

DOMINATION GRAPHS OF NEAR-REGULAR
TOURNAMENTS
AND THE DOMINATION-COMPLIANCE GRAPH

by

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and the Domination-Compliance Graph

Thesis directed by Professor J. Richard Lundgren

ABSTRACT

The *competition graph* $C(D)$ of a digraph D is the graph on the same vertex set as D with an edge between vertices x and y if and only if there is a vertex $z \neq x, y$ such that (x, z) and (y, z) are arcs in D . Competition graphs were first introduced in 1968 by J.E. Cohen in conjunction with his study of food webs. Most recently, in 1994 Fisher, *et al.*, studied competition graphs of tournaments. In their examination of competition graphs of tournaments, Fisher, *et al.*, introduced the domination graph of a tournament. The *domination graph* $dom(T)$ of a tournament T is the graph on the same vertex set as T with edges between vertices which together beat every other vertex in T . Since the introduction of domination graphs, Fisher, *et al.*, have successfully characterized domination graphs of arbitrary tournaments. In this work, we concentrate on a series of problems closely related to the work on the domination graph of a tournament. First, we address the question, *Which tournaments have connected domination graphs?* We answer this question and

present characterizations for all tournaments that have connected domination graphs. Next we examine a graph that is closely related to the domination graph. The *domination-compliance graph* $DC(T)$ of a tournament T is the graph on the same vertex set as T with edges between pairs of vertices that together either beat every other vertex in T or are beaten by every other vertex in T . Our primary goal for this work is to find a characterization for the domination-compliance graph of a tournament. We will present some initial results on this topic. Finally, we examine the domination graph for near-regular tournaments. We characterize all connected graphs and all forests of nontrivial paths that are the domination graphs of near-regular tournaments. In addition, we develop several large classes of near-regular tournaments which have interesting structural properties.

This abstract accurately represents the content of the candidate's thesis. I recommend its publication.

Signed _____
J. Richard Lundgren

DEDICATION

I would like to dedicate this work to my wife, son, mother, father, three brothers and baby sister. Thanks for your love and support.

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1. Introduction

The focus of this dissertation is the study of two graph theoretic concepts: domination graphs of near-regular tournaments and the domination-compliance graph of a tournament. These two concepts are closely related and have their origins in competition graphs.

A digraph D is a set $V(D)$ of vertices and a set $A(D)$ of ordered pairs of distinct vertices called arcs. The **competition graph** $\mathbf{C}(D)$ of a digraph D is the graph on the same vertex set as D with an edge between vertices x and y if and only if there is a vertex $z \neq x, y$ such that (x, z) and (y, z) are arcs. Competition graphs were first introduced and studied in 1968 by J. E. Cohen [11], in conjunction with his study of food webs in ecology. Competition graphs have several important applications, such as frequency assignments in large communication networks, simplification or modification of large scale computer models, communications over noisy channels, military communications, and radio and television transmissions [36, 37]. In the last 20 years, an extensive amount of literature has been devoted to the study of competition graphs. Some samples of this literature include [7, 12, 20, 21, 24, 25, 26, 29, 30, 32, 35].

Generalizations of the competition graph have also been introduced and studied. These include the resource graph, the niche graph and the competition/resource graph. The **resource graph** $\mathbf{R}(D)$ of a digraph D is the

graph on the same vertex set as D with an edge between vertices x and y if and only if there exists a vertex $z \neq x, y$ such that (z, x) and (z, y) are arcs. Resource graphs have been considered by Sugihara [40] and Wang [41]. The **niche graph** $\mathbf{N}(\mathbf{D})$ of a digraph D is the graph on the same vertex set as D with an edge between vertices x and y if and only if there exists a vertex $z \neq x, y$ such that either (z, x) and (z, y) are arcs or (x, z) and (y, z) are arcs. Niche graphs were first introduced by Cable, *et al.*, in [8]. Since their introduction, niche graphs have been studied by Anderson, *et al.*, [1, 2, 3], Bowser and Cable [4, 5], and Fishburn and Gehrlein [13, 14, 15]. The **competition/resource graph** $\mathbf{CR}(\mathbf{D})$ of a digraph D is the graph on the same vertex set as D with an edge between vertices x and y if and only if there is an edge joining x and y in both the competition graph and resource graph of D . Competition/resource graphs have been studied by Jones, *et al.*, [23], Kim, *et al.*, [27], Seager [39], and Scott [38].

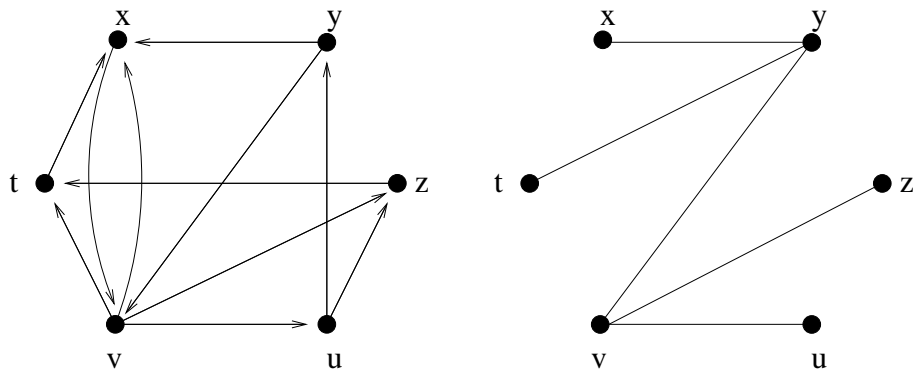


Figure 1.1. A digraph and its competition graph.

In recent years, the literature on competition graphs has focused on the study of competition graphs for a specified class of digraphs [17, 20, 28]. Of these, the most recent has been the study of competition graphs of tournaments. A tournament is a digraph without loops in which every pair of distinct vertices is joined by exactly one arc. In 1994, Fisher, Lundgren, Merz and Reid [17] were the first to consider competition graphs of tournaments. While examining the competition graph of a tournament, Fisher, *et al.*, introduced the concept of the domination graph. The **domination graph** $\text{dom}(\mathbf{T})$ of a tournament T is the graph on the same vertex set as T with an edge between vertices x and y if for all vertices $z \neq x, y$ either (x, z) is an arc or (y, z) is an arc. We will say that two such vertices form a **dominant pair**. See Figure 1.2.

The domination graph of a tournament is closely related to the competition graph of a tournament in the following way: the domination graph of a tournament T is the complement of the competition graph of the reversal of T , where the reversal of a tournament T is the tournament obtained from T by reversing all the arcs. Thus, results obtained for the domination graph of a tournament yield results for the competition graph of a tournament. The motivating factor behind studying domination graphs over competition graphs is that the domination graph of a tournament generally has fewer edges than the competition graph of a tournament, which makes it easier to state and prove

results. As a consequence, Fisher, *et al.*, [17] determined all possible domination graphs of tournaments as stated in Theorem 1.1. A vertex in a graph is a pendant vertex if it is the endpoint of exactly one edge in the graph. A cycle of length n is a graph with vertex set x_1, x_2, \dots, x_n and edges $[x_i, x_{i+1}]$, for $1 \leq i < n - 1$, and $[x_n, x_1]$. An odd-spiked cycle is a graph such that removal of all pendant vertices yields a cycle with odd length. Note that every odd cycle is an odd-spiked cycle. A path with n vertices is a graph with vertices x_1, x_2, \dots, x_n and edges $[x_i, x_{i+1}]$ for all i with $1 \leq i \leq n - 1$. A tree is a connected graph with no cycles. A caterpillar is a tree such that the removal of all pendant vertices yields a path (possibly the trivial path on one vertex). Note that every path is a caterpillar.

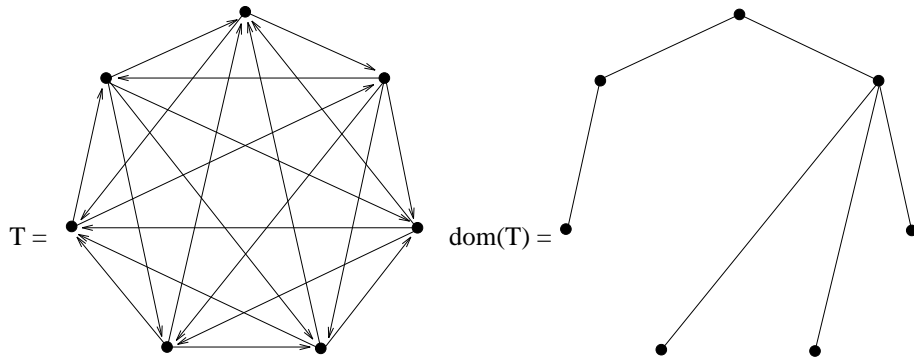


Figure 1.2. A tournament and its domination graph.

Theorem 1.1 (Fisher et al. [17]) *Let T be a tournament. Then $\text{dom}(T)$ is either an odd-spiked cycle with or without isolated vertices, or a forest of caterpillars.*

Much further work has been devoted to domination graphs. In [18], Fisher, *et al.*, extended their results in [17] to oriented graphs. Also in [16, 19], Fisher, *et al.*, investigated and answered the following two questions: *Which connected graphs are the domination graphs of tournaments* and *Which forests of nontrivial caterpillars are the domination graphs of tournaments?* Cho, Kim and Lundgren have recently considered domination graphs of regular tournaments [10]. Doherty and Lundgren [9] are currently examining domination graphs of path tournaments. Path tournaments will be defined in Chapter 5.

As with competition graphs, generalizations of the domination graph have also been introduced and studied. The **compliance graph** $\text{com}(\mathbf{T})$ of a tournament T is the graph on the same vertex set as T with an edge between vertices x and y if for all vertices $z \neq x, y$ either (z, x) is an arc or (z, y) is an arc. The compliance graph was first introduced by Jimenez and Lundgren and will be discussed in Chapter 3. In addition, the compliance graph of a tournament is related to the domination graph in the following way: the compliance graph of a tournament T is the domination graph of the reversal of T , where the reversal of a tournament T is the tournament obtained from T by reversing all the arcs in T . Note that this relationship gives us that a graph is the compliance graph of a tournament if and only if it is also the domination graph of a tournament. Thus, since domination graphs of tournaments have been characterized, we get that the compliance graphs of tournaments

are also characterized. The **mixed-pair graph** $\mathbf{mp}(\mathbf{T})$ of a tournament T is the graph on the same vertex set as T with an edge between vertices x and y if for all vertices $z \neq x, y$ either (x, z) and (z, y) are arcs or (z, x) and (y, z) are arcs. The mixed-pair graph of a tournament has been examined by Bowser, Cable, and Lundgren [6]. The **domination-compliance graph** $\mathbf{DC}(\mathbf{T})$ of a tournament T is the graph on the same vertex set as T with edges between vertices x and y if either there is an edge joining x and y in the domination graph of T or there is an edge joining x and y in the compliance graph of T . The domination-compliance graph of a tournament is studied in Chapter 3.

1.1 Overview

As noted earlier, Fisher, *et al.*, examined the question, *Which connected graphs are the domination graphs of tournaments?* In [19], they determined all connected graphs that are the domination graphs of tournaments. Since all connected domination graphs of tournaments have been determined, we found it natural to ask the following: *Which tournaments have connected domination graphs?* We address and answer this question in Chapter 2. The work presented in Chapter 2 will also prove to be of fundamental importance to the chapters that follow Chapter 2.

Which graphs are the domination-compliance graphs of tournaments? Chapter 3 addresses this question. In Section 3.2, we exhibit an upper bound on the number of edges in the domination-compliance graph of a tournament.

In Section 3.3, we examine consequences of the upper bound on the number of edges and exhibit some forbidden subgraphs. Lastly, in Section 3.5, we provide characterizations for the domination-compliance graphs of reducible and regular tournaments.

In Chapters 4 through 7, we examine domination graphs of near-regular tournaments. In particular, we address the question: *Which graphs are domination graphs of near-regular tournaments?* We do so by examining the following three questions: *Which connected graphs are the domination graphs of near regular tournaments, Which forests of paths are the domination graphs of near-regular tournaments and Which forests of caterpillars are the domination graphs of near-regular tournaments?* The first two questions are the focus of Chapters 4 and 5, respectively. The last question will be addressed in Chapters 6 and 7.

Finally in Chapter 8, we present avenues for future work with respect to the topics presented in this dissertation.

1.2 Definitions and Notation

A **graph** G is a set $\mathbf{V}(G)$ of elements called **vertices** and a list $\mathbf{E}(G)$ of pairs of those vertices called **edges**. We will denote an edge $e \in E(G)$ with its endpoints $x, y \in V(G)$ as $[x, y]$ and say that x and y are **adjacent** or **neighbors** in G . The **degree** of a vertex $x \in V(G)$ is the number of edges that have x as an endpoint. A vertex is **isolated** if it has degree 0. A vertex

is **pendant** if it has degree 1.

A **subgraph** of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An **induced subgraph** of a graph G , induced by a vertex set $S \subseteq V(G)$, is a subgraph H with vertex set S and edge set $E(H) = \{[x, y] \in E(G) : x, y \in S\}$.

A **cycle C_n of length n** is a graph with vertex set x_1, x_2, \dots, x_n and edges $[x_i, x_{i+1}]$, for $1 \leq i < n - 1$, and $[x_n, x_1]$. An **odd-spiked cycle** is a graph such that removal of all pendant vertices yields a cycle with odd length. Note that every odd cycle is an odd-spiked cycle. A **path P_n with n vertices** is a graph with vertices x_1, x_2, \dots, x_n and edges $[x_i, x_{i+1}]$ for all i with $1 \leq i \leq n - 1$.

A **tree** is a connected graph with no cycles. A **caterpillar** is a tree such that the removal of all pendant vertices yields a path (possibly the trivial path on one vertex). This path will be referred to as the **spine** of the caterpillar, and the number of vertices on the spine is the **length** of the caterpillar. A **forest** is a graph whose connected components are trees.

A **proper coloring** of a graph G is an assignment of one color to each vertex such that if two vertices are adjacent, then they are assigned different colors.

A **directed graph or digraph D** is a set $V(D)$ of vertices and a set $A(D)$ of ordered pairs of distinct vertices called **arcs**. We denote an arc from

vertex x to vertex y by either $(x, y) \in A(D)$ or $x \rightarrow y$ and say that x **beats** y . For all vertices $x \in V(D)$ the **out-set** $\mathbf{O}_D(\mathbf{x})$ or $\mathbf{O}(\mathbf{x})$ of x is the set of vertices that x beats. Similarly, the **in-set** $\mathbf{I}_D(\mathbf{x})$ or $\mathbf{I}(\mathbf{x})$ is the set of vertices that beat x . Let $\mathbf{d}_D^+(\mathbf{x})$ or $\mathbf{d}^+(\mathbf{x}) = |\mathbf{O}(\mathbf{x})|$ be the **out-degree** of x in D . We will also refer to the vertices in $O(x)$ ($I(x)$) as **out-neighbors** (**in-neighbors**) of x .

A **tournament** is a digraph without loops in which every pair of distinct vertices is joined by exactly one arc. An **n-tournament** is a tournament with n vertices. A **transmitter** is a vertex in an n -tournament with out-degree $n - 1$. Let S be a subset of $V(T)$. Then $\mathbf{T}(S)$ will denote the subtournament induced by the set of vertices in S . The **reversal** $\tilde{\mathbf{T}}$ of a tournament T is the tournament obtained from T by reversing all the arcs. An n -tournament is **regular** if n is odd and every vertex has out-degree $\frac{n - 1}{2}$. An n -tournament is **near-regular** if n is even and every vertex has out-degree either $\frac{n}{2}$ or $\frac{n}{2} - 1$. A tournament is **reducible** if its vertex set can be partitioned into nonempty sets A and B such that every vertex in A beats every vertex in B . A tournament is **strongly connected** if it is not reducible.

For more basic graph theory definitions see West [42]. Also for more on tournaments see Moon [31], Beineke and Reid [34], and Reid [33].

2. Tournaments with Connected Domination Graphs

2.1 Introduction

In this chapter, we characterize those tournaments with connected domination graphs. In the introductory chapter, we presented Theorem 1.1 which determines all possible domination graphs of a tournament. Due to this theorem, one can see that the only possible connected domination graphs of tournaments are odd-spiked cycles and caterpillars. However, Fisher, *et al.*, discovered that not every caterpillar is the domination graph of a tournament. In [19], Fisher, *et al.*, determined that the only caterpillars that are domination graphs of tournaments are those that have three or more vertices pendant to at least one end of the spine. We summarize their result below. A **star** is a tree with one vertex adjacent to all others and a **nonstellar caterpillar** (a caterpillar that is not a star) is a caterpillar that has at least two vertices on its spine.

Theorem 2.1 (Fisher, et al., [19]) *A connected graph is the domination graph of a tournament if and only if it is an odd-spiked cycle, a star, or a nonstellar caterpillar with three or more vertices pendant to at least one end of the spine.*

Knowing which connected graphs occur as the domination graphs of tournaments, we find it natural to ask: *Which tournaments have these graphs as their domination graphs?* In the remainder of this chapter, we will consider this question. In particular, we shall determine all tournaments that have stars, odd-spiked cycles, and caterpillars with three or more vertices pendant to at least one end of the spine, as their domination graphs.

2.2 Tournaments with connected domination graph

We open this section with an important digraph that will prove useful not only in this chapter but throughout this dissertation.

Define an **orientation** of an undirected graph to be an assignment of order to each of the edge pairs that results in a digraph. Then the **domination digraph** $\mathcal{D}(T)$ of a tournament T is an orientation of $\text{dom}(T)$ induced by T , i.e., $\mathcal{D}(T)$ is the digraph on the same vertex set as T with an arc (x, y) if x and y form a dominant pair and x beats y in T . The domination digraph of a tournament will be an important tool in this chapter for reasons to be explained below and which are largely due to the following lemma.

Lemma 2.2 (Fisher, et al., [18]) *Let T be a tournament and let $x \in V(T)$. In $\mathcal{D}(T)$, we have that $|I(x)| \leq 1$ and at most one vertex in $O(x)$ can have nonzero out-degree.*

Stated differently, Lemma 2.2 says that in the domination digraph a vertex can have at most one in-neighbor and at most one out-neighbor with out-neighbors. Now as we mentioned above, the domination digraph and in particular Lemma 2.2 will be an important tool in this, as well as in the forthcoming chapters. In this chapter it will be used as follows: Let G be a connected graph that is known to be the domination graph of a tournament. Then for all tournaments that have G as their domination graph, their corresponding domination digraphs must be an orientation of G . Since Lemma 2.2 provides restrictions on the structure of the domination digraph, then by abiding to these restrictions we can determine all possible orientations of G that would give a valid domination digraph. Therefore, Lemma 2.2 is being used to determine all possible domination digraphs for all tournaments that have G as their domination graph. Then by using all these possible domination digraphs as a starting point, we will then be able to proceed in constructing all tournaments that have G as their domination graph. The next theorem illustrates this procedure. Let $\mathbf{K}_{1,n}$ denote a star on $n + 1$ vertices.

Theorem 2.3 *Let T be an n -tournament with $n \geq 2$. Then the domination graph of T is a star $K_{1,n-1}$ if and only if T has a transmitter or two distinguished vertices x and y such that*

- i) y beats x ,*

ii) for all vertices $z \neq x, y$, it follows that x beats z and z beats y , and

iii) $T - \{x, y\}$ induces a subtournament without a transmitter.

Proof: First assume that T is a n -tournament whose domination graph is $K_{1, n-1}$. Let x be the vertex with degree $n - 1$ in $K_{1, n-1}$. Consider $\mathcal{D}(T)$. By Lemma 2.2, in the domination digraph a vertex can have in-degree at most 1 and can have at most one out-neighbor with out-neighbors. Thus, we have two possible orientations for $\mathcal{D}(T)$. These are described in the following figure.

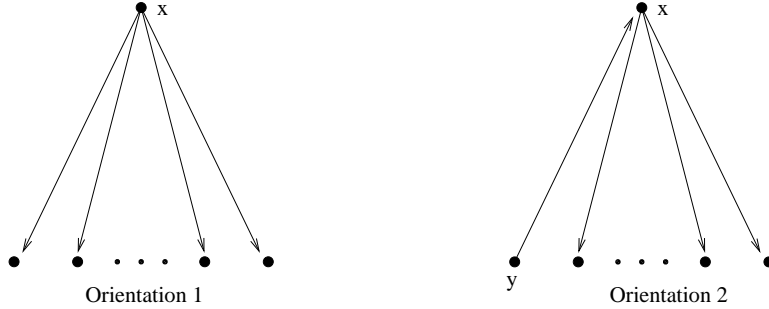


Figure 2.1. Two possible orientations of a domination graph that is a star.

First consider Orientation 1. Note that under this orientation, x has out-degree $n - 1$, so x is a transmitter.

Next consider Orientation 2. Under this orientation, x has out-degree $n - 2$. Also note that there is one vertex y with out-degree 1. Let z be an arbitrary vertex such that $z \neq x, y$. Since $[z, x] \in \text{dom}(T)$ and y beats x , it follows that z beats y . Let $T - \{x, y\}$ denote the subtournament of T on $V(T) \setminus \{x, y\}$. Now suppose that the subtournament $T - \{x, y\}$ contains a transmitter w . Then since x beats every vertex in $V(T) \setminus \{x, y, w\}$ and y

beats x , we have that y and w form a dominant pair. This is a contradiction since $[w, y] \notin \text{dom}(T)$. Thus, $T - \{x, y\}$ must be a subtournament without a transmitter.

For the converse, first assume that T is a tournament that contains a transmitter x . If $y \in V(T) \setminus \{x\}$, then x and y form a dominant pair since x beats every vertex in $V(T) \setminus \{x\}$. Thus, $K_{1, n-1} \subseteq \text{dom}(T)$. To show that $\text{dom}(T) = K_{1, n-1}$, it suffices to show that no other pair of vertices form a dominant pair. Let $y, z \in V(T) \setminus \{x\}$. Since x beats both z and y , it follows that z and y cannot possibly form a dominant pair.

Next assume that T is a tournament with two distinguished vertices x and y such that *i*) y beats x , *ii*) for all vertices $z \neq x, y$, it follows that x beats z and z beats y , and *iii*) $T - \{x, y\}$ induces a subtournament without a transmitter. First we show that $K_{1, n-1}$ is a subgraph of $\text{dom}(T)$. Since (x, z) is an arc for all vertices $z \neq x, y$, we may conclude that $[x, y] \in \text{dom}(T)$. In addition, since (z, y) is also an arc for all $z \neq x, y$, it follows that $[z, x] \in \text{dom}(T)$ for all $z \in V(T) \setminus \{x, y\}$. Thus $K_{1, n-1} \subseteq \text{dom}(T)$. To show that $\text{dom}(T) = K_{1, n-1}$, it suffices to show that no other pair of vertices form a dominant pair. Let z and w be in $V(T) \setminus \{x, y\}$. Since (x, z) and (x, w) are arcs, $[z, w] \notin \text{dom}(T)$. In addition, z and y cannot possibly form a dominant pair. This follows from the fact that $T - \{x, y\}$ does not have a transmitter. So there must exist a vertex u such that u beats z and by construction u beats

y . Therefore, $\text{dom}(T) = K_{1,n-1}$. ■

Another important digraph that will play an important role is U_n . Let \mathbf{U}_n denote the tournament on the set of vertices $\{x_1, x_2, \dots, x_n\}$ with arcs (x_i, x_j) if $i - j$ is odd and negative or even and positive. Notice that U_n is a regular tournament when n is odd and is a near-regular tournament otherwise. We leave the verification of this fact to the reader.

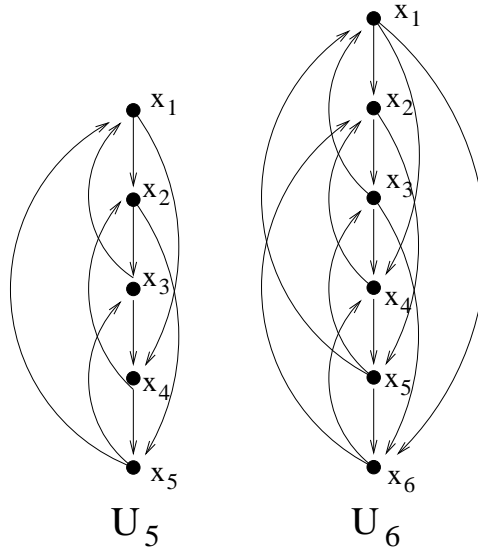


Figure 2.2. The tournaments U_5 and U_6 .

Now in [17], Fisher, *et al.*, determined that if $\text{dom}(T)$ is an odd cycle C_n , then the only possibility for T is U_n . Therefore, we have the next result.

