 0 & 0 & (t+v)^\alpha + (t+v)(t+u)^{\alpha-1} \end{bmatrix}$$

which is non-invertible iff $(t+v)^\alpha + (t+v)(t+u)^{\alpha-1} = 0$ iff $(t+v)^\alpha = (t+v)(t+u)^{\alpha-1}$ iff $(t+v)^{\alpha-1} = (t+u)^{\alpha-1}$. But as α generates $Aut(F)$, $\alpha - 1 \in Perm(F)$. So M is non-invertible iff $t+u = t+v$; this can never occur as t, u, v are distinct. Thus M is invertible; in other words any triple of points from $\mathcal{H}_\alpha$ is an independent set and thus span a plane. Therefore no three points of $\mathcal{H}_\alpha$ lie on a line.

While more detailed surveys of hyperovals can be found in [Che88, Che96] and [Kor91], a brief summary of the known hyperovals in Desarguesian planes is given here. The hyperovals of the form $\mathcal{H}_\alpha$ with $\alpha \in Aut(F)$ are called **translation hyperovals**. The translation hyperovals were classified by

Payne in [Pay71a]. Specifcally if $\alpha \in Aut(F)$, then $\mathcal{H}_\alpha$ is a hyperoval if and only if α is a generator of $Aut(F)$. Observe that for $q = 2^e, e \leq 4$, the only generators of $Aut(F)$ are the squaring map and its inverse. Thus the above construction provides only conics with nuclei. In fact for $e = 1, 2$, or 3 these are the only hyperovals (the case $e = 3$ is given in [Seg57]).

For the case $q = 2^4$, there is the so called Lunelli-Sce hyperoval, first discovered by computer search in [LS58] and later constructed and examined synthetically in [BC99]. This hyperoval can be viewed as the set $\mathcal{H}_\alpha$ where $\alpha : x \mapsto x^{12} + x^{10} + \eta^{11}x^8 + x^6 + \eta^2 x^4 + \eta^9 x^2$ where η is a primitive root of $GF(2^4)$ satisfying $\eta^4 = \eta + 1$. Though this is actually part of a larger class of hyperovals discussed below, it seems appropriate to single-out the Lunelli-Sce hyperoval because of the importance it has had in the study of hyperovals.

In the specific case $q = 2^5$, there is a sporadic example $\mathcal{H}_\alpha$ with $\alpha : x \mapsto x^4 + x^{16} + x^{28} + \xi^{11}(x^{16} + x^{10} + x^{14} + x^{18} + x^{22} + x^{26}) + \xi^{20}(x^8 + x^{20}) + \xi^6(x^{12} + x^{24})$ where ξ is a primitive root of $GF(2^5)$ satisfying $\xi^5 = \xi^2 + 1$ due to O'Keefe and Penttila [OP92].

For $q = 2^e$ with e any odd integer greater than 3, six other classes of hyperovals $\mathcal{H}_\alpha$ are known. It what follows let $\sigma^2 \equiv \gamma^4 \equiv 2 \bmod (q-1)$.

i) $\alpha : x \mapsto x^6$ is due to Segre [Seg62];

ii) $\alpha : x \mapsto x^{\sigma+\gamma}$ is due to Glynn [Gly83];

iii) $\alpha : x \mapsto x^{3\sigma+4}$ is also due to Glynn [Gly83];

iv) $\alpha : x \mapsto x^{\frac{1}{6}} + x^{\frac{3}{6}} + x^{\frac{5}{6}}$ is due to Payne [Pay85b]; and

v) $\alpha : x \mapsto x^{\sigma} + x^{\sigma+2} + x^{3\sigma+4}$ is due to Cherowitzo [Che98].

vi) $\alpha : x \mapsto \frac{x^4+x^3+x^2+x}{(x^2+x+1)^2} + x^{\frac{1}{2}}$ is the Subiaco oval for e odd due to Cherowitzo, Penttila, Pinneri, and Royle [CPPR96].

There are Subiaco ovals for $q = 2^e$ when e is even also due to Cherowitzo, Penttila, Pinneri, and Royle [CPPR96] with notation modifications due to [Pay97a]. In what follows let $\delta = \zeta^{q-1} + \zeta^{1-q}$ where ζ is a primitive root of $GF(q)$ and let $\omega \in F$ satisfy $\omega^2 + \omega + 1 = 0$

When $e \equiv 0 \pmod 4, e \geq 8$ the Subiaco ovals $\mathcal{H}_\alpha$ are given by

$\alpha : x \mapsto \frac{(\delta^4+\delta^2)x^3+\delta^3x^2+\delta^2x}{(x^2+\delta x+1)^2} + (\frac{x}{\delta})^{\frac{1}{2}}$.

When $e \equiv 2 \pmod 4$ there are two classes of Subiaco ovals: one is given by

$\alpha : x \mapsto \frac{\omega^2(x^4+x)}{(x^2+\omega x+1)^2} + x^{\frac{1}{2}}$.

If $e \not\equiv 0 \pmod 5$ the other is given by

$\alpha : x \mapsto \frac{x^3+x^2+\omega^2 x}{(x^2+\omega x+1)^2} + \omega x^{\frac{1}{2}}$;

otherwise it is given by

$\alpha : x \mapsto \frac{\delta^2x^4+\delta^5x^3+\delta^2x^2+\delta^3x}{(x^2+\delta x+1)^2} + \frac{x}{\delta}^{\frac{1}{2}}$. More on these can be found in [Pay97b] and [BLP94]. The Lunelli-Sce hyperoval mentioned above is a special case of a Subiaco hyperoval. The Subiaco ovals were discovered using techniques associated with q-clans, a topic too far afield from this thesis to discuss here. It is worth noting that each q-clan has associated with it a family of hyperovals.

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