Theorem 2.4 (Fisher, et al., [17]) *Let T be a n -tournament where n is odd. Then $\text{dom}(T) = C_n$ if and only if $T \cong U_n$. Furthermore, for all i , $[x_i, x_{i+1}] \in \text{dom}(U_n)$, where $i + 1 = (i \bmod n) + 1$.*

Observe that Theorem 2.4 explicitly tell us what pairs of vertices form dominant pairs in U_n , for n odd. In addition, Theorem 2.2 is implicitly providing information regarding pairs of vertices that do not form dominant pairs. Specifically, it gives us that if $j \not\equiv i+1, i-1 \pmod{n}$, then for vertices x_i and x_j in $V(U_n)$, there must exist a vertex $x_l \in V(U_n)$ such that x_l beats both x_i and x_j . This is true because x_i and x_j do not form a dominant pair in U_n when $j \not\equiv i+1, i-1 \pmod{n}$. This observation will be exploited throughout this work. Now it was mentioned above that U_n is a regular tournament when n is odd. Due to this fact the next result will provide a useful property regarding dominant pairs in U_n .

Lemma 2.5 *Let T be a regular tournament. If vertices $x, y \in V(T)$ form a dominant pair, then $O(x) \cap O(y) = \emptyset$.*

Proof: Let T be a regular n -tournament and $x, y \in V(T)$. Assume x and y form a dominant pair and that x beats y . Then since $[x, y] \in \text{dom}(T)$, we have that $|O(x) \cup O(y)| = n - 1$. Thus, since every vertex in a regular n -tournament has out-degree $\frac{n-1}{2}$, it follows that $|O(x) \cap O(y)| = |O(x)| + |O(y)| - |O(x) \cup O(y)| = \frac{n-1}{2} + \frac{n-1}{2} - (n-1) = 0$. Therefore, $O(x) \cap O(y) = \emptyset$. ■

Lemma 2.5 along with the following result will prove useful in the characterization of those tournaments whose domination graphs are odd-spiked cycles. A proof of the following result can be found in [17].

Lemma 2.6 (Fisher, et al., [17]) *Let S be an induced subdigraph of a digraph D . Then the induced subgraph of $\text{dom}(D)$ on the vertices of S is a subgraph of $\text{dom}(S)$.*

In [17], Fisher, *et al.*, provided the following construction for a tournament T , given an odd-spiked cycle G . Let $\{x_1, x_2, \dots, x_k\}$, where k is odd, represent the set of vertices on the cycle of G and V_i represent the set of vertices pendant at x_i . The tournament T is constructed by first letting $T(\{x_1, x_2, \dots, x_k\}) \cong U_k$ and then orienting arcs from x_i to every vertex in V_i , for all $1 \leq i \leq k$. Next, if x_i beats x_j , then arcs are oriented from vertices in V_i to vertices in V_j , every vertex in V_j beats x_i , and x_j beats every vertex in V_i . Finally, arcs within each pendant set V_i are oriented arbitrarily. Fisher, *et al.*, showed that the tournament T constructed by this method had the property that its domination graph is G . We extend this result by showing that if T is a tournament whose domination graph is an odd-spiked cycle, then T must be constructed by the method just described.

Let Γ_o denote the set of all tournaments T with the following properties. The vertex set of T can be partitioned into the sets $\{x_1, x_2, \dots, x_k\}$, V_1, V_2, \dots , and V_k , with k odd. The arcs in T are as follows:

- The subtournament $T(\{x_1, x_2, \dots, x_k\}) \cong U_k$.
- For all i , arcs are oriented from x_i to vertices in V_i .
- The arcs within each set V_i are arbitrary.

- The remaining arcs in T are as in Figure 2.3. In other words, if x_i beats x_j , then arcs are oriented from vertices in V_i to vertices in V_j , from x_j to vertices in V_i , and from vertices in V_j to x_i .

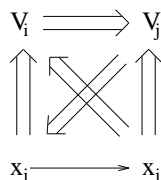


Figure 2.3. This figure determines the arcs between the V_i 's and x_i 's

If a tournament T belongs to Γ_o , we will say that T is a **spiked cycle tournament**. Notice that the construction that was discussed above always produces a spiked cycle tournament.

Theorem 2.7 *Let T be a tournament. Then the domination graph of T is an odd-spiked cycle if and only if T is a spiked cycle tournament.*

Proof: Let T be a tournament whose domination graph is an odd-spiked cycle. Assume the cycle of $dom(T)$ consists of k vertices where k is odd. Label the vertices along the cycle x_1, x_2, \dots, x_k . For $1 \leq i \leq k$, let V_i represent the set of vertices pendant to the vertex x_i , where it is possible that $V_i = \emptyset$. By Lemma 2.2, in $\mathcal{D}(T)$ a vertex x can have in-degree at most 1 and at most one vertex in $O_{\mathcal{D}(T)}(x)$ can have nonzero out-degree. Therefore, all the arcs in $\mathcal{D}(T)$ on the cycle must be oriented either clockwise or counterclockwise and the remaining arcs must be oriented toward each pendant set. So, without loss of generality

assume that $\mathcal{D}(T)$ is as depicted in Figure 2.4.

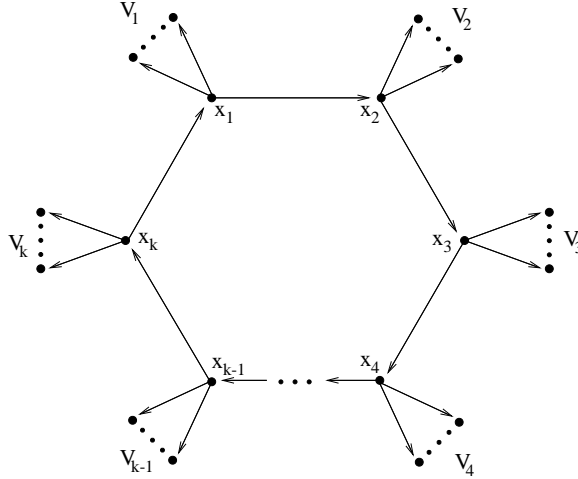


Figure 2.4. One of two possible orientations for the domination digraph of an odd spiked cycle.

We proceed to show that T is a spiked cycle tournament. We begin by showing that the set of vertices $\{x_i\}$ induce a subtournament isomorphic to U_k . By Lemma 2.6, we have that C_k is a subgraph of the domination graph of the subtournament induced by the set of vertices $\{x_1, x_2, \dots, x_k\}$. Then by Theorem 1.1, the domination graph of the subtournament $T(\{x_1, x_2, \dots, x_k\})$ must be C_k . So, we may conclude by Theorem 2.4, that the subtournament induced by the set of vertices $\{x_1, x_2, \dots, x_k\}$ is isomorphic to U_k .

Next we show that the arcs in T between the pendant sets and the vertices on the cycle must be as prescribed by Figure 2.3. For the following we will take $i + 1$ to mean $(i \bmod k) + 1$. Now suppose that $(x_i, x_j) \in A(T)$.

If $V_i = V_j = \emptyset$, there is nothing to prove. If $V_i \neq \emptyset$ and $V_j \neq \emptyset$, let $w \in V_i$ and $z \in V_j$. Since x_j and z form a dominant pair and (x_i, x_j) is an arc, we have $(z, x_i) \in A(T)$. Then since w and x_i form a dominant pair and (z, x_i) is an arc, $(w, z) \in A(T)$. But then since z and x_j form a dominant pair and (w, z) is an arc, $(x_j, w) \in A(T)$. If $V_i \neq \emptyset$ and $V_j = \emptyset$, let $w \in V_i$. Since (x_i, x_j) is an arc and x_j and x_{j+1} form a dominant pair, it follows that $(x_{j+1}, x_i) \in A(T)$. Then since x_i and w form a dominant pair and (x_{j+1}, x_i) is an arc, we have that $(w, x_{j+1}) \in A(T)$. But then since x_j and x_{j+1} form a dominant pair and (w, x_{j+1}) is an arc, $(x_j, w) \in A(T)$, as desired. Finally assume $V_i = \emptyset$ and $V_j \neq \emptyset$. Let $z \in V_j$ be arbitrary. Since z and x_j form a dominant pair and (x_i, x_j) is an arc, $(z, x_i) \in A(T)$. Therefore, it follows that the arcs in T between the pendant sets and the vertices on the cycle are as prescribed in Figure 2.3.

Finally we show that the arcs within each pendant set may be oriented arbitrarily. Since x_i beats every vertex in V_i , no two vertices in V_i can form a dominant pair in T , regardless of the structure of the arcs within V_i . Thus, since no two vertices in V_i are adjacent in $\text{dom}(T)$, the arcs within V_i can be oriented arbitrarily.

For the converse, suppose that T is a spiked cycle tournament. We will proceed by showing that $\text{dom}(T)$ is an odd-spiked cycle. Without loss of generality assume that $T(\{x_1, x_2, \dots, x_k\}) = U_k$.

First we show that $[x_i, x_{i+1}] \in \text{dom}(T)$ for all i , with $1 \leq i \leq k-1$, and that $[x_k, x_1] \in \text{dom}(T)$. Since $T(\{x_1, x_2, \dots, x_k\}) \cong U_k$, we know by Theorem 2.4 that $[x_i, x_{i+1}]$, for $1 \leq i \leq k-1$, and $[x_k, x_1]$ form dominant pairs in the subtournament induced by the set of vertices $\{x_1, x_2, \dots, x_k\}$. Thus, the pairs $\{x_i, x_{i+1}\}$, for $1 \leq i \leq k-1$, and $\{x_k, x_1\}$ beat every vertex in $\{x_1, x_2, \dots, x_k\}$. Therefore, in order to show that these pairs form dominant pairs in T , it suffices to show that they also beat every vertex in V_j , for all j . Consider the pair of vertices x_i and x_{i+1} , for $1 \leq i \leq k-1$. Let $z \in V_l$, with $1 \leq l \leq k$. We will show that z is beaten by either x_i or x_{i+1} . If $l = i, i+1$ there is nothing to prove. So assume that $l < i$. If l is odd, then since $T(\{x_1, x_2, \dots, x_k\}) = U_k$, we have that x_l beats all vertices x_j for $j > l$ and j even. Since i is either odd or even, x_l beats either x_i or x_{i+1} . Then by Figure 2.3, it follows that in either case if x_l beats x_i , then x_i beats z and if x_l beats x_{i+1} , then x_{i+1} beats z . If l is even, then by a similar argument either x_i beats z or x_{i+1} beats z . Next assume that $l > i+1$. If l is odd, then again since $T(\{x_1, x_2, \dots, x_k\}) = U_k$, it follows that x_l beats x_j for $j < l$ and j odd. Since i must either be even or odd, we have that x_l beats either x_i or x_{i+1} . Again by Figure 2.3, in either case if x_l beats x_i , then x_i beats z and if x_l beats x_{i+1} , then x_{i+1} beats z . If l is even, then by a similar argument either x_i beats z or x_{i+1} beats z . Thus, x_i and x_{i+1} form a dominant pair in T for all i , with $1 \leq i \leq k-1$. A similar argument is used to show that $[x_k, x_1] \in \text{dom}(T)$.

Next let $z \in V_i$ be arbitrary. We will show that $[z, x_i] \in \text{dom}(T)$. Since T is a spiked cycle tournament, we have by Figure 2.3 that any vertex $w \in V(T) \setminus V_i$ that beats x_i is beaten by z . Thus, since x_i beats every vertex in V_i , we may conclude that $[z, x_i] \in \text{dom}(T)$.

Lastly, we need to show that no other pair of vertices forms a dominant pair. We have shown that the set of vertices $\{x_1, x_2, \dots, x_k\}$ induces an odd cycle in the domination graph of T . If there exists another pair of vertices that forms a dominant pair, then this dominant pair would create a second cycle in $\text{dom}(T)$. But Theorem 1.1 tells us that the domination graph of a tournament can have at most one cycle. Thus, there cannot exist any other dominant pairs. Therefore, $\text{dom}(T)$ is an odd-spiked cycle. ■

Recall that a caterpillar is a tree such that the removal of all the vertices of degree one results in a path, which we refer to as the spine. As was pointed out earlier, the extreme ends of the spine have at least one vertex pendant to each of them in the caterpillar. Now Theorem 2.1 states that a caterpillar G is the connected domination graph of a tournament if and only if G has three or more vertices pendant to at least one end of the spine. Therefore, in order to characterize those tournaments that have a caterpillar as their domination graph, we only need to consider those caterpillars which have three or more vertices pendant to at least one end of the spine. We will say that such caterpillars have a **triple end**. In the remainder of this section,

we will determine every tournament that has a caterpillar with a triple end as its domination graph.

Let G be a caterpillar with a triple end. By Theorem 2.1, we may conclude that G is the domination graph of some tournament. So let T be a tournament such that $\text{dom}(T) = G$. If we consider $\mathcal{D}(T)$, we have by Lemma 2.2 that in the domination digraph of T a vertex can have in-degree at most one and can have at most one out-neighbor with out-neighbors. Thus, we can see that there are only two possible orientations up to isomorphism for the domination digraph of T . The two orientations arise from the orientation of the arc between \bar{v} and x and are depicted in Figure 2.5. Regardless of the orientation of $\mathcal{D}(T)$, one aspect of the structure of $\mathcal{D}(T)$ is fixed, as stated in the next result.

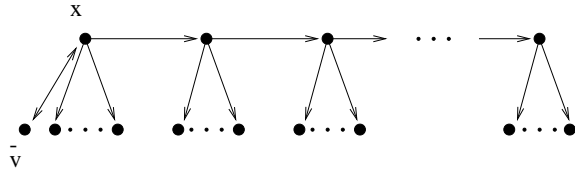


Figure 2.5. This depicts the two possible orientations for the domination digraph of a nonstellar caterpillar with a triple end.

Lemma 2.8 (Fisher, et al., [19]) *Let G be the union of k caterpillars, with $k \geq 1$ and assume that G is the domination graph of a tournament T . Assume H is component of G . Then removing all vertices with out-degree 0 from the subdigraph of $\mathcal{D}(T)$ on $V(H)$ results in a directed path.*

Assume a caterpillar G is the domination graph of a tournament T . Define the **directed spine** of $\mathcal{D}(T)$ to be the directed path that results when all vertices of out-degree 0 are removed from $\mathcal{D}(T)$. We will use the directed spine of $\mathcal{D}(T)$ to label the vertices of G as follows: Let x_1, x_2, \dots, x_k represent the vertices of G that correspond to the vertices on the directed spine of $\mathcal{D}(T)$. Also let V_i represent the set of vertices in G that have out-degree 0 in $\mathcal{D}(T)$ and that are beaten by x_i in $\mathcal{D}(T)$. Figure 2.6 depicts this labeling along with $\mathcal{D}(T)$. Note that the set V_1 may be empty if x_1 corresponds to a pendant vertex in G that beats an end vertex on the spine of G in $\mathcal{D}(T)$. See Figure 2.5.

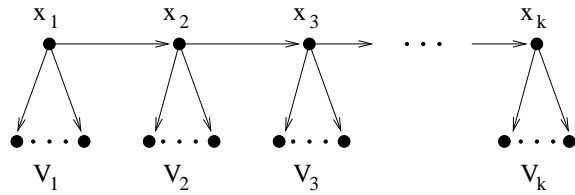


Figure 2.6. A labeling of the domination digraph of a nonstellar caterpillar with a triple end.

On a side note, if a forest of caterpillars G is the domination graph of a tournament and H is a component of G , then we define the directed spine of the subdigraph of $\mathcal{D}(T)$ on $V(H)$ analogously to the above definition. In addition, we also label the vertices of H as we describe above, i.e., by using the directed spine of the subdigraph of $\mathcal{D}(T)$ on $V(H)$.

Now it will be useful to properly color the caterpillars G under consideration with two colors, red and blue. The next result, of Fisher, *et al.*, determines the remaining arcs in Figure 2.6, except for those within the sets V_i .

Lemma 2.9 (Fisher, et al., [19]) *Let G be the union of k caterpillars, with $k \geq 1$, and assume that G is the domination graph of a tournament T . Assume H is a component of G and is properly 2-colored (such a coloring exist and is unique for a caterpillar). If the subdigraph of $\mathcal{D}(T)$ on $V(H)$ is as depicted in Figure 2.6, then the following arcs must be in $T(H)$:*

1. *Arcs that are not within a set V_i , are oriented to the right between vertices of different colors and to the left between vertices of the same color.*
2. *Arcs within each set V_i , for $i \neq k$, may be oriented arbitrarily.*

Let Γ_c denote the set of tournaments T with the following properties. The vertex set of T can be partitioned into the sets $\{x_1, x_2, \dots, x_k\}, V_1, V_2, \dots,$ and V_k , with k even and at least 2. The arcs in T are as follows:

- The subtournament $T(\{x_1, x_2, \dots, x_k\}) \cong U_k$.
- For all i , arcs are oriented from x_i to vertices in V_i .
- The arcs within each V_i , with $1 \leq i \leq k - 1$, are oriented arbitrarily.
- The arcs within V_k yield a subtournament $T(V_k)$ without a transmitter.

Note that this implies that V_k must have at least 3 vertices, because any tournament with fewer than 3 vertices always has a transmitter.

- The remaining arcs in T are as in Figure 2.3. In other words, if x_i beats x_j , then arcs are oriented from vertices in V_i to vertices in V_j , from x_j to vertices in V_i , and from vertices in V_j to x_i .

If a tournament T belongs to Γ_c we will say that T is a **caterpillar tournament**. These tournaments were used by Fisher, *et al.*, in [19] to show that any caterpillar with a triple end is the domination graph of some tournament. We will now show that essentially these are the only tournaments that have a caterpillars (with a triple end) as their domination graphs.

Theorem 2.10 *Let T be a n -tournament with $n \geq 4$. Then the domination graph of T is a caterpillar with a triple end if and only if T is a caterpillar tournament or a tournament with a transmitter.*

Proof: Let T be a tournament for which $dom(T)$ is a caterpillar with a triple end. If $dom(T)$ is a star, then by Theorem 2.3, we may conclude that T is either a caterpillar tournament or a tournament with a transmitter. Therefore, assume that $dom(T)$ is a nonstellar caterpillar. Now Lemma 2.8 provides the existence of the directed spine in $\mathcal{D}(T)$. Thus, we may assume without loss of generality that $\mathcal{D}(T)$ is as depicted in Figure 2.6. In addition, assume that the vertices of G are labeled as in Figure 2.6. Properly color G with the colors red and blue such that x_1 is assigned the color red. We proceed by showing that T is a caterpillar tournament.

First we show that k must be even. Suppose that k is odd. Note

that x_1 and x_k are both assigned the color red. Then a consequence of Lemma 2.9 is that x_1 beats all the blue vertices to the right and x_k beats all the red vertices to the left. Thus, whether or not the triple end exists at x_1 or at x_k , we have that x_1 and x_k form a dominant pair in T , a contradiction since $[x_1, x_k] \notin \text{dom}(T)$. Therefore, k must be even.

Next we show that G must have a triple end at x_k and that $T(V_k)$ must induce a subtournament without a transmitter. So assume that $|V_k| \leq 2$. Let \hat{v} be the vertex in V_k such that \hat{v} is a transmitter in the subtournament $T(V_k)$. Such a vertex exists for all tournaments with at most 2 vertices. Since k is even, x_k is colored blue and so we have that \hat{v} is colored red. Then it follows by Lemma 2.9 that x_1 beats all vertices to the right that are colored blue and \hat{v} beats all vertices to the left that are colored red. Note that every vertex in V_k is beaten by \hat{v} , since \hat{v} is a transmitter in $T(V_k)$. In addition, note that every vertex in V_1 is beaten by x_1 . Therefore, it follows that x_1 and \hat{v} form a dominant pair, a contradiction since $[x_1, \hat{v}] \notin \text{dom}(T)$. Therefore, we may conclude that $|V_k|$ must be at least 3. In addition, by the same argument $T(V_k)$ must induce a subtournament without a transmitter.

Finally, with the proper coloring assigned to the vertices of G , Lemma 2.9 determines that T is a caterpillar tournament.

For the converse, let T be a n -tournament with $n \geq 4$. First assume that T has a transmitter. Then by Theorem 2.3, it follows that $\text{dom}(T)$ is a

caterpillar with a triple end. In particular, $\text{dom}(T)$ is a star with at least three pendant vertices. Therefore, we may assume that T is a caterpillar tournament on $n \geq 4$ vertices. Without loss of generality assume that $T(\{x_1, x_2, \dots, x_k\}) = U_k$. Note that k must be even. We proceed by showing that $\text{dom}(T)$ is a caterpillar with a triple end.

First we show that $[x_i, x_{i+1}] \in \text{dom}(T)$, for all i with $1 \leq i \leq k - 1$. Consider the pair of vertices x_i and x_{i+1} for some $i \in \{1, 2, \dots, k - 1\}$. First note that every vertex in V_i is beaten by x_i and every vertex in V_{i+1} is beaten by x_{i+1} . Thus to show that $[x_i, x_{i+1}] \in \text{dom}(T)$, we will show that for all $x_l \in \{x_1, x_2, \dots, x_k\}$, with $l \neq i, i + 1$, the vertex x_l together with every vertex in V_l is beaten by either x_i or x_{i+1} . So let $x_l \in \{x_1, x_2, \dots, x_k\}$, with $l \neq i, i + 1$. Assume that $l < i$. If l is odd, then since $T(\{x_1, x_2, \dots, x_k\}) = U_k$, we have that x_l beats all vertices x_j for $j > l$ and j even, and is beaten by all vertices x_m for $m > l$ and m odd. Thus, since i must be either odd or even, x_l beats exactly one of x_i or x_{i+1} . In addition, since T is a caterpillar tournament, we have by Figure 2.3 that whichever vertex x_l beats from x_i and x_{i+1} , that same vertex beats every vertex in V_l . Therefore, in either case whether x_l beats x_i or x_{i+1} , it follows that x_l and every vertex in V_l is beaten by either x_i or x_{i+1} . Similarly, if l is even, then x_l and every vertex in V_l is beaten by either x_i or x_{i+1} . Next assume that $l > i + 1$. If l is odd, then since $T(\{x_1, x_2, \dots, x_k\}) = U_k$, it follows that x_l beats all vertices x_j , for $j < l$ and j odd, and is beaten by all vertices

x_m , for $m < l$ and m even. Thus, since i must be either odd or even, x_l beats exactly one of x_i or x_{i+1} . In addition, since T is a caterpillar tournament, we have by Figure 2.3 that whichever vertex x_l beats from x_i and x_{i+1} , that same vertex beats every vertex in V_l . Therefore, in either case whether x_l beats x_i or x_{i+1} , we may conclude that x_l and every vertex in V_l is beaten by either x_i or x_{i+1} . Similarly, if l is even, then x_l and every vertex in V_l is beaten by either x_i or x_{i+1} . Therefore, since i was arbitrary in $\{1, 2, \dots, k-1\}$, it follows that x_i and x_{i+1} form a dominant pair in T , for all i with $1 \leq i \leq k-1$.

Next let $z \in V_i$ be arbitrary. We will show that $[z, x_i] \in \text{dom}(T)$. Since T is a caterpillar tournament, we have by Figure 2.3 that any vertex $w \in V(T) - V_i$ that beats x_i is beaten by z . Thus, since x_i beats every vertex in V_i , it follows that $[z, x_i] \in \text{dom}(T)$. Therefore, since $|V_k|$ is at least 3, we may conclude that a caterpillar with a triple end is a subgraph of $\text{dom}(T)$. See Figure 2.7.

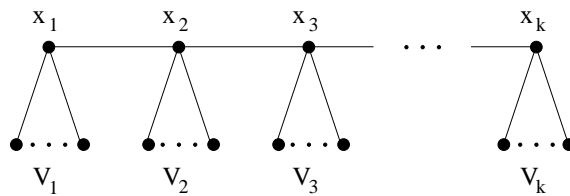


Figure 2.7. A subgraph of the domination graph of a caterpillar tournament.

To show that $dom(T)$ is a caterpillar with a triple end, it suffices to show that no other pair of vertices can form a dominant pair in T . Notice that we have shown the existence of $n - 1$ dominant pairs in T . By Theorem 1.1, we have that the domination graph of any tournament on n vertices can have at most n dominant pairs. Thus, there can exist at most one more dominant pair. If there exists another dominant pair in T , then by Theorem 1.1 it must be one such that the domination graph is an odd-spiked cycle. Note that there are four possibilities. These are

- (1) $[x_1, v]$, for a vertex $v \in V_k$, if V_1 is nonempty,
- (2) $[w, x_k]$, for a vertex $w \in V_1$, if V_1 is nonempty,
- (3) $[x_2, x_k]$, if V_1 is empty, or
- (4) $[x_1, v]$, for a vertex $v \in V_k$, if V_1 is empty.

The above are the only possibilities because these are the only possible edges that will create an odd-spiked cycle. We proceed by showing that none of these can occur. First assume that V_1 is nonempty. Since x_1 beats every vertex in V_1 and beats x_k , we have that $[w, x_k] \notin dom(T)$, for any $w \in V_1$. Also since $T(V_k)$ induces a subtournament without a transmitter, it follows that for every vertex $v \in V_k$ there exists a vertex $v' \in V_k$ such that v' beats v . Thus, since every vertex in V_k beats x_1 , we conclude that $[x_1, v] \notin dom(T)$, for any $v \in V_k$. Next assume that V_1 is empty. Since x_1 beats both x_2 and x_k , it follows that $[x_2, x_k] \notin dom(T)$. Also since $T(V_k)$ induces a subtournament

without a transmitter, then for every vertex $v \in V_k$ there exists a vertex $v' \in V_k$ such that v' beats v . Thus, since every vertex in V_k beats x_1 , it follows that $[x_1, v] \notin \text{dom}(T)$, for any $v \in V_k$. Therefore, $\text{dom}(T)$ is a caterpillar with a triple end. ■

Observe that the domination graph of a caterpillar tournament has an even spine, if $V_1 \neq \emptyset$ and an odd spine, if $V_1 = \emptyset$.

On a final note, if we are given a connected graph G that is known to be the domination graph of a tournament, then we can use Theorems 2.3, 2.4, 2.7 and 2.10 to construct a tournament that has G as its domination graph. But note that this tournament need not be unique. For example, using Theorem 2.3, we can construct the two nonisomorphic tournaments in Figure 2.8 that have $K_{1,3}$ as their domination graph.

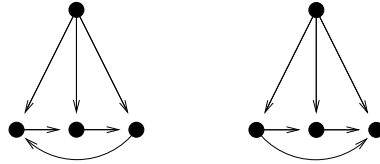


Figure 2.8. Two nonisomorphic tournaments on four vertices that have $K_{1,3}$ as their domination graph.

Therefore, it would be interesting to determine all connected graphs that are the domination graphs of unique tournaments. We leave this final problem as a possible avenue in which the research on this topic can be continued.

3. The Domination-Compliance Graph of a Tournament

3.1 Introduction

In this chapter, we introduce and study a generalization of the domination graph, the domination-compliance graph of a tournament. Vertices x and y are a **compliant pair** in a tournament T if every other vertex $z \neq x, y$ beats either x or y , i.e., x and y are a compliant pair if $I(x) \cup I(y) \cup \{x, y\} = V(T)$ or equivalently if $O(x) \cap O(y) = \emptyset$. The **compliance graph** $\text{com}(T)$ of a tournament T is the graph on the same vertex set as T with edges between vertices that form a compliant pair. It is easy to see that the domination graph is related to the compliance graph in that $\text{com}(T) = \text{dom}(\tilde{T})$, where \tilde{T} is the reversal of T . This relationship has the consequence that every graph that is a domination graph is also a compliance graph.

The **domination-compliance graph** $\text{DC}(T)$ of a tournament T is the graph on the same vertex set as T with edges between vertices which form either a dominant pair or a compliant pair. Observe that $\text{DC}(T) = \text{dom}(T) \cup \text{com}(T)$.

A graph which is closely related to the domination-compliance graph is the competition/resource graph. Recall that the competition graph $C(T)$

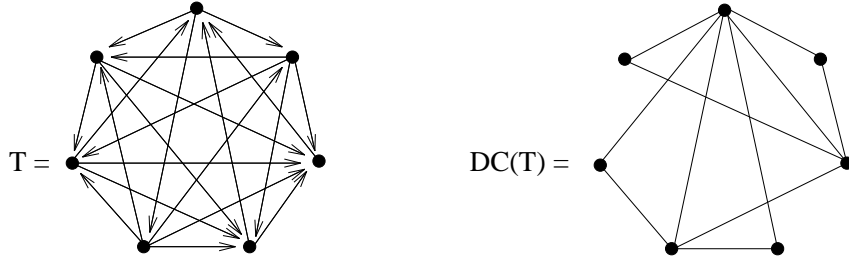


Figure 3.1. A 7-tournament and its domination-compliance graph.

of a tournament T is the graph on the same vertex set as T in which vertices x and y are adjacent if and only if there exists a vertex $z \neq x, y$ such that (x, z) and $(y, z) \in A(T)$. Also recall that the resource graph $R(T)$ of tournament T is the graph on the same vertex set as T in which vertices x and y are adjacent if and only if there exist a vertex $z \neq x, y$ such that (z, x) and $(z, y) \in A(T)$. It is not hard to see that the domination graph of a tournament T is the complement of the resource graph of T and that the compliance graph of a tournament T is the complement of the competition graph of T . This is true only when the digraph is a tournament. As a result of this relationship, competition and resource graphs of tournaments have been characterized. See Fisher, et al., [17, 18, 19] for more about the competition and resource graphs of tournaments. The competition/resource graph of a tournament is the graph on the same vertex set as T in which vertices x and y are adjacent if and only if x and y are adjacent in both the competition graph and the resource graph. Observe that $CR(T) = C(T) \cap R(T)$.

Competition/resource graphs have been studied for arbitrary digraphs, but not specifically for tournaments. See [23, 27, 39].

The next result illustrates the relationship between the domination-compliance graph and the competition/resource graph. Let $(\cdot)^c$ represent the complement operation.

Lemma 3.1 (Jimenez and Lundgren [22]) *If T is a tournament, then*

$$(CR(T))^c = DC(T).$$

Proof: By DeMorgan's Theorem, $(CR(T))^c = (C(T) \cap R(T))^c$

$$= (C(T))^c \cup (R(T))^c$$

$$= com(T) \cup dom(T)$$

$$= DC(T). \blacksquare$$

From Lemma 3.1, we see that results for the domination-compliance graph of a tournament correspond to results for the competition/resource graph of a tournament. However, since the domination-compliance graph of a tournament generally has fewer edges than the competition/resource graph, it is more convenient to state and prove results on the domination-compliance graph of a tournament. In addition, $dom(T) \cap com(T)$, also known as the mixed pair graph of a tournament, has been studied and characterized in [6] by Bowser, Cable and Lundgren. Therefore, since the domination graph, compliance graph, competition graph, resource graph and the mixed pair graph have all been studied

for tournaments, the only items remaining to be studied are the domination-compliance graph of tournaments and the competition/resource graphs of tournaments. In this chapter, we begin an examination of these last two items. We will exhibit an upper bound on the number of edges in the domination-compliance graph of a tournament. In addition, we will give characterizations for the domination-compliance graphs of two specific classes of tournaments.

3.2 Domination-Compliance Graph of Tournaments

Which graphs can be the domination-compliance graphs of tournaments? We will start to partially answer this question by stating and proving an upper bound on the maximum number of edges in the domination-compliance graph. This, in turn, will lead to forbidden subgraph results.

Lemma 3.2 *Let T be a regular tournament and $x, y \in V(T)$. Then $[x, y] \in \text{dom}(T)$ if and only if $[x, y] \in \text{com}(T)$.*

Proof: Let T be a regular n -tournament and $x, y \in V(T)$. Suppose that x and y form a dominant pair. Then since T is a regular tournament, it follows by Lemma 2.5 that $O(x) \cap O(y) = \emptyset$. Thus, $[x, y] \in \text{com}(T)$.

Next assume that x and y form a compliant pair in T and that x beats y . Then since $[x, y] \in \text{com}(T)$, we have that $|I(x) \cup I(y)| = n - 1$. Thus, since T is regular, $|I(x) \cap I(y)| = |I(x)| + |I(y)| - |I(x) \cup I(y)| = \frac{n-1}{2} + \frac{n-1}{2} - (n-1) = 0$. Therefore, it follows that $I(x) \cap I(y) = \emptyset$ and so $[x, y] \in \text{dom}(T)$. ■

Recall that U_n is the tournament on vertex set x_1, x_2, \dots, x_n with $(x_i, x_j) \in A(U_n)$ if $i - j$ is either odd and negative or even and positive. Also recall that U_n is a regular tournament when n is odd. Let \mathbf{P}_n^* be the graph with vertices x_1, x_2, \dots, x_n and edges $[x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n], [x_1, x_{n-1}]$ and $[x_2, x_n]$. See Figure 3.2. Note that for $n = 1, 2, 3$, the graph P_n^* is the path on one, two, and three vertices, respectively.

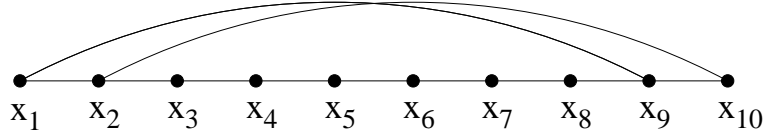


Figure 3.2. The graph P_{10}^* .

Lemma 3.3 (Jimenez and Lundgren [22]) *If $T \cong U_n$, then*

$$DC(T) \cong \begin{cases} P_n^* & \text{if } n \text{ is even} \\ C_n & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Let T be an n -tournament isomorphic to U_n , $n \geq 1$. First assume that n is odd. Since U_n is a regular tournament, Theorem 2.4 and Lemma 3.2 imply that $DC(T) \cong C_n$.

Next assume n is even. The result is trivial if $n = 2$, so assume that $n \geq 4$. First we show that $[x_i, x_{i+1}] \in DC(T)$, for all $1 \leq i \leq n - 1$. Let $x_l \in V(T)$, with $l \neq i, i + 1$. Assume that $l < i$. If l is odd, then x_l is beaten by all vertices x_j with $j > l$ and j odd. Since i is either odd or even, it follows

that either x_i or x_{i+1} beats x_l . Similarly, if l is even, then x_l is beaten by either x_i or x_{i+1} . Next assume that $l > i + 1$. If l is odd, then x_l is beaten by all vertices x_j with $j < l$ and j even. Since i is either odd or even, then either x_i or x_{i+1} beats x_l . Similarly, if l is even, then x_l is beaten by either x_i or x_{i+1} . Therefore, it follows that x_i and x_{i+1} form a dominant pair in T . Since the domination graph is contained in the domination-compliance graph, $[x_i, x_{i+1}] \in DC(T)$, for $1 \leq i \leq k - 1$.

Next, we show that $[x_1, x_{n-1}]$ and $[x_2, x_n] \in DC(T)$. Since x_1 beats x_j , for j even, and x_{n-1} beats x_j , for $j < n - 1$ and j odd, it follows that $[x_1, x_{n-1}] \in dom(T) \subseteq DC(T)$. Similarly, since x_n is beaten by x_j , for j odd, and x_2 is beaten by x_j , for $j > 2$ and j even, it follows that $[x_2, x_n] \in com(T) \subseteq DC(T)$.

Finally, we need to show that no other pair of vertices forms an edge in $DC(T)$. Assume that $j \neq i + 1, i - 1$ and $i, j \in \{2, 3, \dots, n - 1\}$. We will show that x_i and x_j are not adjacent in $DC(T)$. If i and j are odd, then x_i and x_j beat x_1 and are beaten by x_2 . If i and j are even, then x_i and x_j beat x_{n-1} and are beaten by x_n . If i and j are of opposite parity, then assume without loss of generality that $i < j$. Since $|i - j| \geq 3$, we conclude that x_i and x_j beat x_{i+1} and are beaten by x_{i+2} .

Next assume that $i = 1, j > 2$ and $j \neq n - 1$. We will show that x_1 and x_j are not adjacent in $DC(T)$. If j is even, then x_1 and x_j beat x_2 and

are beaten by x_3 . If j is odd, then x_1 and x_j beat x_n and are beaten by x_{n-1} .

Similarly, x_n is not adjacent to x_j for $j < n - 1$ and $j \neq 2$. ■

The converse of this last lemma is not true. For example, consider the tournament on seven vertices and its corresponding domination-compliance graph in Figure 3.3.

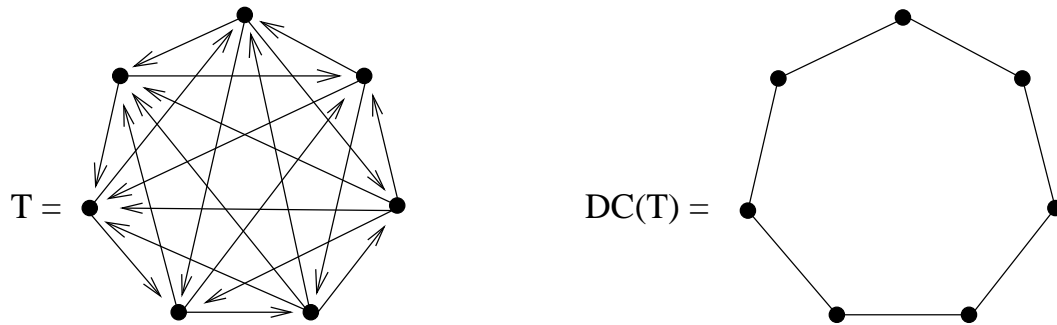


Figure 3.3. $DC(T)$ is C_7 , but T is not isomorphic to U_7 .

The next result is analogous to Lemma 2.6.

Lemma 3.4 (Jimenez and Lundgren [22]) *Let R be an induced subdigraph of a digraph D . Then the induced subgraph of $DC(D)$ on the vertices of R is a subgraph of $DC(R)$.*

Proof: Let $x, y \in V(R)$ where $[x, y] \in DC(D)$. Then either $[x, y] \in dom(D)$ or $[x, y] \in com(D)$. Since $V(R) \subseteq V(D)$, we have that either $[x, y] \in dom(R)$ or $[x, y] \in com(R)$. Thus, $[x, y]$ is an edge in $DC(R)$. ■

Theorem 2.7 from Chapter 2 will prove useful in the verification of the next result.

Lemma 3.5 (Jimenez and Lundgren [22]) *Let T be a tournament whose domination graph is an odd-spiked cycle. Assume that $\{x_1, x_2, \dots, x_k\}$ are the set of vertices on the cycle of $\text{dom}(T)$. In addition, assume that V_j , for $j = 1, 2, \dots, k$, is the set of vertices pendant to x_j in $\text{dom}(T)$. Then the following are the only possible edges in $\text{com}(T)$.*

- i) Let $v \in V_i$. Then $[x_i, v] \in \text{com}(T)$ if and only if $d_{T(V_i)}^+(v) = 0$.*
- ii) Let $v \in V_i$ and $w \in V_j$. Then $[v, w] \in \text{com}(T)$ if and only if $j \equiv i + 1 \pmod{k}$ and $d_{T(V_j)}^+(w) = 0$ or $j \equiv i - 1 \pmod{k}$ and $d_{T(V_i)}^+(v) = 0$.*
- iii) $[x_i, x_j] \in \text{com}(T)$ if and only if $j \equiv i + 1 \pmod{k}$ and $V_i = \emptyset$ or $j \equiv i - 1 \pmod{k}$ and $V_j = \emptyset$.*
- iv) Let $v \in V_i$ and assume $i \neq j$. Then $[v, x_j] \in \text{com}(T)$ if and only if $j \equiv i + 2 \pmod{k}$ and $V_{i+1} = \emptyset$.*

Proof: Let T be a tournament whose domination graph is an odd-spiked cycle. Then it follows by Theorem 2.7 that T must be a spiked cycle tournament. So assume without loss of generality that $T(\{x_1, x_2, \dots, x_k\}) = U_k$. Note that the vertices x_i and the sets V_j are related by Figure 2.3. Therefore, since every vertex in V_i beats every vertex x_j that beats x_i , we have that no pair of vertices in V_i , for all i , can be an edge in $\text{com}(T)$.

- i) Let $v \in V_i$. We begin by showing that if there exists a vertex $z \in V(T)$ that both x_i and v beat, then z must be an element of V_i . Assume*

$z \in O(v) \cap O(x_i)$, for some $z \in V(T)$. First, z cannot be an element of V_j , with $j \neq i$, because if x_i beats z , then since T is a spiked cycle tournament, it follows from Figure 2.3 that z must beat v . Second, z cannot be any vertex x_j , with $j \neq i$, because if x_i beats x_j , then again since T is a spiked cycle tournament, we have by Figure 2.3 that x_j beats v . Therefore, z must be in V_i . Thus, $d_{T(V_i)}^+(v) = 0$ (i.e, v beats no vertex in V_i) if and only if $O(v) \cap O(x_i) = \emptyset$. Therefore, we have that $d_{T(V_i)}^+(v) = 0$ is equivalent to $O(x_i) \cap O(v) = \emptyset$ which in turn is equivalent to $[x_i, v] \in com(T)$.

ii) Let $v \in V_i$ and $w \in V_j$. First we show that if $j \not\equiv i+1, i-1 \pmod{k}$, then $[v, w] \notin com(T)$. A consequence of Theorem 2.4 is that the only dominant pairs in U_k are $[x_i, x_{i+1}]$, for $1 \leq i \leq k-1$, and $[x_1, x_k]$. Thus, if $j \not\equiv i+1, i-1 \pmod{k}$, then there exist a vertex x_l , $l \neq i, j$, in $T(\{x_1, x_2, \dots, x_k\} = U_k$ such that x_i and x_j are beaten by x_l . But then since T is a spiked cycle tournament and x_l beats both x_i and x_j , we have by Figure 2.3 that both v and w beat x_l and so it is not possible for v and w to form a compliant pair. Therefore, we may conclude that $[v, w] \notin com(T)$ for $j \not\equiv i+1, i-1 \pmod{k}$.

Now we show that $[v, w] \in com(T)$ if and only if $j \equiv i+1 \pmod{k}$ and $d_{T(V_j)}^+(w) = 0$ or $j \equiv i-1 \pmod{k}$ and $d_{T(V_i)}^+(v) = 0$. Note that it suffices to show that it is true for exactly one of these last two conditions. So without loss of generality assume that $j \equiv i+1 \pmod{k}$. Let $z \in O(v) \cap O(w)$ for

some $z \in V(T)$. First, z cannot be in V_l with $l \neq j$, because if v dominates z , then by Figure 2.3 it follows that z beats x_i . Then since z beats x_i and $[x_i, x_j] \in \text{dom}(T)$, we have that x_j beats z . But then by Figure 2.3, we may conclude that z beats w . Second, z cannot be any vertex x_l with $l \neq i, j$, because if both v and w beat z , then by Figure 2.3 we have that x_l beats both x_i and x_j , a contradiction since $[x_i, x_j] \in \text{dom}(T)$. Thus, it follows that z must be an element of V_j . Note that since x_i beats x_j , we have by Figure 2.3 that every vertex in V_i beats every vertex in V_j . Therefore, since v beats every vertex in V_j , we may conclude that $d_{T_{V_j}}^+(w) = 0$ (i.e, w beats no vertex in V_j) if and only if $O(v) \cap O(w) = \emptyset$. From these observations, we have that $[v, w] \in \text{com}(T)$ is equivalent to $O(v) \cap O(w) = \emptyset$ which is equivalent to $j \equiv i + 1 \pmod{k}$ and $d_{T_{V_j}}^+(w) = 0$ or $j \equiv i - 1 \pmod{k}$ and $d_{T_{V_i}}^+(v) = 0$.

iii) First we show that if $j \not\equiv i + 1, i - 1 \pmod{k}$, then $[x_i, x_j] \notin \text{com}(T)$. A consequence of Theorem 2.4 is that the only dominant pairs in $T(\{x_1, x_2, \dots, x_k\}) = U_k$ are $[x_i, x_{i+1}]$, with $1 \leq i \leq k - 1$, and $[x_1, x_k]$. In addition, Lemma 3.2 implies that in a regular tournament a pair of vertices forms a dominant pair if and only if they form a compliant pair. Thus, since U_k is a regular tournament, we may conclude that the only compliant pairs in U_k are $[x_i, x_{i+1}]$ with $1 \leq i \leq k - 1$ and $[x_1, x_k]$. Thus, if a pair of vertices in $\{x_1, x_2, \dots, x_k\}$ cannot form a compliant pair in $T(\{x_1, x_2, \dots, x_k\})$, they

certainly cannot form a compliant pair in T . Therefore, $[x_i, x_j] \notin \text{com}(T)$ when $j \not\equiv i + 1, i - 1 \pmod{k}$.

Next we show that $[x_i, x_j] \in \text{com}(T)$ if and only if $j \equiv i + 1 \pmod{k}$ and $V_i = \emptyset$ or $j \equiv i - 1 \pmod{k}$ and $V_j = \emptyset$. Note that it suffices to show that it is true for exactly one of these two conditions. So without loss of generality assume that $j \equiv i + 1 \pmod{k}$. Let $z \in O(x_i) \cap O(x_j)$. First, z cannot be in V_l , with $l \neq i$, because if x_i beats z , then by Figure 2.3 it follows that x_l beats x_i . Then since $[x_i, x_j] \in \text{dom}(T)$, we have that x_j beats x_l . But then by Figure 2.3, we can see that z beats x_j . Second, z cannot be any vertex x_l on the cycle, because x_i and x_j are compliant pair in the subtournament $T(\{x_1, x_2, \dots, x_k\}) = U_k$ and so it is not possible for both of them to beat a vertex in $\{x_1, x_2, \dots, x_k\}$. Therefore, z must be in V_i . Note that since x_i and x_j form a dominant pair in T and $j \equiv i + 1 \pmod{k}$, we can see by Figure 2.3 that x_j beats every vertex in V_i . Thus, $O(x_i) \cap O(x_j) = \emptyset$ if and only if $V_i = \emptyset$. Therefore, we may conclude that $[x_i, x_j] \in \text{com}(T)$ is equivalent to $O(x_i) \cap O(x_j) = \emptyset$ which is equivalent to $j \equiv i + 1 \pmod{k}$ and $V_i = \emptyset$ or $j \equiv i - 1 \pmod{k}$ and $V_j = \emptyset$.

iv) Let $v \in V_i$ and $j \not\equiv i, i + 2 \pmod{k}$. We begin by showing that v and x_j cannot be a compliant pair. This will be accomplished by showing that if x_i beats x_j , then both v and x_j beat x_{j-2} and if x_j beats x_i , then both

v and x_j beat x_{j+1} . For the following, all arithmetic is done modulo k .

Assume that x_j beats x_i . Note that $j - 2 \neq i$ since $j \not\equiv i + 2 \pmod{k}$. Then since x_{j-1} and x_j form a compliant pair in $T(\{x_1, x_2, \dots, x_k\})$, it follows that x_j and x_{j-1} cannot both beat x_i . Thus, x_{j-1} is beaten by x_i . Then since $[x_{j-2}, x_{j-1}] \in \text{dom}(T)$, we have that x_{j-2} beats x_i . But then we can see by Figure 2.3 that v beats x_{j-2} . In addition, since x_{j-2} beats x_{j-1} and $[x_{j-1}, x_j] \in \text{dom}(T)$, we may conclude that x_j beats x_{j-2} . Therefore, both v and x_j beat x_{j-2} and so they cannot be a compliant pair in T .

Next assume that x_i beats x_j . Note that $j \not\equiv i - 1 \pmod{k}$ since x_{i-1} beats x_i . Thus, since $[x_j, x_{j+1}] \in \text{dom}(T)$, we have that x_{j+1} beats x_i . Then by Figure 2.3 it follows that v beats x_{j+1} . But x_j also beats x_{j+1} and so v and x_j cannot be a compliant pair in T .

We finish this proof by showing that $[v, x_j] \in \text{com}(T)$ if and only if $j \equiv i + 2 \pmod{k}$ and $V_{i+1} = \emptyset$. So assume that $j \equiv i + 2 \pmod{k}$. Note that $j - 1 \equiv i + 1 \pmod{k}$. Thus, $[x_i, x_{j-1}] \in \text{dom}(T)$. Let $z \in O(v) \cap O(x_{j-1})$. We will show that z must be in V_{i+1} . First, z cannot be in V_l , with $l \neq i + 1$, because if x_j beats z , then it follows by Figure 2.3 that x_l beats x_j . Then since $[x_{j-1}, x_j] \in \text{dom}(T)$, we have that x_{j-1} beats x_l . Then since x_i and x_{j-1} are a compliant pair in $T(\{x_1, x_2, \dots, x_k\})$, we may conclude that x_l must beat x_i . But then by Figure 2.3 it follows that v beats x_l and from the same figure z

beats v . Second, z cannot be a vertex x_l , with $l \neq i, j$, because if x_j beats x_l , then since x_{j-1} and x_j form a compliant pair in $T(\{x_1, x_2, \dots, x_k\})$, we have that x_l must beat x_{j-1} . Then since $[x_i, x_{j-1}] \in \text{dom}(T)$, it follows that x_i beats x_l . But then by Figure 2.3 we may conclude that x_l beats v . Therefore, z must be in V_{i+1} . Note that since x_i beats x_{j-1} and x_{j-1} beats x_j , we have by Figure 2.3 that both v and x_j beat every vertex in V_{j-1} . Therefore, $O(x_j) \cap O(v) = \emptyset$ if and only if $V_{i+1} = \emptyset$. And so we may conclude that $[v, x_j] \in \text{com}(T)$ is equivalent to $O(x_j) \cap O(v) = \emptyset$ which is equivalent to $j \equiv i + 2 \pmod{k}$ and $V_{i+2} = \emptyset$. ■

Theorem 3.6 (Jimenez and Lundgren [22]) *Let T be an n -tournament.*

The maximum number of edges in $DC(T)$ is $2(n-1)$ and this bound is the best possible for $n \geq 4$.

Proof: Suppose $|E(DC(T))| > 2(n-1)$. A consequence of Theorem 1.1 is that the domination graph of a tournament with n vertices can have at most n edges. Thus, since $\text{com}(T) = \text{dom}(\tilde{T})$, we can see that the compliance graph of a tournament with n vertices can have at most n edges. Now since $DC(T) = \text{dom}(T) \cup \text{com}(T)$ and $|E(DC(T))| > 2(n-1)$, then at least one of $\text{dom}(T)$ or $\text{com}(T)$ must have n edges. Since $\text{com}(T) = \text{dom}(\tilde{T})$, we may assume without loss of generality that $|E(\text{dom}(T))| = n$. Then the only way that the domination-compliance graph can have more than $2(n-1)$ edges is if we have one of the following configurations:

- (1) $|E(\text{dom}(T))| = n$, $|E(\text{com}(T))| = n$, and $|E(\text{dom}(T) \cap \text{com}(T))| = 0$.
- (2) $|E(\text{dom}(T))| = n$, $|E(\text{com}(T))| = n$, and $|E(\text{dom}(T) \cap \text{com}(T))| = 1$.
- (3) $|E(\text{dom}(T))| = n$, $|E(\text{com}(T))| = n - 1$, and
 $|E(\text{dom}(T) \cap \text{com}(T))| = 0$.

Now since $|E(\text{dom}(T))| = n$, then $\text{dom}(T)$ must be an odd-spiked cycle. If $\text{dom}(T)$ is an odd cycle, then it follows by Theorem 2.4 that T must be isomorphic to U_n , with n odd. But since U_n is a regular tournament when n is odd, we may conclude by Lemma 3.2 that $DC(T) = \text{dom}(T) = \text{com}(T)$, which does not have more than $2(n-1)$ edges. Thus, if the above configurations exist, then $\text{dom}(T)$ must be an odd-spiked cycle with at least one pendant vertex.

Case 1. Suppose the existence of Configuration 1 or Configuration 3. Since $|E(\text{dom}(T))| = n$, we have by the above observation that $\text{dom}(T)$ must be an odd-spiked cycle with at least one pendant vertex. Thus, by Theorem 2.7, we have that T must be a spiked cycle tournament. Let x_1, x_2, \dots, x_k represent the vertices on the cycle of $\text{dom}(T)$ and V_i be the set of vertices pendant at x_i . Without loss of generality assume that $T(\{x_1, x_2, \dots, x_k\}) = U_k$. Now since $|E(\text{dom}(T) \cap \text{com}(T))| = 0$, no edges of the form $[x_i, x_{i+1}]$ in $\text{dom}(T)$ can be edges in $\text{com}(T)$. By *iv*) of Lemma 3.5, this will hold true if and only if $V_i \neq \emptyset$. Thus, it follows that for all i , the set V_i is nonempty. In addition, no edges of the form $[v, x_i]$, for $v \in V_i$, can be edges in $\text{com}(T)$. By *i*) of Lemma 3.5, this will hold true if and only if no vertex in V_i has out-degree 0 in $T(V_i)$.

But now since every pendant set must be nonempty, it follows by *iv)* of Lemma 3.5 that no vertex in $\{x_1, x_2, \dots, x_k\}$ can form an edge with any other vertex of T in $com(T)$. Thus, $\{x_1, x_2, \dots, x_k\}$ is an independent set in $com(T)$. Then since $com(T) = dom(\tilde{T})$, it follows by Theorem 1.1 that $com(T)$ must be an odd-spiked cycle with at least k isolated vertices or a forest of caterpillars with at least k isolated vertices. In either case, $com(T)$ can have at most $n - k$ edges. Therefore, since k must be at least three, it follows that $com(T)$ can have at most $n - 3$ edges, contrary to our assumption that $com(T)$ has either n edges (Configuration 1) or $n - 1$ edges (Configuration 3). Hence, we may conclude that Configuration 1 and 3 cannot exist.

Case 2. Suppose the existence of Configuration 2. Since $|E(dom(T))| = n$, we may conclude by the above observations that $dom(T)$ is an odd-spiked cycle with at least one pendant vertex. So assume T to be as in Case 1. Now since $|E(dom(T) \cap com(T))| = 1$, then we must have that exactly one of the edges of the form $[x_i, x_{i+1}]$ or $[z, x_i]$, where $z \in V_i$, must be in $com(T)$.

First suppose $[x_i, x_{i+1}]$, for some i , is an edge in $com(T)$. Then by *iii)* of Lemma 3.5, we have that $V_i = \emptyset$. Now since $|E(dom(T) \cap com(T))| = 1$, no other edges of the form $[x_j, x_{j+1}]$, with $j \neq i$, in $dom(T)$ can be edges in $com(T)$. By *iv)* of Lemma 3.5, this will hold true if and only if $V_j \neq \emptyset$. Thus, it follows that for all $j \neq i$, V_j is nonempty. In addition, no edges of the form $[w, x_j]$, for $w \in V_j$, can be edges in $com(T)$. By *i)* of Lemma 3.5, this will

hold true if and only if no vertex in V_j has out-degree 0 in $T(V_j)$. But now since every pendant set, other than V_i , must be nonempty, it follows by *iv*) of Lemma 3.5 that no vertex in $\{x_1, x_2, \dots, x_k\} \setminus \{x_i, x_{i+1}\}$ can form an edge with any other vertex of T in $com(T)$. Thus, $\{x_1, x_2, \dots, x_k\} \setminus \{x_i, x_{i+1}\}$ is an independent set in $com(T)$. But then since $com(T) = dom(\tilde{T})$, it follows by Theorem 1.1 that $com(T)$ must be an odd-spiked cycle with at least $k - 2$ isolated vertices or a forest of caterpillars with at least $k - 2$ isolated vertices. In either case, $com(T)$ can have at most $n - (k - 2)$ edges. Therefore, since k must be at least three, it follows that $com(T)$ can have at most $n - 1$ edges, contrary to our assumption that $com(T)$ has n edges.

Next suppose that $[v, x_i]$, where $v \in V_i$, is an edge in $com(T)$. By *i*) of Lemma 3.5, we must have that v has out-degree 0 in the subtournament $T(V_i)$. Now since $|E(dom(T) \cap com(T))| = 1$, no edges of the form $[x_j, x_{j+1}]$ in $dom(T)$ can be edges in $com(T)$. By *iv*) of Lemma 3.5, this will hold true if and only if $V_j \neq \emptyset$. Thus, it follows that for all j , the set V_j is nonempty. In addition, no other edges of the form $[w, x_j]$, for $w \in V_j$ and $j \neq i$, can be edges in $com(T)$. By *i*) of Lemma 3.5, this will hold true if and only if no vertex in V_j has out-degree 0 in $T(V_j)$, for $j \neq i$. But now since every pendant set must be nonempty, it follows by *iv*) of Lemma 3.5 that no vertex in $\{x_1, x_2, \dots, x_k\} \setminus \{x_i\}$ can form an edge with any other vertex of T in $com(T)$. Thus, $\{x_1, x_2, \dots, x_k\} \setminus \{x_i\}$ is an independent set in $com(T)$. But

then since $com(T) = dom(\tilde{T})$, we have by Theorem 1.1 that $com(T)$ must be an odd-spiked cycle with at least $k - 1$ isolated vertices or a forest of caterpillars with at least $k - 1$ isolated vertices. In either case, $com(T)$ can have at most $n - (k - 1)$ edges. Therefore, since k must be at least three, it follows that $com(T)$ can have at most $n - 2$ edges, contrary to our assumption that $com(T)$ has n edges. Therefore, it follows that Configuration 2 cannot exist.

Since we have shown that none of the configurations we exhibited can exist, it follows that it is not possible to have $|E(DC(T))| > 2(n - 1)$. Therefore, $DC(T)$ has at most $2(n - 1)$ edges.

To show that this bound is best possible for $n \geq 4$, let T be the tournament obtained from U_{n-1} by adding a vertex x_n with arcs (x_i, x_n) , for all i with $1 \leq i \leq n - 1$. For this tournament, $n - 1$ edges are obtained from $dom(T)$ and $n - 1$ edges are obtained from $com(T)$. See Figure 3.4. ■

3.3 Consequences of the upper bound and forbidden subgraphs

Theorem 3.6 gives the maximum number of edges in the domination-compliance graph of a tournament. As a result, it is straightforward to deduce results about graph parameters for the domination-compliance graph and the competition/resource graph. In addition, we will also deduce some results concerning forbidden subgraphs. Below are some examples.

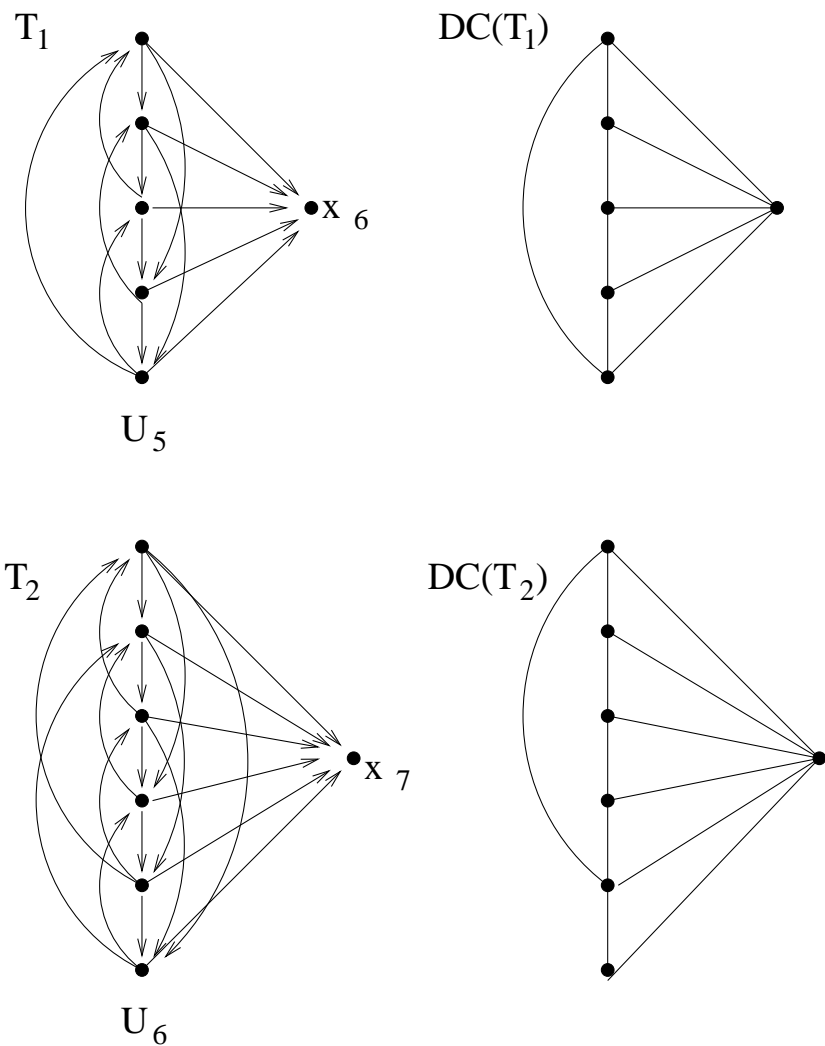


Figure 3.4. Two tournaments on six and seven vertices that have the maximum number of edges in $DC(T)$.

Corollary 3.7 *The minimum number of edges in the competition/resource graph of an n -tournament is $\binom{n}{2} - 2(n - 1)$.*

Let G be a graph. A subset of vertices of G forms a **clique** if there are edges between every pair of vertices in the subset. We define the **clique number** $\omega(G)$ of G to be the maximum size of a clique in G . A subset of vertices of G forms an **independent set** if there are no edges between any two vertices in the subset. We define the **independence number** $\alpha(G)$ of G to be the maximum size of an independent set in G . Clearly, $\alpha(G) = \omega(\overline{G})$ where \overline{G} is the complement of G .

Corollary 3.8 *Let T be a tournament. Then the clique number of $DC(T)$ is at most 4.*

Proof: Let T be an n -tournament. The result is trivial for $n \leq 4$, so assume that $n \geq 5$. Suppose that K_5 is a subgraph of $DC(T)$. Then by Lemma 3.4, there must exist a 5-tournament T' such that $DC(T') = K_5$. But this cannot possibly happen since the maximum possible number of edges in $DC(T')$ is 8 and K_5 has ten edges. Therefore, K_5 is not a subgraph of $DC(T)$, for any n -tournament T with $n \geq 5$. ■

Corollary 3.9 *Let T be a tournament. Then the independence number of the competition/resource graph of T is at most 4.*

Theorem 3.10 *If T is a tournament on six vertices, then $K_{3,3}$ is not a subgraph of $DC(T)$.*

Theorem 3.10 can be verified by one of two methods. The first involves enumerating all 56 nonisomorphic tournaments on six vertices and verifying that $K_{3,3}$ is not a subgraph of $DC(T)$. The second method is by proof and can be found in Appendix A.

Corollary 3.11 *Let T be a tournament. Then $K_{3,3}$ is not a subgraph of $DC(T)$.*

Proof: Let T be an n -tournament. The result is trivial if $n \leq 5$, so assume that $n \geq 6$. If $K_{3,3}$ is a subgraph of $DC(T)$, then by Lemma 3.4 there must exist a tournament T' on six vertices such that $K_{3,3}$ is a subgraph of $DC(T')$. But this contradicts Theorem 3.10. Therefore, $K_{3,3}$ is not a subgraph of $DC(T)$.

■

3.4 Characterization for reducible and regular tournaments

In this chapter, we have been investigating the domination-compliance graph of a tournament. Our goal of this investigation is to arrive at a result that characterizes all graphs that are the domination-compliance graphs of tournaments. At this point, we have not been able to achieve such a result. But if we restrict T to be either a reducible or regular tournament, we can get such a result.

Recall that a tournament T is reducible if it is possible to partition its

vertex set into two nonempty sets A and B in such a way that every vertex in A beats every vertex in B . By definition, reducible tournaments are precisely those which are not strongly connected. A regular tournament is one in which every vertex has the same out-degree. Note that if T is a regular tournament, then T has an odd number of vertices. Let \mathbf{I}_n be the graph with n isolated vertices.

Lemma 3.12 *Let T be a reducible tournament with a partition of its vertex set into nonempty sets A and B such that every vertex in A beats every vertex in B . If T has no transmitter, then $\text{dom}(T) = \text{dom}(T(A)) \cup I_{|B|}$.*

Proof: Let T be a reducible tournament with a partition of its vertex set into nonempty sets A and B such that every vertex in A beats every vertex in B . Assume T has no transmitter. We will show that $\text{dom}(T) = \text{dom}(A) \cup I_{|B|}$.

Clearly no pair of vertices in B can be a dominant pair in T since A is nonempty. Since T does not have a transmitter, then for every vertex $x \in V(T)$, we have $d^+(x) < n - 1$. Thus, no two vertices $x \in A$ and $y \in B$ can be a dominant pair. Therefore, if $[x, y] \in \text{dom}(T)$, then both x and y must belong to A . Thus, by Lemma 3.4, we have $[x, y] \in \text{dom}(T(A))$. Conversely, if $[x, y] \in \text{dom}(T(A))$, then $[x, y] \in \text{dom}(T)$ since every vertex in A beats all vertices in B . Therefore, $\text{dom}(T) = \text{dom}(T(A)) \cup I_{|B|}$. ■

Corollary 3.13 *Let T be a reducible tournament with a partition of its vertex*

set into nonempty sets A and B such that every vertex in A beats every vertex in B . If T has no vertex with out-degree 0, then $\text{com}(T) = \text{com}(T(B)) \cup I_{|A|}$.

Lemma 3.14 *If T is an n -tournament with a vertex of out-degree 0, then $\text{com}(T) = K_{1,n-1}$.*

Proof: Let T be an n -tournament that has a vertex x of out-degree 0. Note that for any $y \in V(T) - \{x\}$, we have that $[x, y] \in \text{com}(T)$ since x is beaten by every vertex in $V(T) \setminus \{x\}$. Thus, $K_{1,n-1} \subseteq \text{com}(T)$. To show that $\text{com}(T) = K_{1,n-1}$, it suffices to show that no other pair of vertices form a compliant pair. Let $y, z \in V(T) - \{x\}$. Since both y and z beat x , it follows that $[y, z] \notin \text{com}(T)$.

■

Let G and H be two disjoint graphs. Then define the **join** $\mathbf{G} \vee \mathbf{H}$ of G and H to be the graph obtained from the union of G and H by adding the set of edges $\{[x, y] : x \in V(G), y \in V(H)\}$.

Theorem 3.15 *Let T be a reducible n -tournament. Then $DC(T)$ is either*

- 1) *two disjoint odd-spiked cycles with or without isolated vertices,*
- 2) *a forest of caterpillars,*
- 3) *an odd-spiked cycle and a forest of caterpillars,*
- 4) *$K_2 \vee I_{n-2}$, or*
- 5) *$K_1 \vee G$, where G is an odd-spiked cycle with or without isolated vertices or a forest of caterpillars.*

Proof: Let T be a reducible n -tournament. Since T is reducible, its vertex set can be partitioned into nonempty sets A and B such that every vertex of A beats every vertex of B in T .

First assume that T has no transmitter or vertex of out-degree 0. Thus, both $|A|$ and $|B|$ must be at least 1. By Lemma 3.12, we have that $dom(T) = dom(T(A)) \cup I_{|B|}$. Also by Theorem 1.1, it follows that $dom(T(A))$ is either an odd-spiked cycle with or without isolated vertices or a forest of caterpillars. Next by Corollary 3.13, $com(T) = com(T(B)) \cup I_{|A|}$. It follows by Theorem 1.1 that $com(T(B))$ is either an odd-spiked cycle with or without isolated vertices or a forest of caterpillars. In addition, note that $E(dom(T)) \cap E(com(T)) = \emptyset$. Therefore, since $DC(T) = dom(T) \cup com(T)$, then $DC(T)$ is either two disjoint odd-spiked cycles with or without isolated vertices, a forest of caterpillars, or an odd-spiked cycle and a forest of caterpillars.

Next assume T has both a transmitter x and a vertex y with out-degree 0. We may conclude by Theorem 2.3 that $dom(T) = K_{1,n-1}$. Also by Lemma 3.14, we have that $com(T) = K_{1,n-1}$. In addition, notice that $E(dom(T)) \cap E(com(T)) = \{[x, y]\}$. Since $DC(T) = dom(T) \cup com(T)$, we may conclude that $DC(T) = K_2 \vee I_{n-2}$.

Now assume T has a transmitter x and no vertex with out-degree 0. Let $A = \{x\}$ and $B = V(T) \setminus \{x\}$. It follows by Theorem 2.3 that $dom(T) =$

$K_{1,n-1}$ and by Lemma 3.13 that $com(T) = com(T(B)) \cup I_1$. In addition, by Theorem 1.1 we know that $com(T(B))$ is either an odd-spiked cycle with or without isolated or a forest of caterpillars. Now notice that $E(dom(T)) \cap E(com(T)) = \emptyset$. Thus, since $DC(T) = dom(T) \cup com(T)$, it follows that $DC(T)$ is $K_1 \vee G$, where G is either an odd-spiked cycle with or without isolated vertices or a forest of caterpillars. A similar argument shows that $DC(T)$ is $K_1 \vee G$, where G is either an odd-spiked cycle with or without isolated vertices or a forest of caterpillars when T has a vertex with out-degree 0 and no transmitter. ■

Now suppose that T is a regular n -tournament with n odd. Then a consequence of Lemma 3.2 is that $dom(T) = com(T)$. Thus, since $DC(T) = dom(T) \cup com(T)$, we have that $DC(T) = dom(T) = com(T)$ when T is a regular tournament. Recently, in [10] Cho, Kim, and Lundgren determined that the domination graph of a regular tournament is either an odd cycle or a forest of paths. Thus, we have the following theorem.

Theorem 3.16 *Let T be a regular n -tournament with n odd. Then $DC(T)$ is either an odd cycle or a forest of paths.*

In the case where all the paths are nontrivial, Cho, *et al.*, found a characterization for forests of nontrivial paths which are the domination graphs of regular tournaments. This result is presented in Chapter 4.

We now conclude this chapter by presenting some ways in which the research on the domination-compliance graph can be continued. Recall that reducible tournaments are precisely those tournaments which are not strongly connected. Although we have successfully characterized the domination-compliance graph for reducible tournaments, such a characterization for strongly connected tournaments has not yet been found. Thus, a problem one can consider would be to try to find a characterization for the domination-compliance graph of strongly connected tournaments. Note that if such a characterization is found, then the domination-compliance graph will be characterized for all tournaments.

A second way in which the research on the domination-compliance graph could be continued would be to try to determine if the domination-compliance graph of a tournament is always a planar graph. Define a **planar graph** to be a graph that can be drawn in the plane without crossing edges. The reason we pose this problem is because the work presented in this chapter leads one to believe that that this is in fact true for the following reasons. First, planar graphs have the property that they have at most $3n - 6$ edges, where n is the number of vertices in the graph. For $n \geq 4$, the domination-compliance graph of a tournament on n vertices has at most $2(n - 1)$ edges. Thus, since $2(n - 1) \leq 3n - 6$, for $n \geq 4$, the domination-compliance graph always has

the right number of edges in the sense that it never exceeds the maximum number of edges that a planar graph can have. Second, a planar graph never has K_5 or $K_{3,3}$ as a subgraph. By Corollaries 3.8 and 3.11, we have that the domination-compliance graph of a tournament also has this property. Finally, the domination-compliance graphs for reducible and regular tournaments are planar.

Now notice that the domination graph and the compliance graph can always be properly colored with at most three colors. Thus, a possibly easier (preliminary) problem that one might consider would be to try to determine whether the domination-compliance graph of a tournament is 4-colorable, i.e., that it can be properly colored with at most four colors. It is well known that planar graphs can be properly colored with at most four colors. So if one can show that the domination-compliance cannot be 4-colored, then it will follow that it is not always the case that the domination-compliance graph is a planar graph. On the other hand, showing that the domination-compliance graph is 4-colorable will strengthen our belief that the domination-compliance graph is a planar graph. Therefore, we end this chapter with the following two conjectures.

Conjecture 3.17 *Let T be a tournament. Then $DC(T)$ can be properly colored using at most four colors.*

Conjecture 3.18 *Let T be a tournament. Then $DC(T)$ is a planar graph.*

4. Domination Graphs of Near-Regular Tournaments

4.1 Introduction

The focus of this and the following chapters involves investigating the domination graphs of near-regular tournaments. Recall that an n -tournament T is near-regular if n is even and every vertex has out-degree either $\frac{n}{2}$ or $\frac{n}{2} - 1$. Recently Cho, *et al.*, investigated the domination graphs of regular tournaments. In [10] they showed that the domination graph of a regular tournament is either an odd cycle or a forest of paths. The following theorem summarizes their findings.

Theorem 4.1 (Cho, et al., [10]) *Let T be a regular tournament and assume G is a graph without isolated vertices. Then G is the domination graph of T if and only if G is an odd cycle or G is a forest of m even paths and n nontrivial odd paths such that*

- (i) *if $m = 0, 1, 2, 4$, then n must be odd and $m + n \geq 7$ and*
- (ii) *if $m = 3$ or $m \geq 5$, then n must be odd.*

Since near-regular tournaments make up a large class of structurally nice tournaments, it was natural to proceed with a study analogous to that in [10].

Thus, in this and the following chapters we will investigate the following question, *Which connected graphs are domination graphs of near-regular tournaments?* As we will see, if T is a near-regular tournament, then the domination graph of T is not limited to odd cycles and forests of paths.

In the remainder of the chapter we will investigate which connected domination graphs are the domination graphs of near-regular tournaments. In particular, we will show that stars, odd cycles and caterpillars are not domination graphs of near-regular tournaments. We will then determine exactly which odd-spiked cycles are the domination graphs of near-regular tournaments. The following notation will be used in the remainder of this chapter as well as in the ensuing chapters. Let x be a vertex in a near-regular n -tournament T . If the out-degree of x is $\frac{n}{2}$, then we will say that x is a “+” vertex. If the out-degree of x is $\frac{n}{2} - 1$, then we will say that x is a “-” vertex.

4.2 Connected Domination Graphs of Near-Regular Tournaments

In this section, we characterize those connected graphs that are the domination graphs of near-regular tournaments. We begin by presenting a lemma which will provide useful information regarding dominant pairs in a near-regular tournament.

Lemma 4.2 *Let T be a near-regular n -tournament and $x, y \in V(T)$. If $[x, y] \in$*

$dom(T)$, then at least one of x and y is a “+” vertex. Furthermore, if both are “+” vertices, then $|O(x) \cap O(y)| = 1$, and if exactly one is a “+” vertex, then $|O(x) \cap O(y)| = 0$.

Proof: Let T be a near-regular n -tournament and $x, y \in V(T)$. Assume that x and y form a dominant pair and that x beats y . Then since $[x, y] \in dom(T)$, we have that $|O(x) \cup O(y)| = n - 1$. Thus, both x and y cannot be “-” vertices since if they are, then $|O(x) \cup O(y)| = |O(x)| + |O(y)| - |O(x) \cap O(y)| \leq \left(\frac{n}{2} - 1\right) + \left(\frac{n}{2} - 1\right) = n - 2$. Therefore, at least one of x and y must be a “+” vertex. Furthermore, if both x and y are “+” vertices, $|O(x) \cap O(y)| = |O(x)| + |O(y)| - |O(x) \cup O(y)| = \frac{n}{2} + \frac{n}{2} - (n - 1) = 1$. In addition, if exactly one, say x , is a “+” vertex, then $|O(x) \cap O(y)| = |O(x)| + |O(y)| - |O(x) \cup O(y)| = \frac{n}{2} + \left(\frac{n}{2} - 1\right) - (n - 1) = 0$. ■

Recall from Lemma 2.2 that in the domination digraph a vertex x can have in-degree at most 1, and at most one out-neighbor of x can have out-neighbors. When the tournament is near-regular, we can put further restrictions on the out-degree of a vertex in the domination digraph.

Theorem 4.3 *Let T be a near-regular n -tournament. In $\mathcal{D}(T)$, a “-” vertex can have out-degree at most 1 and a “+” vertex can have out-degree at most 3.*

Proof: Let T be a near-regular n -tournament and let x be an arbitrary vertex in $V(T)$. First assume that x is a “-” vertex and has out-degree at least 2 in

$\mathcal{D}(T)$. Let $y, z \in O_{\mathcal{D}(T)}(x)$. Since x is a “ $-$ ” vertex and forms a dominant pair with both y and z , it follows by Lemma 4.2 that both y and z are “ $+$ ” vertices. Also since T is a tournament, we have that either y beats z or z beats y in T . Without loss of generality assume y beats z . But then $|O(x) \cap O(y)| \geq 1$ since $z \in O(x) \cap O(y)$, which contradicts Lemma 4.2. Therefore, a “ $-$ ” vertex can have out-degree at most 1 in $\mathcal{D}(T)$.

Next suppose that x is a “ $+$ ” vertex and has out-degree at least 4 in $\mathcal{D}(T)$. Let $y, z, t, r \in O_{\mathcal{D}(T)}(x)$. Consider the subtournament T' induced by the set of vertices $\{y, z, t, r\}$. Since T' is a tournament on four vertices there must exist a vertex in T' with out-degree at least 2. Without loss of generality assume y is such a vertex in T' . But then since T' is a subtournament of T , we have that $|O(x) \cap O(y)| \geq 2$, which contradicts Lemma 4.2. Therefore, a “ $+$ ” vertex can have out-degree at most 3 in $\mathcal{D}(T)$. ■

Now Theorem 2.1 tells us that the only possible connected domination graphs of tournaments are odd-spiked cycles, stars and nonstellar caterpillars such that each has a triple end. Trivially, an odd cycle cannot be the domination graph of a near-regular tournament since near-regular tournaments have an even number of vertices. In addition, we will show, using Lemma 4.2 and Theorem 4.3, that if G is the connected domination graph of a near-regular n -tournament with $n \geq 4$, then G cannot be a star or a caterpillar with a triple end.

Lemma 4.4 *If T is a near-regular n -tournament with $n \geq 4$, then $K_{1,n-1}$ is not the domination graph of T .*

Proof: Let T be a near-regular n -tournament with $n \geq 4$ and suppose that $\text{dom}(T) = K_{1,n-1}$. Since $\text{dom}(T) = K_{1,n-1}$, then it follows by Theorem 2.3 that T either has a transmitter or has two distinguished vertices x and y such that

- i) y beats x ,
- ii) for all vertices $z \neq x, y$, we have that x beats z and z beats y , and
- iii) $T - \{x, y\}$ induces a subtournament without a transmitter.

Suppose T has a transmitter x . Then x has out-degree $n - 1$. But every vertex in T must have out-degree either $\frac{n}{2}$ or $\frac{n}{2} - 1$, a contradiction since $n - 1 > \frac{n}{2}$ for $n \geq 4$ and n even.

Next suppose that T has two distinguished vertices x and y with the properties stated above. Since $T - \{x, y\}$ must induce a subtournament without a transmitter, the subtournament $T - \{x, y\}$ must have at least three vertices. Therefore, T must have at least five vertices. Note that in T the vertex x has out-degree $n - 2$. But every vertex in T must have out-degree either $\frac{n}{2}$ or $\frac{n}{2} - 1$. Again we arrive at a contradiction since $n - 2 > \frac{n}{2}$ for $n \geq 5$ and n even. Thus, it follows that $K_{1,n-1}$ is not the domination graph of any near-regular n -tournament with $n \geq 4$. ■

Let H be a caterpillar with k vertices y_1, y_2, \dots, y_k on its spine. Also

let W_i represent the set of vertices pendant at x_i . We will say that H is an **odd gapped pendant restricted (OGPR) caterpillar** if it has the following properties:

- $|V_i| \leq 1$ for all i with $2 \leq i \leq k - 1$.
- The set V_1 can have at most two vertices and the set V_3 can have at most three vertices or vice versa.
- Consecutive vertices on the spine with degree 3 or more must be separated by an odd number of edges.

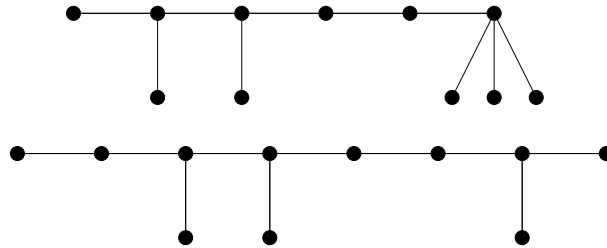


Figure 4.1. Two odd gapped pendant restricted caterpillars.

Odd gapped pendant restricted caterpillars play a significant role with respect to domination graphs of near-regular tournaments. It turns out that if a caterpillar H is the domination graph of a near-regular tournament, then H must be an OGPR caterpillar. This will be shown by proving the stronger result that if a forest G of nontrivial caterpillars is the domination graph of a near-regular tournament, then each component of G must be an odd gapped

pendant restricted caterpillar. In order to do so we need the following result.

Lemma 4.5 (Fisher, et al., [16]) *Let G be the union of k nontrivial caterpillars R_1, R_2, \dots, R_k and suppose G is the domination graph of a tournament T . Let G be properly 2-colored. Also let v_i and v_j be any two vertices in $V(R_i)$ and $V(R_j)$, respectively, that have the same color. Then if v_i beats v_j in T and if v and w are vertices of R_i and R_j , respectively, we have that v beats w in T if and only if v and w are the same color.*

If a caterpillar H is a component of the domination graph tournament T , then recall that the directed spine of the subdigraph of $\mathcal{D}(T)$ on $V(H)$ is defined to be the directed path that results when all vertices with out-degree 0 are removed from the subdigraph of $\mathcal{D}(T)$ on $V(H)$.

Theorem 4.6 *Let a forest G of nontrivial caterpillars be the domination graph of a near-regular tournament T . If H is a component of G , then H must be an odd gapped pendant restricted caterpillar.*

Proof: Assume a forest G of nontrivial caterpillars is the domination graph of a near-regular tournament T . Let H be a component of G . Consider the subdigraph of $\mathcal{D}(T)$ on $V(H)$. Lemma 2.8 guarantees the existence of the directed spine in the subdigraph of $\mathcal{D}(T)$ on $V(H)$. Thus, we may assume without loss of generality that the subdigraph of $\mathcal{D}(T)$ on $V(H)$ is as depicted in Figure 4.2. In addition, assume that the vertices of H are labeled as in Figure 4.2.

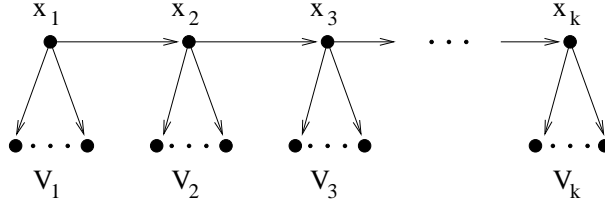


Figure 4.2. The subdigraph of the domination digraph on the vertex set of a component of $\text{dom}(T)$.

In Figure 4.2, the sets V_i represent the set of vertices that are pendant to x_i in H . Also note that the set V_1 maybe empty, in which case x_1 is a pendant vertex in H and the spine of H consists of the vertices x_2, x_3, \dots, x_k . Assume G is properly 2-colored. We begin by making the following observation regarding dominant pairs in $V(H)$. First, we have that $O_T(x_i) \cap O_T(x_{i+1}) = V_i$ for $1 \leq i \leq k - 1$. To see this, note that every vertex in V_i has the same color as x_{i+1} . So it follows by Lemma 2.9 that x_{i+1} beats every vertex in V_i . Hence, $V_i \subseteq O_T(x_i) \cap O_T(x_{i+1})$. Let $w \in V(T)$. Assume w belongs to $V(H')$ where H' is a component of G different from H . Then since every component in G is nontrivial and the vertices x_i and x_{i+1} are colored differently, we have by Lemma 4.5 that either x_i beats w or x_{i+1} beats w , but not both. If w is in $V(H)$, then again since x_i and x_{i+1} are colored differently, by Lemma 2.9 either x_i beats w or x_{i+1} beats w , but not both. Therefore, it follows that $O_T(x_i) \cap O_T(x_{i+1}) = V_i$, for $1 \leq i \leq k - 1$. Second, let $v \in V_i$. Then by a similar argument, it follows that if $w \in O_T(x_i) \cap O_T(v)$, then w must be in

V_i . With these observations in hand, we proceed by showing that H is an odd gapped pendant restricted caterpillar.

First we show that $|V_i| \leq 1$ for all i with $1 \leq i \leq k - 1$ and also that V_k has at most three vertices. Suppose that $|V_i| = 2$ for some i , with $1 \leq i \leq k - 1$. Since we know that $O_T(x_i) \cap O_T(x_{i+1}) = V_i$, we may conclude that $|O_T(x_i) \cap O_T(x_{i+1})| \geq 2$. But this contradicts Theorem 4.2 since $[x_i, x_{i+1}] \in \text{dom}(T)$. Therefore, $|V_i| \leq 1$ for all i with $1 \leq i \leq k - 1$. Now since x_k beats every vertex of V_k in the domination digraph, we have by Lemma 4.3 that V_k can have at most three vertices.

Note that if $V_i \neq \emptyset$ for some i with $1 \leq i \leq k - 1$, then since $O_{\mathcal{D}(T)}(x_i) = |\{x_{i+1}\} \cup V_i| = 2$, it follows by Lemma 4.3 that x_i must be a “+” vertex in T . Since we know that $|O(x_i) \cap O(x_{i+1})| = |V_i| = 1$ and $[x_i, x_{i+1}] \in \text{dom}(T)$, we may conclude by Lemma 4.2 that x_{i+1} must also be a “+” vertex. Lastly, if V_k has two or three vertices, then it follows by Lemma 4.3 that x_k must be a “+” vertex in T . We will use these facts to show that there must be an odd number of edges between two consecutive vertices on the spine with degree at least 3. Note that these vertices correspond to all vertices x_i that have out-degree at least 2 and in-degree 1 in $\mathcal{D}(T)$. Also note that the only vertex that can have out-degree greater than 2 and in-degree 1 is x_k .

Assume that $i, j \in \{1, 2, \dots, k\}$ with $i < j$ and satisfying the following properties:

- The vertex x_i has out-degree 2 and in-degree 1 in $\mathcal{D}(T)$. Note that this means $|V_i| = 1$.
- The vertex x_j has out-degree 2 (possibly greater than 2 if $j = k$) and in-degree 1 in $\mathcal{D}(T)$. Note that this means that $|V_j| \geq 1$.
- Every vertex x_l between x_i and x_j has out-degree and in-degree exactly 1 in $\mathcal{D}(T)$. Note that this means that for all l with $i < l < j$, we have $V_l = \emptyset$.

So assume that there is at least one vertex along the spine between x_i and x_j . By the comment in the preceding paragraph, the vertices x_i, x_{i+1} and x_j are all “+” vertices in T . In addition, since $O_T(x_j) \cap O_T(x_{j-1}) = V_{j-1} = \emptyset$ and $[x_{j-1}, x_j] \in \text{dom}(T)$, it follows by Lemma 4.2 that x_{j-1} must be a “-” vertex in T . Then since T is a near-regular tournament, we have that every vertex in T must be either a “+” or a “-” vertex. Thus, there must be at least two vertices on the spine of H between x_i and x_j . So suppose that there is an odd number of vertices along the spine between x_i and x_j . Then there will be two consecutive vertices x_l and x_{l+1} , $i < l < l + 1 < j$, such that either both are “+” vertices in T or both are “-” vertices in T . It follows by Lemma 4.2 that both cannot be “-” vertices in T since $[x_l, x_{l+1}] \in \text{dom}(T)$. In addition, both cannot be “+” vertices in T since $[x_l, x_{l+1}] \in \text{dom}(T)$ and we must have that $|O_T(x_l) \cap O_T(x_{l+1})| = 1$. But this last condition is not possible since $|O_T(x_l) \cap O_T(x_{l+1})| = V_l$ and $V_l = \emptyset$. Thus, we may conclude

that there must be an even number of vertices along the spine between x_i and x_j , which is equivalent to having x_i and x_j separated by an odd number of edges. Therefore, it follows that H must be an odd gapped pendant restricted caterpillar. ■

Lemma 4.7 *Let G be a caterpillar with a triple end. If T is a near-regular tournament, then G is not the domination graph of T .*

Proof: Let G be a caterpillar that is the domination graph of a near-regular tournament T . Suppose that G has a triple end. Then we may conclude by Theorem 4.6 that G is an odd gapped pendant restricted caterpillar. In addition, since $\text{dom}(T) = G$, we have by Theorem 2.10 that T is a caterpillar tournament. Now consider $\mathcal{D}(T)$. Lemma 2.8 guarantees the existence of the directed spine, thus we may assume that $\mathcal{D}(T)$ is as depicted in Figure 4.3.

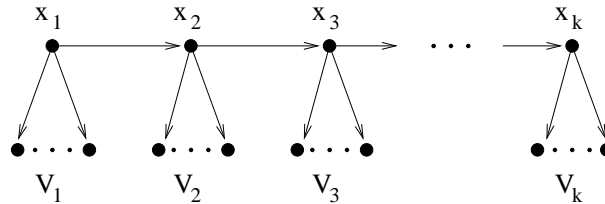


Figure 4.3. The domination digraph of T .

With respect to Figure 4.3, since T is a caterpillar tournament, then the spine of G consists of the vertices x_1, x_2, \dots, x_k if V_1 is nonempty and consists of the vertices x_2, x_3, \dots, x_k if V_1 is empty. Also since G is a caterpillar with a triple end, it follows from the proof of Theorem 4.6 that the triple end

must exist at x_k .

Now since T is a caterpillar tournament and G is an odd gapped pendant restricted caterpillar we have the following in $\mathcal{D}(T)$:

- (1) The integer k is even.
- (2) The sets V_i , for $1 \leq i \leq k - 1$, can have at most one vertex.
- (3) The set V_k has exactly three vertices and $T(V_k)$ is a subtournament without a transmitter.
- (4) Let $i < j$. If x_i and x_j are two consecutive vertices on the spine of G that have in-degree 1 and out-degree at least 2 in $\mathcal{D}(T)$, then they must be separated by an odd number of edges in G . Note that these vertices correspond to those vertices on the spine of G that have degree at least 3.

So using $\mathcal{D}(T)$, we will proceed by showing that T cannot be a near-regular tournament, which is a contradiction.

Case 1: Suppose that $V_1 \neq \emptyset$. Then the spine of G consists of the vertices x_1, x_2, \dots, x_k . To derive a contradiction, we will show that more than half of the vertices of T are “+” vertices. To show this, we first need to establish a series of results.

Result 1: $V_1 = 1$.

Since T is a caterpillar tournament, x_2 beats every vertex in V_1 . Thus, since $[x_1, x_2] \in \text{dom}(T)$ and x_1 beats every vertex in V_1 , it follows by Lemma

4.2 that $|V_1| = 1$.

Result 2: *Consecutive vertices on the spine with out-degree 2 are separated by an odd number of edges. In addition, half of the vertices between two such vertices are “+” vertices in T .*

Since we already have that consecutive vertices on the spine with in-degree 1 and out-degree at least 2 in $\mathcal{D}(T)$ must be separated by an odd number of edges, it suffices to show that x_1 and x_i are separated by an odd number of edges, where i is the smallest integer greater than 1 such that x_i has out-degree at least 2 in $\mathcal{D}(T)$. For this i , we may conclude by Lemma 4.3 that x_i must be a “+” vertex in T since x_i has out-degree greater than 1 in $\mathcal{D}(T)$. See Figure 4.3. To show that x_1 and x_i are separated by an odd number of edges, we may assume that $i > 2$, because the result is trivial if $i = 2$.

Now since T is a caterpillar tournament, $O(x_i) \cap O(x_{i+1}) = V_i$ for all i with $1 \leq i \leq k - 1$. Thus, since $[x_1, x_2] \in \text{dom}(T)$ and $|O(x_1) \cap O(x_2)| = |V_1| = 1$, it follows by Lemma 4.2 that both x_1 and x_2 must be “+” vertices in T . Also since $[x_{i-1}, x_i] \in \text{dom}(T)$, the vertex x_i is a “+” vertex, and $|O(x_{i-1}) \cap O(x_i)| = |V_{i-1}| = 0$, it follows from Lemma 4.2 that x_{i-1} must be a “-” vertex in T . Therefore, there must be at least two vertices between x_1 and x_i . So suppose that there are an odd number of vertices between x_1 and x_i . Then since every vertex in T is either a “+” or a “-” vertex, we will have two consecutive vertices x_l and x_{l+1} , with $1 < l < l + 1 < i$, such that both

are either “ $-$ ” vertices or “ $+$ ” vertices. Since $[x_l, x_{l+1}] \in \text{dom}(T)$, by Lemma 4.2 they cannot both be “ $-$ ” vertices. In addition, it follows by Lemma 4.2 that they both cannot be “ $+$ ” vertices since $|O(x_l) \cap O(x_{l+1})| = |V_l| = 0$. Therefore, there must be an even number of vertices between x_1 and x_i on the spine, which is equivalent to having x_1 and x_i separated by an odd number of edges.

Note from the argument just presented, it follows that if x_i and x_j with $i < j$ are two consecutive vertices on the spine that have out-degree at least 2 in $\mathcal{D}(T)$, then the vertices between them must alternate between being “ $+$ ” and “ $-$ ” vertices. Thus since there are an even number of vertices between two such vertices x_i and x_j , we have that half of the vertices between them must be “ $+$ ” vertices and half must be “ $-$ ” vertices.

Result 3: *Every vertex in V_k is a “ $+$ ” vertex in T .*

Let $w \in V_k$. Since $|V_k| = 3$ and $T(V_k)$ induces a tournament without a transmitter, it follows that $T(V_k) = U_3$, i.e., the subtournament on V_k is a 3-cycle. Thus, w beats exactly one vertex in V_k . Therefore, since $[x_k, w] \in \text{dom}(T)$ and x_k beats every vertex in V_k , it follows by Lemma 4.2 that $|O(x_k) \cap O(w)| = 1$. But by the same lemma, w is a “ $+$ ” vertex.

Now using Results 1 through 3 we are ready to derive our contradiction. Let r be the number of vertices on the spine that have out-degree at least 2 in $\mathcal{D}(T)$. Note that vertices with out-degree 2 in $\mathcal{D}(T)$ have exactly one

pendant neighbor. Thus, since $|V_k| = 3$, we have that $|V(T)| = k + (r - 1) + 3$ (k vertices are on the spine and $r + 3$ vertices are pendant). Next, since k is even and consecutive vertices on the spine that have out-degree 2 in $\mathcal{D}(T)$ have an even number of vertices between, then it follows that r must be even, i.e., there must be an even number of vertices on the spine that have out-degree at least 2 in $\mathcal{D}(T)$. Now note that by Lemma 4.3, every vertex on the spine with out-degree 2 in $\mathcal{D}(T)$ must be a “+” vertex in T . From Result 2, we have that half of the vertices between consecutive vertices on the spine with out-degree 2 in $\mathcal{D}(T)$ must be “+” vertices in T . In addition, from Result 3, every vertex in V_k is a “+” vertex in T . So counting the number of “+” vertices in T , at least $\frac{k-r}{2} + r + 3$ of the vertices in T are “+” vertices ($r + \frac{k-r}{2}$ of them occur on the spine and 3 occur in V_k). But this is a contradiction since $\frac{k-r}{2} + r + 3 > \frac{k + (r - 1) + 3}{2} = \frac{|V(T)|}{2}$ and a near-regular tournament must have the property that exactly half of its vertices are “+” vertices.

Case 2. Suppose that V_1 is empty. Then the spine of G consists of the vertices x_2, x_3, \dots, x_k . Let i be the smallest integer such that the vertex x_i has in-degree 1 and out-degree at least 2 in $\mathcal{D}(T)$. Note that $i > 1$. By the choice of i , all vertices x_j , with $j < i$, have out-degree 1 in $\mathcal{D}(T)$. Therefore, $V_j = \emptyset$ for all $j < i$.

Subcase 2.1 Suppose that i is even. Note that there are an odd number of vertices x_j with $j < i$. Also note that there are an even number

of vertices between consecutive vertices on the spine that have out-degree at least 2 in $\mathcal{D}(T)$. Therefore, since k is even, there must be an odd number of vertices on the spine that have out-degree at least 2 in $\mathcal{D}(T)$. So counting the number of vertices in T , we have an even number of vertices in $\{x_1, x_2, \dots, x_k\}$ and since $|V_i| \leq i$, for $i < k$ and $|V_k| = 3$, we have an odd number of vertices in $V_1 \cup V_2 \cup \dots \cup V_k$. But this implies that T has an odd number of vertices. This is a contradiction since T is near-regular and must have an even number of vertices.

Subcase 2.2 Suppose that i is odd. From Results 1 through 3, we have the following:

- (1) Every vertex with out-degree at least 2 in $\mathcal{D}(T)$ is a “+” vertex in T .
- (2) Half the vertices between consecutive vertices on the spine with out-degree at least 2 are “+” vertices in T .
- (3) Every vertex in V_k is a “+” vertex in T .

So let r be the number of vertices on the spine that have in-degree 1 and out-degree at least 2 in $\mathcal{D}(T)$. Since i is odd, there is an even number of vertices x_j with $j < i$. Thus, since k is even and there must be an even number of vertices between consecutive vertices on the spine that have in-degree 1 and out-degree at least 2 in $\mathcal{D}(T)$, it follows that r must be an even integer. Thus, as in Case 1, we have that $|V(T)| = k + (r - 1) + 3$.

Now note that if we show that half of the vertices in $\{x_1, x_2, \dots, x_{i-1}\}$

are “+” vertices, then we shall be able to produce the same contradiction as in Case 1. Thus, to finish this proof, it suffices to show that half of the vertices in $\{x_1, x_2, \dots, x_{i-1}\}$ are “+” vertices.

We begin by observing that since T is a caterpillar tournament, $O(x_{i-1}) \cap O(x_i) = V_{i-1} = \emptyset$. Then since $[x_{i-1}, x_i] \in \text{dom}(T)$ and x_i is a “+” vertex in T , it follows by Lemma 4.2 that x_{i-1} must be a “-” vertex in T . Then since $[x_{i-2}, x_{i-1}] \in \text{dom}(T)$ and $O(x_{i-2}) \cap O(x_{i-1}) = V_{i-2} = \emptyset$, we have by Lemma 4.2 that x_{i-2} must be a “+” vertex in T . Then since $[x_{i-3}, x_{i-2}] \in \text{dom}(T)$ and $O(x_{i-3}) \cap O(x_{i-2}) = V_{i-3} = \emptyset$, it follows by Lemma 4.2 that x_{i-3} must be a “-” vertex. Continuing in this fashion, we conclude that x_j is a “+” vertex in T , for j odd and $1 \leq j \leq i-1$ and is a “-” vertex in T , for j even and $1 \leq j \leq i-1$. Therefore, since $i-1$ is even we have that half of the vertices in $\{x_1, x_2, \dots, x_{i-1}\}$ are “+” vertices in T . Hence it follows as in Case 1 that more than half of the vertices in T are “+” vertices, which is a contradiction.

Therefore, it follows that a caterpillar G with a triple end cannot be the domination graph of a near-regular tournament. ■

Due to Lemmas 4.4 and 4.7, if a connected graph G is the domination graph of a near-regular n -tournament with $n \geq 4$, then G must be an odd-spiked cycle. It is not too difficult to see that not all odd-spiked cycles are the domination graphs of near-regular tournaments. For example consider the

following odd-spiked cycle.

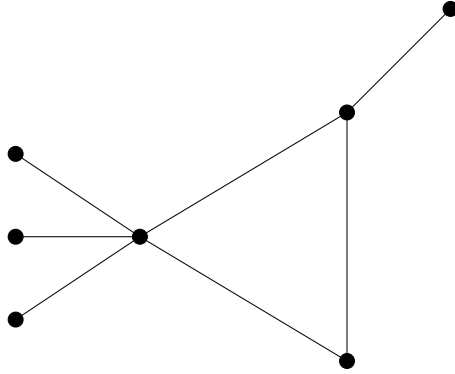


Figure 4.4. An odd-spiked cycle which is not the domination graph of a near-regular tournament.

Theorem 2.7 provides us with the means to construct the set of all tournaments that have this odd-spiked cycle as their domination graph. But none of these tournaments is a near-regular tournament. We will show that whether an odd-spiked cycle is the domination graph of a near-regular tournament will depend on the structure of the odd-spiked cycle.

Define an **odd gapped single spiked odd cycle** to be an odd-spiked cycle with the following properties:

- Each vertex on the cycle can have at most one pendant neighbor.
- Consecutive vertices on the cycle that have degree 3, or equivalently that have a pendant neighbor, must be separated by an odd number of edges. Note that this is equivalent to having an even number of vertices between consecutive vertices on the cycle which have degree 3.

Observe that an odd gapped single spiked odd cycle must always have an odd number of pendant vertices.

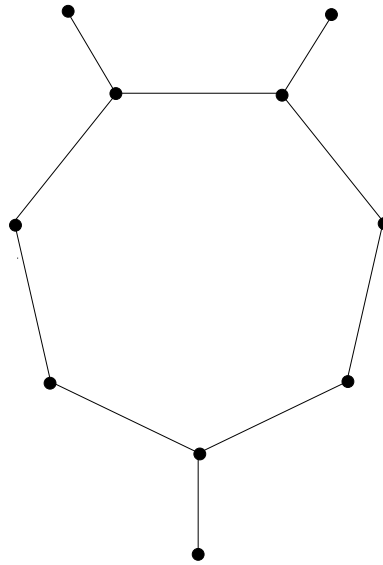


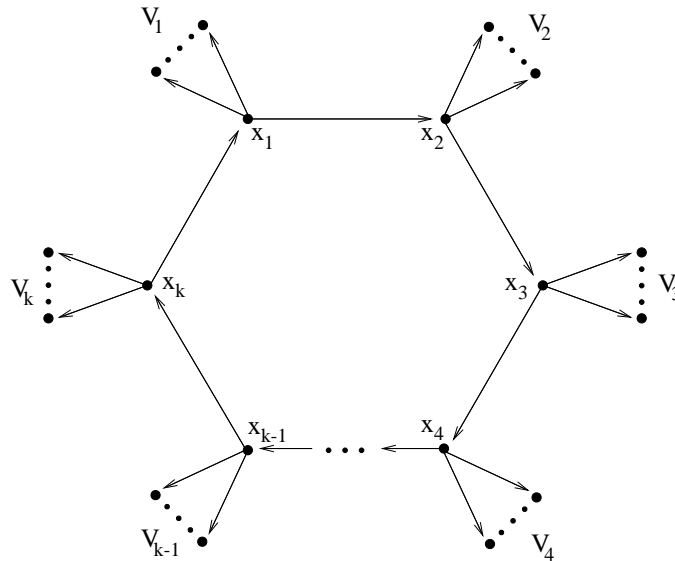
Figure 4.5. An odd gapped single spiked odd cycle.

Odd gapped single spiked odd cycles will play an important role in determining all connected domination graphs of near-regular tournaments. In fact, we will show that the only connected domination graphs of near-regular tournaments are essentially odd gapped single spiked odd cycles. In addition, we will show that every odd gapped single spiked odd cycle is the domination graph of a near-regular tournament. This, together with the fact that an odd cycle, a star, and a caterpillar with a triple end cannot be the domination graph of a near-regular tournament, will give us the main result of this chapter. Therefore, we proceed to derive the main result of Chapter 4, which is: A

connected graph G on $n \geq 2$ vertices is the domination graph of a near-regular tournament if and only if G is either K_2 or an odd gapped single spiked odd cycle.

Lemma 4.8 *Let G be an odd-spiked cycle. If G is the domination graph of a near-regular n -tournament T with $n \geq 4$, then G must be an odd gapped single spiked odd cycle.*

Proof: Let G be an odd-spiked k -cycle that is the domination graph of a near-regular n -tournament T with $n \geq 4$. Then it follows by Theorem 2.7 that T must be a spiked cycle tournament. So let x_1, x_2, \dots, x_k be the vertices on the cycle of G with $T(\{x_1, x_2, \dots, x_k\}) = U_k$ and let V_i , for $1 \leq i \leq k$, represent the set of pendant vertices at x_i . The remaining arcs in T , except for those within each V_i , are determined by Figure 2.3. In particular, note that $\mathcal{D}(T)$ is as shown below.



We proceed by showing that G must be an odd gapped single spiked odd cycle.

For the following arguments let $i + 1$ denote $(i \bmod k) + 1$.

First we show that $|V_i| \leq 1$ for $1 \leq i \leq k$. Assume that $|V_i| \geq 2$ for some i . Since T is a caterpillar tournament, Figure 2.3 determines that x_{i+1} beats every vertex in V_i . Thus, it follows that $|O(x_i) \cap O(x_{i+1})| \geq 2$. But this is a contradiction to Lemma 4.2, since $[x_i, x_{i+1}] \in \text{dom}(T)$. Therefore, that $|V_i| \leq 1$ for all $1 \leq i \leq k$.

Next we show that any two consecutive vertices along the cycle that have a pendant neighbor must be separated by an odd number of edges. First of all, if there is exactly one vertex x_i on the cycle that has a pendant neighbor, then it is easy to verify that there is an odd number of edges on the cycle when the cycle is traversed from x_i to itself.

So let x_i and x_j , with $i \neq j$, be two consecutive vertices along the cycle of G such that

- (1) x_i and x_j have a pendant neighbor,
- (2) there is a path from x_i to x_j on the cycle in $\mathcal{D}(T)$, and
- (3) every vertex between x_i and x_j on the path from x_i to x_j in $\mathcal{D}(T)$ has no pendant neighbors.

By Lemma 4.3, we can see that x_i and x_j must be “+” vertices in T since they both have out-degree 2 in $\mathcal{D}(T)$. If $j = i + 1$, then trivially x_i and x_j are separated by an odd number of edges. So assume that there is at least

one vertex between x_i and x_j on the path from x_i to x_j in $\mathcal{D}(T)$. Since T is a caterpillar tournament, $O_T(x_i) \cap O_T(x_{i+1}) = V_i$. Thus, since $|V_i| = 1$ and $[x_i, x_{i+1}] \in \text{dom}(T)$, it follows by Lemma 4.2 that x_{i+1} must be a “+” vertex in T . Similarly, since $V_{j-1} = \emptyset$ and $[x_{j-1}, x_j] \in \text{dom}(T)$, it follows by Lemma 4.2 that x_{j-1} must be a “-” vertex in T . Since every vertex in T is either a “+” vertex or “-” vertex, there must be at least two vertices between x_i and x_j on the path from x_i to x_j in $\mathcal{D}(T)$. If the number of vertices along the path from x_i to x_j in $\mathcal{D}(T)$ is odd and at least 3, then there will be two consecutive vertices x_l and x_{l+1} on this path between x_i and x_j such that both are either “-” vertices or both are “+” vertices. By Lemma 4.2, they both cannot be “-” vertices since they form a dominant pair in T . They also cannot be “+” vertices. To see this, observe that since $[x_l, x_{l+1}] \in \text{dom}(T)$, it follows by Lemma 4.2 that $|O(x_l) \cap O(x_{l+1})| = 1$. But then since T is a caterpillar tournament, $O_T(x_l) \cap O_T(x_{l+1}) = V_l$. Thus, we have a contradiction since $V_l = \emptyset$. Therefore, it follows that there must be an even number of vertices between x_i and x_j on the path from x_i to x_j in $\mathcal{D}(T)$, which is equivalent to having x_i and x_j separated by an odd number of edges. Hence, G is an odd gapped single spiked odd cycle. ■

Having established Lemma 4.8, naturally we wanted to know if all odd gapped single spiked odd cycles are the domination graphs of near-regular tournaments. We will show that the answer is yes. But before we do so, we

present the following two useful results.

Theorem 4.9 *Let $\{i_1, i_2, \dots, i_k\} \subseteq \{1, 2, \dots, n\}$ such that $i_1 < i_2 < \dots < i_k$.*

If the indices i_1, i_2, \dots, i_k alternate in parity, then the subtournament of U_n with vertices $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ is isomorphic to U_k .

Proof: Let $T = U_n$ and $\{i_1, i_2, \dots, i_k\} \subseteq \{1, 2, \dots, n\}$ such that $i_1 < i_2 < \dots < i_k$. Suppose that the indices i_1, i_2, \dots, i_k alternate in parity. To show that $T(\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}) \cong U_k$, it suffices to show that x_{i_l} beats x_{i_j} in the subtournament $T(\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\})$ if $l - j$ is either odd and negative or even and positive.

Assume $l - j$ is odd and negative. Then since the indices i_1, i_2, \dots, i_k alternate in parity, we have that i_l and i_j are of opposite parity. Also since $l - j$ is negative, $i_l < i_j$. But then since i_l and i_j are of opposite parity, $i_l - i_j$ is odd. In addition, since $i_l < i_j$, it follows that $i_l - i_j$ is negative. Thus, by definition of U_n , we have that x_{i_l} beats x_{i_j} in U_n . Therefore, since $T(\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\})$ is a subtournament of U_n , it follows that x_{i_l} beats x_{i_j} in $T(\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\})$.

Next assume $l - j$ is even and positive. Then since the indices i_1, i_2, \dots, i_k alternate in parity, i_l and i_j are of the same parity. Also since $l - j$ is even, $i_l > i_j$. But since i_l and i_j are of the same parity, $i_l - i_j$ is even. In addition, since $i_l > i_j$ we have that $i_l - i_j$ is positive. Therefore, we may again conclude that x_{i_l} beats x_{i_j} in U_n . Thus, x_{i_l} beats x_{i_j} in the subtournament $T(\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\})$. ■

In addition to being useful for proving the next result, Theorem 4.9 gives us a nice method for extracting subtournaments of U_n that are isomorphic to U_k , where $k \leq n$.

Lemma 4.10 *Let G be an odd gapped single spiked odd cycle and assume G is the domination graph of a tournament T . Let $\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$ be the set of pendant vertices such that $i_1 < i_2 < \dots < i_k$. Then the subtournament induced by the set of pendant vertices is isomorphic to U_k .*

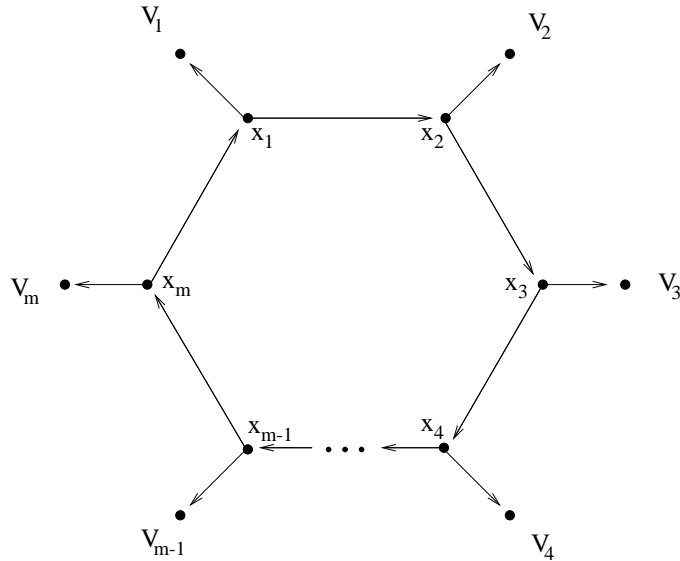
Proof: Let G be an odd gapped single spiked odd cycle and suppose G is the domination graph of a tournament T . Then since $\text{dom}(T) = G$, it follows by Theorem 2.7 that T is a caterpillar tournament. Also since G is an odd gapped single spiked odd cycle, each vertex on the cycle of G can have at most one pendant neighbor. So let x_1, x_2, \dots, x_m , with m odd, be the vertices on the cycle of G such that $T(\{x_1, x_2, \dots, x_m\}) \cong U_m$. In addition, let $\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$ be the set of pendant vertices, with $i_1 < i_2 < \dots < i_k$. To show that $T(\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}) \cong U_k$, it suffices to show that v_{i_j} beats v_{i_l} if $j - l$ is either odd and negative or even and positive. By definition, since G is an odd gapped single spiked odd cycle, we have that i_1, i_2, \dots, i_k must alternate in parity. So assume that $j - l$ is either odd and negative or even and positive. Then since the indices i_1, i_2, \dots, i_k alternate in parity, we have by Lemma 4.9 that x_{i_j} beats x_{i_l} . Therefore, since T is a caterpillar tournament and x_{i_j} beats x_{i_l} , we

may conclude by Figure 2.3 that v_{i_j} beats v_{i_i} . Thus, $T(\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}) \cong U_k$.

■

Lemma 4.11 *If G is an odd gapped single spiked cycle, then it is the domination graph of a near-regular tournament.*

Proof: Suppose G is an odd gapped single spiked cycle. Let x_1, x_2, \dots, x_m , with m odd, be the vertices on the cycle of G and let $v_{i_1}, v_{i_2}, \dots, v_{i_k}$, with k odd, be the pendant neighbors of those vertices, such that $i_1 < i_2 < \dots < i_k$. With this labeling of the vertices of G , construct a spiked cycle tournament that has G as its domination graph (as described in Chapter 2). Without loss of generality assume $T(\{x_1, x_2, \dots, x_m\}) = U_m$. Note that the remaining arcs in T are determined by Figure 2.3. In particular, we have that $\mathcal{D}(T)$ is as depicted in the following figure.



Note that in the figure above, the pendant sets V_i have at most one vertex since G is an odd gapped single spiked odd cycle. We claim that T is near-regular. To show this it suffices to show that every vertex has out-degree either $\frac{n}{2}$ or $\frac{n}{2} - 1$ where $n = m + k$.

First we will show that every pendant vertex has out-degree $\frac{n}{2} - 1$ and that its corresponding neighbor on the cycle has out-degree $\frac{n}{2}$. Let $v_{i_r} \in \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$, with $1 \leq r \leq k$, and x_{i_r} be its neighbor on the cycle. Since $T(\{x_1, x_2, \dots, x_m\}) = U_m$ and U_m is a regular tournament, x_{i_r} beats $\frac{m-1}{2}$ vertices in $\{x_1, x_2, \dots, x_m\}$ and is beaten by $\frac{m-1}{2}$ vertices in $\{x_1, x_2, \dots, x_m\}$. Note that since T is a spiked cycle tournament, $O_T(x_{i_r}) \cap O_T(v_{i_r}) = \emptyset$. Thus, since $[x_{i_r}, v_{i_r}] \in \text{dom}(T)$ and x_{i_r} is beaten by $\frac{m-1}{2}$ vertices on the cycle, v_{i_r} beats the $\frac{m-1}{2}$ vertices on the cycle that beat x_{i_r} . Next since G is an odd gapped single spiked odd cycle, the indices i_1, i_2, \dots, i_k alternate in parity. Also since G is the domination graph of T , it follows by Lemma 4.10 that $T(\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}) \cong U_k$. Thus, since U_k is a regular tournament, v_{i_r} beats $\frac{k-1}{2}$ vertices in $\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$ and is beaten by the other $\frac{k-1}{2}$ vertices in $\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$. Again since $O_T(x_{i_r}) \cap O_T(v_{i_r}) = \emptyset$ and $[x_{i_r}, v_{i_r}] \in \text{dom}(T)$, we have that x_{i_r} beats $\frac{k-1}{2}$ vertices in $\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$. Therefore, since x_{i_r} beats v_{i_r} , we conclude that $|O_T(x_{i_r})| = \frac{m-1}{2} + \frac{k-1}{2} + 1 = \frac{m+k}{2} = \frac{n}{2}$ and $|O_T(v_{i_r})| = \frac{m-1}{2} + \frac{k-1}{2} = \frac{m+k}{2} - 1 = \frac{n}{2} - 1$, as desired.

Next we will show that the remaining vertices in T have out-degree

either $\frac{n}{2}$ or $\frac{n}{2} - 1$.

First assume that $k = 1$, in other words, that G has exactly one pendant vertex. For this case, note that T has $n = m + 1$ vertices. So without loss of generality assume that x_1 is the vertex on the cycle that has a pendant neighbor v_1 . Let x_j , with $j \neq 1$, be any other vertex on the cycle. Since $T(\{x_1, x_2, \dots, x_m\}) = U_m$, we know that x_j beats exactly $\frac{m-1}{2}$ vertices in $\{x_1, x_2, \dots, x_m\}$. Now if x_j beats x_i , then since T is a spiked cycle tournament, we have by Figure 2.3 that v_1 beats x_j . And so, x_j has out-degree $\frac{m-1}{2} = \frac{n}{2} - 1$. If x_j is beaten by x_i , then again since T is a spiked cycle tournament, it follows by Figure 2.3 that x_j beats v_1 . And so, x_j has out-degree $\frac{m-1}{2} + 1 = \frac{n}{2}$. Since x_j was an arbitrary vertex on the cycle without a pendant neighbor, we conclude that all vertices on the cycle without pendant neighbors have the appropriate out-degree in T .

Next suppose that G has more than one pendant vertex. So let x_i and x_j , with $i \neq j$, be two consecutive vertices along the cycle of G such that

- (1) x_i and x_j have a pendant neighbor,
- (2) there is a path from x_i to x_j on the cycle in $\mathcal{D}(T)$, and
- (3) every vertex between x_i and x_j on the path from x_i to x_j in $\mathcal{D}(T)$, has no pendant neighbors.

For the following, we will take $r - i$ to mean $(r - i) \bmod m$. So let x_r be a vertex between x_i and x_j on the path from x_i to x_j in $\mathcal{D}(T)$. We will show

that if $r - i$ is odd, then $O_T(x_r) = \frac{n}{2}$ and if $r - i$ is even, then $|O(x_r)| = \frac{n}{2} - 1$. If $j \equiv i + 1 \pmod{m}$, then the result is trivial since we have already shown that every vertex with a pendant neighbor has out-degree $\frac{n}{2}$. So assume that there are at least two vertices between x_i and x_j on the path from x_i and x_j . We proceed by induction on $r - i$. Let $r \equiv i + 1 \pmod{m}$, so $r - i$ is odd. Since T is a caterpillar tournament and $[x_i, x_{i+1}] \in \text{dom}(T)$, we have that $O(x_i) \cap O(x_{i+1}) = \{v\}$ where v is the pendant vertex at x_i . But then since $[x_i, x_{i+1}] \in \text{dom}(T)$, it follows that $|O(x_i) \cup O(x_{i+1})| = n - 1$. Therefore, $|O(x_{i+1})| = |O(x_i) \cup O(x_{i+1})| - |O(x_i)| + |O(x_i) \cap O(x_{i+1})| = (n - 1) - \frac{n}{2} + 1 = \frac{n}{2}$. Next let $r \equiv i + 2 \pmod{m}$. Note that $r - i$ is even. Since $[x_{i+1}, x_{i+2}] \in \text{dom}(T)$ and T is a spiked cycle tournament, $O_T(x_{i+1}) \cap O_T(x_{i+2}) = V_{i+1} = \emptyset$. But since $[x_{i+1}, x_{i+2}] \in \text{dom}(T)$, we conclude that $|O_T(x_{i+1}) \cup O_T(x_{i+2})| = n - 1$. Therefore, since we have shown that $|O(x_{i+1})| = \frac{n}{2}$, it follows that $|O(x_{i+2})| = |O(x_{i+1}) \cup O(x_{i+2})| - |O(x_{i+1})| + |O(x_{i+1}) \cap O(x_{i+2})| = (n - 1) - \frac{n}{2} + 0 = \frac{n}{2} - 1$.

Assume the result holds for r , with $r - i \geq 2$. We will show that the result holds for $r + 1$. If $(r + 1) - i$ is odd, then $r - i$ is even. By the induction hypothesis, $|O(x_r)| = \frac{n}{2} - 1$. Since T is a caterpillar tournament and $[x_r, x_{r+1}] \in \text{dom}(T)$, it follows that $O(x_r) \cap O(x_{r+1}) = V_r = \emptyset$. But then since $[x_r, x_{r+1}] \in \text{dom}(T)$, we have that $|O(x_r) \cup O(x_{r+1})| = n - 1$. Therefore, $|O(x_{r+1})| = |O(x_r) \cup O(x_{r+1})| - |O(x_r)| + |O(x_r) \cap O(x_{r+1})| = (n - 1) -$

$\left(\frac{n}{2} - 1\right) + 0 = \frac{n}{2}$. Next if $r + 1 - i$ is even, then $r - i$ is odd. By the induction hypothesis, $|O(x_r)| = \frac{n}{2}$. Since T is a caterpillar tournament and $[x_r, x_{r+1}] \in \text{dom}(T)$, we may conclude that $O(x_r) \cap O(x_{r+1}) = V_r = \emptyset$. But then since $[x_r, x_{r+1}] \in \text{dom}(T)$, it follows that $|O(x_r) \cup O(x_{r+1})| = n - 1$. Therefore, $|O(x_{r+1})| = |O(x_r) \cup O(x_{r+1})| - |O(x_r)| + |O(x_r) \cap O(x_{r+1})| = (n - 1) - \frac{n}{2} + 0 = \frac{n}{2} - 1$. The result follows by induction. ■

Theorem 4.12 *Let G be a connected graph on $n \geq 2$ vertices. Then G is the domination graph of a near-regular tournament if and only if G is either K_2 or an odd gapped single spiked odd cycle.*

Proof: If $n = 2$, then there is exactly one tournament on two vertices which is near-regular, clearly its domination graph is K_2 . So let $n \geq 4$. Then by Lemmas 4.4 and 4.7, we may conclude that if G is the domination graph of a near-regular tournament, then G cannot be a star or a caterpillar. Thus, by Theorem 1.1, G must be an odd-spiked cycle. It follows by Lemma 4.8 that G must be an odd gapped single spiked odd cycle.

Conversely, for $n \geq 4$, we have by Lemma 4.11 that if G is an odd gapped single spiked odd cycle, then G is the domination graph of a near-regular tournament. ■

Therefore, we have successfully characterized all the connected domination graphs for near-regular tournaments. Although a caterpillar is never

the domination graph of a near-regular tournament, we have been able to determine by Lemma 4.6, that if a caterpillar is a component of the domination graph of a near-regular tournament, then it must be an odd gapped pendant restricted caterpillar. We will use this fact in Chapters 6 and 7 to construct infinite families of near-regular tournaments with the property that each has a forest of nontrivial caterpillars as its domination graph.

5. Near-Regular Path Tournaments

5.1 Introduction

In this chapter, we study and characterize all forests of paths that are domination graphs of near-regular tournaments. In [19], Fisher, *et al.*, determined which forests of caterpillars are possible as domination graphs of arbitrary tournaments. In the special case that each caterpillar is a nontrivial path, we have the following theorem.

Theorem 5.1 (Fisher, et al., [19]) *The union of n nontrivial paths is the domination graph of a tournament if and only if $n = 4$ or $n \geq 6$.*

Note that Theorem 5.1 holds for tournaments in general. Recall Theorem 4.1 which gives a characterization for all forests of paths that are domination graphs of regular tournaments. In particular, notice that not all combinations of m even paths and n nontrivial odd paths with $m + n = 4$ or $m + n \geq 6$, are domination graphs of regular tournaments. However, when the tournament is near-regular, we will show that any combination of four or at least six nontrivial paths, except four odd paths, is the domination graph of a near-regular tournament. Therefore, in the near-regular case, we get closer to the original result (Theorem 5.1). In the remainder of this chapter, an even path refers to a path on an even number of vertices and an odd path refers to a path on an

odd number of vertices.

5.2 Forests of m even paths and n nontrivial odd paths.

Let $\wp(m, n)$ denote the set of all graphs that are the union of m even paths and n odd paths (possibly trivial). Let $\wp^*(m, n)$ denote the set of all graphs that are the union of m even paths and n nontrivial odd paths. For the remainder of this chapter, we will focus on answering the question, *For which integers m and n is a graph $G \in \wp^*(m, n)$ the domination graph of a near-regular tournament?* We start by introducing some notation which will simplify the ensuing work.

Suppose $G \in \wp^*(m, n)$ is the domination graph of a tournament T . Let $G = E_1 \cup E_2 \cup \dots \cup E_m \cup O_1 \cup O_2 \cup \dots \cup O_n$, where E_i denotes the i th even path and O_j denotes the j th odd path. The vertex set of E_i is $\{u_{i1}, u_{i2}, \dots, u_{im_i}\}$ and the vertex set of O_j is $\{v_{j1}, v_{j2}, \dots, v_{jn_j}\}$. Define the following subtournaments of T as follows:

$T(B)$ with the vertices $\{u_{11}, u_{21}, \dots, u_{m1}, v_{11}, v_{21}, \dots, v_{n1}\}$,

$T(E)$ with the vertices $\{u_{11}, u_{21}, \dots, u_{m1}\}$,

$T(O)$ with the vertices $\{v_{11}, v_{21}, \dots, v_{n1}\}$,

$T(E_i)$ with the vertices $\{u_{i1}, u_{i2}, \dots, u_{im_i}\}$, and

$T(O_j)$ with the vertices $\{v_{j1}, v_{j2}, \dots, v_{jn_j}\}$.

We will refer to the set of vertices $\{u_{11}, u_{21}, \dots, u_{m1}, v_{11}, v_{21}, \dots, v_{n1}\}$ as the **base** vertices. In particular, u_{i1} is the base vertex that corresponds to the i th

even path E_i and v_{j1} is the base vertex that corresponds to the j th odd path O_j . In the remainder of this chapter, we will let R_i denote an arbitrary path when it is not specified whether it is an even or odd path. In addition, we will let $\{x_{i1}, x_{i2}, \dots, x_{is_i}\}$ denote the vertex set of R_i . Now by properly coloring each path R_i with two colors, red and blue, we are able to derive the following results.

Lemma 5.2 *Let $G \in \wp^*(m, n)$ be the domination graph of a tournament T and let $T(E_i)$ be the subtournament on the vertex set $\{u_{i1}, u_{i2}, \dots, u_{im_i}\}$ of the even path E_i and T_j be the subtournament on the vertex set $\{x_{j1}, x_{j2}, \dots, x_{js_j}\}$ of a path R_j different from E_i . Then for all vertices $x_{jk} \in V(T_j)$, $|O(x_{jk}) \cap V(T(E_i))| = |I(x_{jk}) \cap V(T(E_i))| = \frac{m_i}{2}$.*

Proof: Properly 2-color the vertices E_i and R_j using colors red and blue such that u_{i1} and x_{j1} are colored red.

Assume that u_{i1} beats x_{j1} in T . Let $x_{jk} \in V(T_j)$. If k is odd, then x_{jk} is colored red. Since u_{i1} and x_{j1} have the same color, we have by Lemma 4.5 that x_{jk} beats all the vertices in $T(E_i)$ that are colored blue and is beaten by all the vertices in $T(E_i)$ that are colored red. Since $T(E_i)$ has exactly $\frac{m_i}{2}$ red vertices and $\frac{m_i}{2}$ blue vertices, $|O(x_{jk}) \cap V(T(E_i))| = |I(x_{jk}) \cap V(T(E_i))| = \frac{m_i}{2}$. If k is even, then x_{jk} is colored blue. Again by Lemma 4.5, the vertex x_{jk} beats all the vertices in $T(E_i)$ that are colored red and is beaten by all the vertices

in $T(E_i)$ that are colored blue. Again it follows that $|O(x_{jk}) \cap V(T(E_i))| = |I(x_{jk}) \cap V(T(E_i))| = \frac{m_i}{2}$.

Next assume that x_{j1} beats u_{i1} . By a similar argument, we may conclude that $|O(x_{jk}) \cap V(T(E_i))| = |I(x_{jk}) \cap V(T(E_i))| = \frac{m_i}{2}$. ■

Lemma 5.3 *Let $G \in \wp^*(m, n)$ be the domination graph of a tournament T and let $T(O_i)$ be the subtournament on the vertex set $\{v_{i1}, v_{i2}, \dots, v_{in_i}\}$ of the odd path O_i and T_j be the subtournament on the vertex set $\{x_{j1}, x_{j2}, \dots, x_{js_j}\}$ of a path R_j different from O_i . If $(v_{i1}, x_{j1}) \in A(T)$, then for k odd, we have $|O(x_{jk}) \cap V(T(O_i))| = \frac{n_i - 1}{2}$ and for k even, we have $|O(x_{jk}) \cap V(T(O_i))| = \frac{n_i + 1}{2}$. If $(x_{j1}, v_{i1}) \in A(T)$, then for k odd, we have $|O(x_{jk}) \cap V(T(O_i))| = \frac{n_i + 1}{2}$ and for k even, we have $|O(x_{jk}) \cap V(T(O_i))| = \frac{n_i - 1}{2}$.*

Proof: Properly 2-color the vertices of O_i and R_j with colors red and blue such that v_{i1} and x_{j1} get colored red. Then $T(O_i)$ has exactly $\frac{n_i + 1}{2}$ red vertices and $\frac{n_i - 1}{2}$ blue vertices.

Assume v_{i1} beats x_{j1} in T . Let $x_{jk} \in V(T_j)$. If k is odd, then x_{jk} is colored red. Since v_{i1} and x_{j1} are colored red and v_{i1} beats x_{j1} , then it follows by Lemma 4.5 that x_{jk} beats all the vertices in $T(O_i)$ that are colored blue. Since $T(O_i)$ has exactly $\frac{n_i - 1}{2}$ blue vertices, $|O(x_{jk}) \cap V(T(O_i))| = \frac{n_i - 1}{2}$. If k is even, then x_{jk} is colored blue. Again by Lemma 4.5, we have that x_{jk} beats all the vertices in $T(O_i)$ that are assigned the color red. Since $T(O_i)$ has exactly $\frac{n_i + 1}{2}$ red vertices, $|O(x_{jk}) \cap V(T(O_i))| = \frac{n_i + 1}{2}$.

Next assume that x_{j_1} beats v_{i_1} and let $x_{j_k} \in V(T_j)$. If k is odd, then x_{j_k} is assigned the color red. Since v_{i_1} and x_{j_1} are colored red and x_{j_1} beats v_{i_1} , it follows by Lemma 4.5 that x_{j_k} beats all the vertices in $T(O_i)$ that are assigned the color red. But we know that $T(O_i)$ has exactly $\frac{n_i + 1}{2}$ red vertices, and so $|O(x_{j_k}) \cap V(T(O_i))| = \frac{n_i + 1}{2}$. If k is even, then x_{j_k} is assigned the color blue. Again by Lemma 4.5, we have that x_{j_k} beats all the vertices in $T(O_i)$ that are assigned the color blue. Since $T(O_i)$ has exactly $\frac{n_i - 1}{2}$ blue vertices, $|O(x_{j_k}) \cap V(T(O_i))| = \frac{n_i - 1}{2}$. ■

Now assume that $G \in \wp^*(m, n)$ is the domination graph of a tournament T and let R_i be any component of G with vertex set $\{x_{i_1}, x_{i_2}, \dots, x_{i_{s_i}}\}$. By Lemma 2.2, a vertex can have at most one in-neighbor and at most one out-neighbor that has out-neighbors in $\mathcal{D}(T)$. Thus, R_i must have one of the two orientations of Figure 5.1 in $\mathcal{D}(T)$.

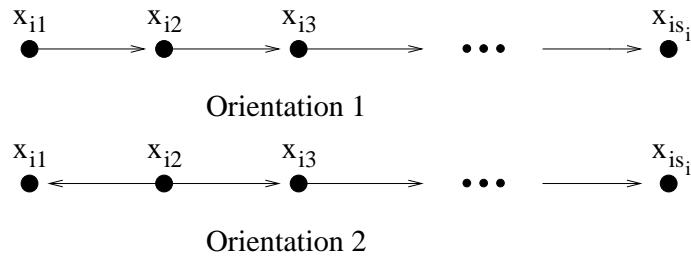


Figure 5.1. Two possible orientations for a path in the domination digraph.

The next result determines the corresponding subtournament of a path R_i given that it has one of the two orientations of Figure 5.1 in the

domination digraph. Let \mathbf{U}'_n denote the subtournament obtained from U_n by reversing the arc (x_1, x_2) .

Lemma 5.4 *Let $G \in \wp^*(m, n)$ be the domination graph of a near-regular tournament T and let R_j be a path of G with vertices $x_{j1}, x_{j2}, \dots, x_{js_j}$. If R_j has Orientation 1 (Figure 5.1) in $\mathcal{D}(T)$, then the subtournament $T(R_j) \cong U_k$. If R_j has Orientation 2 (Figure 5.1) in $\mathcal{D}(T)$, then the subtournament $T(R_j) \cong U'_k$.*

Proof: Suppose that R_j has Orientation 1 in $\mathcal{D}(T)$. For simplicity, we will let $x_{ji} = x_i$, for all $x_{ji} \in V(R_j)$. Now for $1 \leq i \leq k - 2$, since x_i beats x_{i+1} and $[x_{i+1}, x_{i+2}] \in \text{dom}(T)$, it follows that x_{i+2} beats x_i . Then for $1 \leq i \leq k - 3$, since x_{i+3} beats x_{i+1} and $[x_i, x_{i+1}] \in \text{dom}(T)$, it follows that x_i beats x_{i+3} . Thus, for $1 \leq i \leq k - 4$, since x_i beats x_{i+3} and $[x_{i+3}, x_{i+4}] \in \text{dom}(T)$, we have that x_{i+4} beats x_i . Continuing in this fashion, we conclude that x_i beats x_j if $i - j$ is odd and negative or even and positive. Thus by definition the subtournament $T(\{x_1, x_2, \dots, x_k\}) \cong U_k$.

Next suppose that R_j has Orientation 2 in $\mathcal{D}(T)$. Using an argument similar to the above we have that for $i, j \in \{2, 3, \dots, k\}$, the vertex x_i beats x_j if $i - j$ is odd and negative or even and positive. To show that $T(R_j) \cong U'_k$, it suffices to show that x_1 beats x_i for i even and $i \geq 4$, and is beaten by x_j for j odd and $j \geq 3$.

Since $[x_1, x_2] \in \text{dom}(T)$ and x_i beats x_2 for i even and $i \geq 4$, we

conclude that x_1 beats x_i , for i even and $i \geq 4$.

Next we show that x_i is beaten by x_j for j odd and $j \geq 3$. If $k = 3$, then since T is a tournament, we have that either x_1 beats x_3 or x_3 beats x_1 . In either case, it is easy to see that $T(R_j) \cong U'_3$. So assume that $k > 3$. Let j be odd with $j \geq 3$. If $j = k$, then since $k > 3$, we have that $j - 1$ is even and $j - 1 \geq 4$. Thus, since $[x_{j-1}, x_j] \in \text{dom}(T)$ and x_2 beats x_j , it follows that x_{j-1} beats x_2 . Then since $[x_1, x_2] \in \text{dom}(T)$ and x_{j-1} beats x_2 , we have that x_1 beats x_{j-1} . But then since $[x_{j-1}, x_j] \in \text{dom}(T)$ and x_1 beats x_{j-1} , it follows that x_j beats x_1 . So assume $j < k$. Then since $[x_j, x_{j+1}] \in \text{dom}(T)$ and x_2 beats x_j , we have that x_{j+1} beats x_2 . Then since $[x_1, x_2] \in \text{dom}(T)$ and x_{j+1} beats x_2 , it follows that x_1 beats x_{j+1} . But then since $[x_j, x_{j+1}] \in \text{dom}(T)$ and x_1 beats x_{j+1} , we have that x_j beats x_1 . Therefore, we may conclude that x_1 is beaten by x_j for j odd with $j \geq 3$. ■

Theorem 5.5 *Let $G \in \wp^*(m, n)$ be the domination graph of a near-regular tournament T and let R_j be a path of G with vertex set $\{x_{j_1}, x_{j_2}, \dots, x_{j_{s_j}}\}$. If R_j has Orientation 2 in Figure 5.1, then in T we have that x_{j_1} is a “−” vertex, x_{j_2} is a “+” vertex, x_{j_i} is a “+” vertex for i odd and $i \geq 3$ and x_{j_l} is a “−” vertex for l even and $l \geq 4$.*

Proof: Suppose that R_j has Orientation 2 of Figure 5.1 in $\mathcal{D}(T)$. Let R_i be any other component of G different from R_j . In addition, let $\{x_{i_1}, x_{i_2}, \dots, x_{i_{s_i}}\}$ denote the vertex set of R_i . First, we show that dominant pairs in R_j cannot

both beat any vertex in R_i . Properly 2-color the vertices of R_i and R_j with the colors red and blue such x_{i1} and x_{j1} are both colored red. Let x_{ir} be any vertex in $V(R_i)$. If x_{j1} beats x_{i1} in T , then by Lemma 4.5 vertices in R_j beat vertices of the same color in R_i and vertices in R_i beat vertices of the opposite color in R_j . Thus, for $1 \leq l \leq k - 1$, since x_{jl} and $x_{j(l+1)}$ are colored with opposite colors, we have that if x_{l1} beats x_{ir} , then x_{ir} beats $x_{l(i+1)}$. By a similar argument, if x_{i1} beats x_{j1} , then dominant pairs in R_j cannot both beat any vertex in R_i . Therefore, it follows that for all $1 \leq l \leq k - 1$, there is no vertex in $V(R_j)$ that both x_{jl} and $x_{j(l+1)}$ beat in T and so $O(x_{jl}) \cap O(x_{j(l+1)})$ must be contained in $V(R_j)$.

We continue by showing that the vertices of R_i have the degrees in T as stated above. To simplify the rest of this proof, we will let $x_{ji} = x_i$ for all $x_{ji} \in V(R_j)$. Since R_j has Orientation 2 in $\mathcal{D}(T)$, it follows by Lemma 5.4 that $T(R_j) = U'_{s_j}$. From above, we have that for all $1 \leq i \leq k - 1$, $O(x_i) \cap O(x_{i+1})$ must be contained in $V(R_j)$. Thus, since x_2 and x_3 are the only vertices of U'_{s_j} such that there exists a vertex, namely x_1 , in U'_{s_j} that they both beat, it follows that $O(x_1) \cap O(x_2)$ is empty, $O(x_2) \cap O(x_3) = \{x_1\}$, and $O(x_i) \cap O(x_{i+1}) = \emptyset$ for $3 \leq i \leq s_j$. Note that since x_2 has out-degree 2 in $\mathcal{D}(T)$, then we may conclude by Lemma 4.3 that x_2 must be a “+” vertex in T . But then since $[x_1, x_2] \in \text{dom}(T)$ and $O(x_1) \cap O(x_2) = \emptyset$, we have by

Lemma 4.2 that x_1 must be a “ $-$ ” vertex in T . Then since $[x_2, x_3] \in \text{dom}(T)$ and $O(x_2) \cap O(x_3) = \{x_1\}$, it follows by Lemma 4.2 that x_3 must be a “ $+$ ” vertex in T . Then since $[x_3, x_4] \in \text{dom}(T)$ and $O(x_3) \cap O(x_4) = \emptyset$, by Lemma 4.2 the vertex x_4 must be a “ $-$ ” vertex in T . Then since $[x_4, x_5] \in \text{dom}(T)$ and $O(x_4) \cap O(x_5) = \emptyset$, again it follows by Lemma 4.2 that x_5 is a “ $+$ ” vertex in T . Continuing in this fashion, we have that x_i is a “ $+$ ” vertex in T , for i odd with $i \geq 3$, and x_j is a “ $-$ ” vertex in T , for j even with $j \geq 4$. ■

The next result establishes that the subtournament $T(O)$ must have a certain structural property.

Lemma 5.6 *Let $G \in \wp(m, n)$. Assume G is the domination graph of a near-regular tournament T . Then n is an even integer and $T(O)$ is a near-regular tournament.*

Proof: Let $G \in \wp(m, n)$ be the domination graph of a near-regular tournament T . Since T is near-regular, the number of vertices in T is even. Thus it follows that n must be even.

So suppose that n is an even integer such that $n \geq 2$ and assume that $T(O)$ is not near-regular. Let $v_{l_1} \in V(T(O))$. By Lemma 5.2, the base vertex v_{l_1} beats exactly half of the vertices in $T(E_i)$ for $1 \leq i \leq m$. So let v_{j_1} be any other vertex in $T(O)$. By Lemma 4.3, if v_{l_1} beats v_{j_1} , then v_{l_1} beats $\frac{n_j + 1}{2}$ vertices in $T(O_j)$ and if $(v_{j_1}, v_{l_1}) \in A(T)$, then v_{l_1} beats $\frac{n_j - 1}{2}$ vertices in $T(O_j)$. Since O_l must have one of the orientations of Figure 5.1, we may

conclude by Lemma 5.4 that either $T(O_l) \cong U_{n_l}$ or $T(O_l) \cong U'_{n_l}$. Note that if $T(O_l) \cong U_{n_l}$, then v_{l1} beats $\frac{n_l - 1}{2}$ vertices in $T(O_l)$ and if $T(O_l) \cong U'_{n_l}$, then v_{l1} beats $\frac{n_l - 1}{2} - 1$ vertices in $T(O_l)$.

Now since $T(O)$ is not near-regular then there must exist a vertex $v_{k1} \in V(T(O))$, with $1 \leq k \leq n$, such that $d_{T(O)}^+(v_{k1}) > \frac{n}{2}$ or $d_{T(O)}^+(v_{k1}) < \frac{n}{2} - 1$ (i.e., v_{k1} beats either more than half of the vertices in $T(O)$ or beats one less than half of the vertices in $T(O)$). First assume that $d_{T(O)}^+(v_{k1}) = t > \frac{n}{2}$. If $T(O_k) \cong U_{n_k}$, then

$$\begin{aligned} d_{T(O)}^+(v_{k1}) &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i: (v_{k1}, v_{i1}) \in A(T)} \frac{n_i + 1}{2} + \sum_{i: (v_{i1}, v_{k1}) \in A(T)} \frac{n_i - 1}{2} \\ &\quad + \frac{n_k - 1}{2} \\ &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \frac{t}{2} - \frac{n - t - 1}{2} - \frac{1}{2} \\ &= \frac{1}{2}|V(T)| + t - \frac{n}{2} > \frac{1}{2}|V(T)|. \end{aligned}$$

The last inequality follows since $t - \frac{n}{2} > 0$. Thus, we have a contradiction.

If $T(O_k) \cong U'_{n_k}$, then

$$d_{T(O)}^+(v_{k1}) = \sum_{i=1}^m \frac{m_i}{2} + \sum_{i: (v_{k1}, v_{i1}) \in A(T)} \frac{n_i + 1}{2} + \sum_{i: (v_{i1}, v_{k1}) \in A(T)} \frac{n_i - 1}{2}$$

$$\begin{aligned}
& + \frac{n_k - 1}{2} - 1 \\
= & \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \frac{t}{2} - \frac{n - t - 1}{2} - \frac{1}{2} - 1 \\
= & \frac{1}{2}|V(T)| + (t - \frac{n}{2} - 1).
\end{aligned}$$

Note that $t - \frac{n}{2} - 1 \geq 0$. If $t - \frac{n}{2} - 1 = 0$, then v_{k1} is a “+” vertex in T . But by Theorem 5.5 we have that v_{k1} must be a “-” vertex, since O_k has Orientation 2 of Figure 5.1 in $\mathcal{D}(T)$. Thus, we have a contradiction. If $t - \frac{n}{2} - 1 > 0$, then it follows that $d_T^+(v_{k1}) > \frac{|V(T)|}{2}$, again a contradiction.

Next assume that $d_{T(O)}^+(v_{k1}) = t < \frac{n}{2} - 1$. If $T(O_k) \cong U_{n_k}$, then

$$\begin{aligned}
d_T^+(v_{k1}) &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i: (v_{k1}, v_{i1}) \in A(T)} \frac{n_i + 1}{2} + \sum_{i: (v_{i1}, v_{k1}) \in A(T)} \frac{n_i - 1}{2} \\
& + \frac{n_k - 1}{2} \\
= & \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \frac{t}{2} - \frac{n - t - 1}{2} - \frac{1}{2} \\
= & \frac{1}{2}|V(T)| + (t - \frac{n}{2}) < \frac{|V(T)|}{2} - 1.
\end{aligned}$$

This last inequality follows since $t < \frac{n}{2} - 1$. Thus, we have a contradiction.

If $T(O_k) \cong U'_{n_k}$, then

$$\begin{aligned}
d_T^+(v_{k1}) &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i: (v_{k1}, v_{i1}) \in A(T)} \frac{n_i + 1}{2} + \sum_{i: (v_{i1}, v_{k1}) \in A(T)} \frac{n_i - 1}{2} \\
&\quad + \frac{n_k - 1}{2} - 1 \\
&= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \frac{t}{2} - \frac{n - t - 1}{2} - \frac{1}{2} - 1 \\
&= \frac{1}{2}|V(T)| + (t - \frac{n}{2} - 1) < \frac{|V(T)|}{2} - 1.
\end{aligned}$$

This is a contradiction, so it follows that $T(O)$ must be a near-regular tournament. ■

Suppose $G \in \wp^*(m, n)$ is the union of $p = (m+n)$ paths R_1, R_2, \dots, R_p and T is a tournament whose vertex set is $V(G)$. Let the vertex set of R_i be $\{x_{i1}, x_{i2}, \dots, x_{is_i}\}$. For each vertex $x_{ij} \in V(R_i)$, we will color x_{ij} red, if j is odd and color x_{ij} blue, if j is even. Note that if $G = \text{dom}(T)$, then since R_i must have one of the orientations in Figure 5.1, by Lemma 5.5 either $T(R_i) = U_{s_i}$ or $T(R_i) = U'_{s_i}$. We will assume for the remainder of this work that $T(R_i) = U_{s_i}$ for all i with $1 \leq i \leq p$. We can then construct a tournament T with $\text{dom}(T) = G$ in the following manner. Given a structure for $T(E)$,

$T(O)$, and $T(B)$ and a proper 2-coloring (as prescribed above) for G , Lemma 4.5 determines the remaining arcs in T . If T is constructed in this fashion, we will say that T is a **path tournament**. Furthermore, if T is near-regular, we will say that T is a **near-regular path tournament**. We will show that whether or not T is a near-regular tournament such that $\text{dom}(T) = G$ will depend on the structure of $T(E)$, $T(O)$, and $T(B)$.

Lemma 5.7 *Let $G \in \wp^*(m, n)$ be the union of m even paths E_1, E_2, \dots, E_m and n nontrivial odd paths O_1, O_2, \dots, O_n . Also let T be a path tournament with $\text{dom}(T) = G$. Suppose that*

- (1) $T(O)$ is a near-regular tournament,
- (2) each vertex in $T(E)$ beats $\frac{n}{2}$ or $\frac{n}{2} - 1$ of the vertices of $T(O)$ in T ,
- (3) $T(E_i) = U_{m_i}$ for all $1 \leq i \leq m$, and
- (4) $T(O_j) = U_{n_j}$ for all $1 \leq j \leq n$.

Then T is a near-regular tournament.

Proof: Let R_j be a path of G with $V(R_j) = \{x_{j1}, x_{j2}, \dots, x_{js_j}\}$ and let $x_{jl} \in V(R_j)$. Also let R_i be a path different from R_j with $V(R_i) = \{x_{i1}, x_{i2}, \dots, x_{is_i}\}$. By Lemma 5.2, the vertex x_{jl} beats $\frac{s_i}{2}$ vertices of $V(R_i)$ in T if R_i is an even path. If R_i is an odd path and x_{j1} beats x_{i1} in T , then it follows by Lemma 5.3 that x_{jl} beats $\frac{s_i + 1}{2}$ vertices of $V(R_i)$ in T if l is odd and beats $\frac{s_i - 1}{2}$ vertices of $V(R_i)$ in T if l is even. If R_i is an odd path and

x_{i1} beats x_{j1} in T , then again by Lemma 5.3, we have that x_{jl} beats $\frac{s_i - 1}{2}$ vertices of $V(R_i)$ in T if l is odd and beats $\frac{s_i + 1}{2}$ vertices of $V(R_i)$ in T if l is even. Note that by assumption $T(R_j) = U_{s_j}$. So if R_j is an odd path, then x_{jl} beats $\frac{s_j - 1}{2}$ vertices of $V(R_j)$ in T . In addition, if R_j is an even path, then x_{jl} beats $\frac{s_j}{2}$ vertices of $V(R_j)$ in T if l is odd and beats $\frac{s_j}{2} - 1$ vertices of $V(R_j)$ in T if l is even. With this information we proceed by showing that every vertex in T has out-degree either $\frac{|V(T)|}{2}$ or $\frac{|V(T)|}{2} - 1$.

Let O_j be an arbitrary odd path and let $v_{jk} \in V(O_j)$. First assume that k is odd. Then

$$\begin{aligned} d_T^+(v_{jk}) &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i: (v_{j1}, v_{i1}) \in A(T)} \frac{n_i + 1}{2} \\ &\quad + \sum_{i: (v_{i1}, v_{j1}) \in A(T)} \frac{n_i - 1}{2} + \frac{n_j - 1}{2}. \end{aligned}$$

If v_{j1} beats $\frac{n}{2}$ of the vertices in $T(O)$, then

$$\begin{aligned} d_T^+(v_{jk}) &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \left(\frac{n}{2}\right) \frac{1}{2} - \left(\frac{n}{2} - 1\right) \frac{1}{2} - \frac{1}{2} \\ &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} \\ &= \frac{|V(T)|}{2}. \end{aligned}$$

If v_{j_1} beats $\frac{n}{2} - 1$ of the vertices in $T(O)$, then

$$\begin{aligned}
d_T^+(v_{jk}) &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \left(\frac{n}{2} - 1\right) \frac{1}{2} - \left(\frac{n}{2}\right) \frac{1}{2} - \frac{1}{2} \\
&= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} - 1 \\
&= \frac{|V(T)|}{2} - 1.
\end{aligned}$$

Next assume k is even. Then

$$\begin{aligned}
d_T^+(v_{jk}) &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i : (v_{j_1}, v_{i_1}) \in A(T)} \frac{n_i - 1}{2} \\
&\quad + \sum_{i : (v_{i_1}, v_{j_1}) \in A(T)} \frac{n_i + 1}{2} + \frac{n_j - 1}{2}.
\end{aligned}$$

If v_{j_1} beats $\frac{n}{2}$ of the vertices in $T(O)$, then

$$\begin{aligned}
d_T^+(v_{jk}) &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} - \left(\frac{n}{2}\right) \frac{1}{2} + \left(\frac{n}{2} - 1\right) \frac{1}{2} - \frac{1}{2} \\
&= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} - 1 \\
&= \frac{|V(T)|}{2} - 1.
\end{aligned}$$

If v_{j1} beats $\frac{n}{2} - 1$ of the vertices in $T(O)$, then

$$\begin{aligned}
d_T^+(v_{jk}) &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} - \left(\frac{n}{2} - 1\right) \frac{1}{2} + \left(\frac{n}{2}\right) \frac{1}{2} - \frac{1}{2} \\
&= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} \\
&= \frac{|V(T)|}{2}.
\end{aligned}$$

Thus, it follows that every vertex in $V(O_j)$ has out-degree either $\frac{|V(T)|}{2}$ or $\frac{|V(T)|}{2} - 1$ in T . Since O_j was arbitrary, this results holds true for all odd paths.

Next let E_j be an arbitrary even path and let $u_{jk} \in V(E_j)$. Assume that k is odd. Note that u_{jk} beats $\frac{m_j}{2}$ of the vertices in $V(E_j)$ in T . Then

$$\begin{aligned}
d_T^+(u_{jk}) &= \sum_{i=1, i \neq j}^m \frac{m_i}{2} + \sum_{i: (u_{j1}, v_{i1}) \in A(T)} \frac{n_i + 1}{2} \\
&\quad + \sum_{i: (v_{i1}, u_{j1}) \in A(T)} \frac{n_i - 1}{2} + \frac{m_j}{2}.
\end{aligned}$$

If u_{j1} beats $\frac{n}{2}$ of the vertices of $T(O)$, then

$$\begin{aligned}
d_T^+(u_{jk}) &= \sum_{i=1, i \neq j}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \left(\frac{n}{2}\right) \frac{1}{2} - \left(\frac{n}{2}\right) \frac{1}{2} \\
&= \frac{|V(T)|}{2}.
\end{aligned}$$

If u_{j1} beats $\frac{n}{2} - 1$ of the vertices of $T(O)$, then

$$\begin{aligned} d_T^+(u_{jk}) &= \sum_{i=1, i \neq j}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \left(\frac{n}{2} - 1\right) \frac{1}{2} - \left(\frac{n}{2} + 1\right) \frac{1}{2} \\ &= \frac{|V(T)|}{2} - 1. \end{aligned}$$

Next assume that k is even. Note that v_{jk} beats $\frac{m_j}{2} - 1$ of the vertices in $V(E_j)$ in T . Thus,

$$\begin{aligned} d_T^+(u_{jk}) &= \sum_{i=1, i \neq j}^m \frac{m_i}{2} + \sum_{i: (u_{j1}, v_{i1}) \in A(T)} \frac{n_i - 1}{2} \\ &\quad + \sum_{i: (v_{i1}, u_{j1}) \in A(T)} \frac{n_i + 1}{2} + \left(\frac{m_j}{2} - 1\right). \end{aligned}$$

If u_{j1} beats $\frac{n}{2}$ of the vertices of $T(O)$, then

$$\begin{aligned} d_T^+(u_{jk}) &= \sum_{i=1, i \neq j}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} - \left(\frac{n}{2}\right) \frac{1}{2} + \left(\frac{n}{2}\right) \frac{1}{2} - 1 \\ &= \frac{|V(T)|}{2} - 1. \end{aligned}$$

If u_{j1} beats $\frac{n}{2} - 1$ of the vertices of $T(O)$, then

$$d_T^+(u_{jk}) = \sum_{i=1, i \neq j}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} - \left(\frac{n}{2} - 1\right) \frac{1}{2} + \left(\frac{n}{2} + 1\right) \frac{1}{2} - 1$$

$$= \frac{|V(T)|}{2}.$$

Thus every vertex in $V(E_j)$ has out-degree either $\frac{|V(T)|}{2}$ or $\frac{|V(T)|}{2} - 1$ in T . Since E_j was arbitrary, the result follows for all even paths.

Therefore, we have shown that every vertex in T has out-degree either $\frac{|V(T)|}{2}$ or $\frac{|V(T)|}{2} - 1$ and so it follows that T is a near-regular tournament.

■

Two vertices x and y are **paired** in a digraph D if there exists a vertex w such that either (w, x) and (w, y) are arcs or (x, w) and (y, w) are arcs. Two vertices x and y are **distinguished** in a digraph D if there exists a vertex w such that either (x, w) and (w, y) are arcs or (w, x) and (y, w) are arcs. A digraph is **well-covered** if every pair of vertices is paired and distinguished. Well-covered tournaments play a significant role in the theory of domination graphs of tournaments. Let G be the union of k nontrivial caterpillars R_1, R_2, \dots, R_k . In [16], Fisher, *et al.*, showed by way of Theorem 2.3 that if there exists a well-covered tournament on k vertices, then G is the domination graph of some tournament. The proof of Theorem 2.3 in [16] can be summarized as follows: Under the assumption that there is a well-covered tournament on k vertices, first a tournament T is constructed on $V(G)$. This is done in the following manner. By Theorem 2.2, in the domination digraph of T

each component R_i of G must have one of the orientations depicted by Figure 2.6. Such an orientation is given to each component R_i of G . Note that in Figure 2.6 there is always a vertex that has in-degree 0. This vertex is fixed for each component R_i and is labeled v_i . We shall refer to v_i as the base vertex for R_i . Next, each component of G is properly 2-colored with colors red and blue such that for each component, its base vertex gets the color red. Then using Lemma 2.9, the remaining arcs in the subtournament $T(R_i)$ are determined for all i . Now the well-covered tournament is used to construct the subtournament on all the base vertices. Then using Lemma 4.5, the remaining arcs in T are determined. Once the tournament has been constructed, the property of being well-covered is used to show that no additional edges appear in $\text{dom}(T)$, i.e., that $\text{dom}(T)$ is actually G .

In addition, Fisher, *et al.*, showed by way of Theorem 2.5 in [16] that if every component of G does not have a triple end or is $K_{1,3}$, then it is also necessary for the subtournament on the base vertices of a tournament T to be well-covered in order to have $\text{dom}(T) = G$. The reasons for this can be summarized as follows: If the subtournament of T on the base vertices is not well-covered, then there must be a pair of vertices v_i and v_j that are not paired or not distinguished. Under the assumption that v_i beats v_j , it is shown in the proof of Theorem 2.5 of [16] that if v_i and v_j are not paired in the subtournament on the base vertices, then v_j forms a dominant pair with a

vertex in R_i , which should not occur. Also if v_i and v_j are not distinguished in the subtournament on all the base vertices, then v_i forms a dominant pair with a vertex in R_j , which also should not occur. Theorems 2.3 and 2.5 of [16], will prove useful for the following two results.

Theorem 5.8 *Let $G \in \wp^*(m, n)$. If G is the domination graph of a near-regular tournament, then G is also the domination graph of a near-regular path tournament.*

Proof: Let G be the union of $p = (m + n)$ paths R_1, R_2, \dots, R_p . Also let s_i denote the number of vertices in R_i . Assume G is the domination graph of a near-regular tournament T . To show that G is the domination graph of a path tournament T^* , it suffices to exhibit a path tournament T^* with $\text{dom}(T^*) = G$ such that for each component R_i of G , the subtournament on $V(R_i)$ is U_{s_i} .

Now if for each path R_i , we have that $T(R_i) = U_{s_i}$, then Lemma 4.5 determines that T is a path tournament, and we are done. So suppose that there exists a path R_i with vertex set $\{x_{i1}, x_{i2}, \dots, x_{is_i}\}$ such that the subtournament of T on $V(R_i)$ is not U_{s_i} . Then note that in $\mathcal{D}(T)$, the path R_i must have Orientation 2 of Figure 5.1, else by Lemma 5.4 we have that $T(R_i) \cong U_{s_i}$. Thus, since R_i has Orientation 2 in $\mathcal{D}(T)$, it follows by Lemma 5.4 that $T(R_i) \cong U'_{s_i}$. Then by Lemma 5.5, we have that x_{i1} is a “ $-$ ” vertex in T and x_{i2} is a “ $+$ ” vertex in T . So reverse the arc between x_{i1} and x_{i2} in

T to get a new tournament T' that differs with T in one arc. Now note that for all vertices other than x_{i1} and x_{i2} , their out-degrees are the same in T' as they were in T . In addition, note that in T' , x_{i1} gains an out arc and x_{i2} loses an out arc. Therefore, since x_{i1} is a “ $-$ ” vertex in T and x_{i2} is a “ $+$ ” vertex in T , every vertex in T' has out-degree either $\frac{|V(T)|}{2}$ or $\frac{|V(T)|}{2} - 1$ and so T' is a near-regular tournament.

Next we show that $\text{dom}(T') = G$. For each path in G , let v_j be the base vertex of R_j , i.e., v_j has in-degree 0 in the subdigraph of $\mathcal{D}(T)$ on $V(R_j)$. Note that $v_i = x_{i2}$, since x_{i2} has in-degree 0 in the subdigraph of $\mathcal{D}(T)$ on $V(H)$. Let B be the set of all v_j . Since G is the domination graph of T , the proof of Theorem 2.5 of [16] implies that the subtournament $T(B)$ must be well-covered. See the discussion above. We now claim that $B' = B \setminus \{x_{i2}\} \cup \{x_{i1}\}$ is a well-covered tournament. To see this, we first show that x_{i1} is paired and distinguished with every vertex in $T(B')$.

Let $v_j \in B'$. Since v_j is distinguished with x_{i2} in $T(B)$, there exists a vertex $v_k \in B$ such that either x_{i2} beats v_k and v_k beats v_j or v_k beats x_{i2} and v_j beats v_k . For the former, if x_{i2} beats v_k and v_k beats v_j , then since $[x_{i1}, x_{i2}] \in \text{dom}(T)$ and x_{i1} is a “ $-$ ” vertex in T , Lemma 4.2 shows that v_k must beat x_{i1} . Thus, x_{i1} is paired with v_j in $T(B')$ since they are both beaten by v_k and $v_k \in B'$. Similarly, if v_k beats x_{i2} and v_j beats v_k , then since

$[x_{i1}, x_{i2}] \in \text{dom}(T)$, we conclude that x_{i1} beats v_k . Again, x_{i1} and v_j are paired in $T(B')$ since they both beat v_k . Next since v_j is paired with x_{i2} in $T(B)$, there must exist a vertex $v_r \in B$ such that both v_j and x_{i2} beat v_r or they are both beaten by v_r . If both v_j and x_{i2} beat v_k , then since $[x_{i1}, x_{i2}] \in \text{dom}(T)$ and x_{i1} is a “ $-$ ” vertex in T , Lemma 4.2 shows that v_k must beat x_{i1} . Therefore, x_{i1} and v_j are distinguished in $T(B')$ since $v_k \in B'$. Similarly, if x_{i2} and v_j are beaten by v_k , then since $[x_{i1}, x_{i2}] \in \text{dom}(T)$, it follows that x_{i1} beats v_k . Again we conclude that x_{i1} and v_j are distinguished in $T(B')$.

Next we need to show that every other pair of vertices in B' is paired and distinguished. Let v_j and v_k be two vertices in B' . Since $T(B)$ is well-covered, v_k and v_j are paired by some vertex v_l in B . If $l \neq i$, i.e., $v_l \neq x_{i2}$, then since v_l is in B' , it follows that v_k and v_j are paired by v_l in $T(B')$. If $l = i$, then we have that v_k and v_j are paired by x_{i2} . Thus, either both v_j and v_k beat x_{i2} or are beaten by x_{i2} . If both are beaten by x_{i2} , then since $[x_{i1}, x_{i2}] \in \text{dom}(T)$ and x_{i1} is a “ $-$ ” vertex in T , by Lemma 4.2 both v_j and v_k must beat x_{i1} . Thus, v_j and v_k are paired in $T(B')$ by x_{i1} . If both v_j and v_k beat x_{i2} , then since $[x_{i1}, x_{i2}] \in \text{dom}(T)$, it follows that x_{i1} beats both v_j and v_k . Again we conclude that v_j and v_k are paired in $T(B')$ by x_{i1} . Next since v_j and v_k are distinguished, there must be a vertex v_r in B such that either v_j beats v_r and v_r beats v_k or v_r beats v_j and v_k beats v_r . If $r \neq i$, i.e.,

$v_r \neq x_{i2}$, then since v_r is a vertex in B' , the vertex v_j and v_k are distinguished in $T(B')$. If $r = i$, then assume without loss of generality that x_{i2} is beaten by v_j and beats v_k . Then since $[x_{i1}, x_{i2}] \in \text{dom}(T)$ and x_{i1} is a “ $-$ ” vertex in T , Lemma 4.2 shows that x_{i1} is beaten by v_k and x_{i1} beats x_j . Thus, v_j and v_k are distinguished by $x_{i1} \in T(B')$. Therefore, it follows that the subtournament $T(B')$ is well-covered.

Now since x_{i1} has in-degree 0 in the subdigraph of $V(T')$ on $V(R_i)$, we have that B' is the set of base vertices for all paths in G . So from the proof of Theorem 2.3 of [16], it follows that $\text{dom}(T') = G$. See the discussion preceding this theorem. Now if for each path R_i of G , we have that $T'(R_i) \cong U_{s_i}$, then it follows by Lemma 4.5 that T' is a path tournament. If this is not the case, then iterate this process until a tournament T^* is obtained with the property that every path R_i of G has $T^*(R_i) \cong U_{s_i}$. Since p is finite, this process must terminate. ■

Theorem 5.9 *Assume $G \in \wp^*(m, n)$. Then, G is the domination graph of a near-regular tournament if and only if there exists a near-regular path tournament T for G such that $T(B)$ is well-covered.*

Proof: Assume G is the domination graph of a near-regular tournament. Then by Theorem 5.8, the graph G is the domination graph of a near-regular path tournament T . So T is a path tournament for G . Furthermore, by Theorem

2.5 of [16], there is a well-covered tournament on $m + n$ vertices. In fact, from the proof of Theorem 2.5 of [16] the tournament $T(B)$ must be well-covered. See the discussion preceding Theorem 5.8.

For the converse suppose that there exists a near-regular path tournament T for G such that $T(B)$ is well covered. It follows from the proof of Theorem 2.3 in [16] that $\text{dom}(T) = G$. See the discussion preceding Theorem 5.8. ■

We will now derive the main result (Theorem 5.21) of Chapter 5 which is: *If $G \in \wp^*(m, n)$, then G is the domination graph of a near-regular tournament if and only if n is even, with $n \neq 4$ and $m + n = 4$ or $m + n \geq 6$.* This result, will be proved by using Theorem 5.10 and Lemmas 5.13, 5.14, 5.15, 5.17, and 5.20 which we will present next.

Theorem 5.10 *Let m and n be integers with n even and $m + n = 4$ or $m + n \geq 6$. Assume that there exist tournaments T_O , T_E and T_B with the following properties:*

- (1) T_O is a near-regular tournament on n vertices,
- (2) T_E is a tournament on m vertices,
- (3) T_O and T_E are subtournaments of T_B such that $V(T_O)$ and $V(T_E)$ partition the vertex set of T_B ,
- (4) in T_B every vertex in T_E beats either $\frac{n}{2}$ or $\frac{n}{2} - 1$ of the vertices in T_O ,

and

(5) T_B is well-covered.

Then for every graph $G \in \wp^*(m, n)$, there exists a near-regular tournament T with $\text{dom}(T) = G$.

Proof: Let a graph $G \in \wp^*(m, n)$ be the union of m even paths E_1, E_2, \dots, E_m and n odd paths O_1, O_2, \dots, O_n . First we construct a path tournament T whose vertex set is $V(G)$. Let $T(E_i) = U_{m_i}$ for $1 \leq i \leq m$ and $T(O_j) = U_{n_j}$ for $1 \leq j \leq n$. Next take $T(O) = T_O$ to be the tournament on the base vertices for the n odd paths, $T(E) = T_E$ to be the tournament on the base vertices for the m even paths, and $T(B) = T_B$ to be the tournament on all base vertices. Then by properly 2-coloring G such that all the base vertices get assigned the color red, we have that Lemma 4.5 determines the remaining arcs in T .

Now since $T(O)$, $T(E)$, and $T(B)$ satisfy Properties 1 through 4, Lemma 5.7 shows that T is a near-regular tournament. In addition, since $T(B)$ satisfies Property 5, i.e., since $T(B)$ is well covered, we conclude by Theorem 5.9 that $\text{dom}(T) = G$. Therefore, since G was an arbitrary graph in $\wp^*(m, n)$, the result follows for all graphs in $\wp^*(m, n)$. ■

Now given integers m and n with n even, $n \neq 4$ and $m + n = 4$ or $m + n \geq 6$, we will use Theorem 5.10 to show that all graphs in $\wp^*(m, n)$ are domination graphs of near-regular tournaments. Before doing so, we present

two special types of tournaments that shall prove useful.

Let $\mathbf{W}_{1,n}$ denote the set of all $(n + 1)$ -tournaments T which have a transmitter x and such that $T - \{x\}$ induces a tournament having every arc in a 3-cycle. Let $\overline{\mathbf{W}_{1,n}}$ denote the set of all $(n + 1)$ -tournaments which have a vertex x of out-degree 0 and such that $T - \{x\}$ induces a subtournament having every arc in a 3-cycle.

Lemma 5.11 *If $T \in W_{1,n}$, then T is a well-covered tournament.*

Proof: Assume that $T \in W_{1,n}$. Note that n must be either 3 or at least 5, since there does not exist a tournament on one, two or four vertices with every arc in a 3-cycle. Let x be the transmitter in T and $u, v \in V(T) \setminus \{x\}$. Assume that u beats v . Since x beats u and v in T , the vertices u and v are paired. Also since $T - \{x\}$ has every arc in a 3-cycle, there exists a vertex $w \in V(T) \setminus \{x\}$ such that v beats w and w beats u . Thus, u and v are distinguished. Thus, since u and v were arbitrary, it follows that every pair of vertices in $V(T) \setminus \{x\}$ are paired and distinguished.

Next we show that x is paired and distinguished with every vertex in $V(T) \setminus \{x\}$. Again let $u \in V(T) \setminus \{x\}$. Since $T - \{x\}$ has every arc in a 3-cycle, there must exist vertices z_1 and z_2 such that z_1 beats u and u beats z_2 . Then since both x and u beat z_2 in T , we conclude that x and u are paired. Also, since x beats z_1 and z_1 beats u , it follows that x and u are distinguished. Thus,

since u was arbitrary, it follows that x is paired and distinguished with every vertex in $V(T) \setminus \{x\}$. Therefore, $\overline{W_{1,n}}$ is well-covered. ■

Note that a tournament T is a well-covered if and only if its reversal is well-covered. Thus, since every tournament in $\overline{W_{1,n}}$ is the reversal of a tournament in $W_{1,n}$, we get the following corollary to Lemma 5.11.

Corollary 5.12 *If $T \in \overline{W_{1,n}}$, then T is a well-covered tournament.*

For the purpose of simplification, for the proofs of Lemmas 5.13, 5.14, 5.15, 5.17, and 5.20, if not specified, we will let $V(T_E) = \{u_1, u_2, \dots, u_m\}$, $V(T_O) = \{v_1, v_2, \dots, v_n\}$ and $V(T_B) = V(T_O) \cup V(T_E)$.

Lemma 5.13 *Let $G \in \wp^*(m, 0)$ with $m = 4$ or $m \geq 6$. Then G is the domination graph of a near-regular tournament.*

Proof: It is well known that there always exists a well-covered tournament on $m = 4$ or $m \geq 6$ vertices. For example, consider the tournaments in $W_{1,m-1}$ for $m = 4$ or $m \geq 6$. So let $T_E = T_B$ be any well-covered tournament on m vertices. Then it follows by Theorem 5.10 that for every graph $G \in \wp^*(m, 0)$, there exists a near-regular tournament T with $dom(T) = G$. ■

Lemma 5.14 *Let $G \in \wp^*(m, 2)$ with $m = 2$ or $m \geq 4$. Then G is the domination graph of a near-regular tournament.*

Proof: For $m = 2$ or $m \geq 4$, we will construct tournaments T_E , T_O and T_B that satisfy Theorem 5.10. Then by Theorem 5.10, it will follow that for every

graph $G \in \wp^*(m, 2)$, there is a near-regular tournament T with $\text{dom}(T) = G$.

Construction for $m = 2$. Let $T_O = U_2$ and $T_E = U_2$. The remaining arcs in T_B are $v_1 \rightarrow \{u_1, u_2\}$, $v_2 \rightarrow u_1$ and $u_2 \rightarrow v_2$. See Figure 5.2. Since every vertex in T_E beats either zero or one of the vertices in T_O , the tournaments T_E , T_O , and T_B satisfy Properties 1 through 4 of Theorem 5.10. Note that T_B belongs to $W_{1,3}$. Thus, by Lemma 5.11, the tournament $T(B)$ is well-covered and so Property 5 of Theorem 5.10 is also satisfied. Therefore, it follows by Theorem 5.10 that for every graph $G \in \wp^*(2, 2)$, there exists a near-regular tournament T with $\text{dom}(T) = G$.

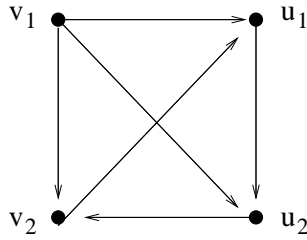


Figure 5.2. The tournament T_B for $m = n = 2$.

Construction for $m = 5$. Let $T_E = U_5$ and $T_O = U_2$. The remaining arcs in T_B are $\{v_1, v_2\} \rightarrow u_1$, and for $j = 2, 3, 4, 5$, we have $u_j \rightarrow v_1$. See Figure 5.3. Then since every vertex in T_E beats either zero or one of the vertices in T_O , the tournaments T_E , T_O , and T_B satisfy Properties 1 through 4 of Theorem 5.10. In addition, Table 5.1 verifies that T_B is a well-covered tournament. Therefore, it follows by Theorem 5.10 that for every graph $G \in \wp^*(5, 2)$, there exists a near-regular tournament T with $\text{dom}(T) = G$.

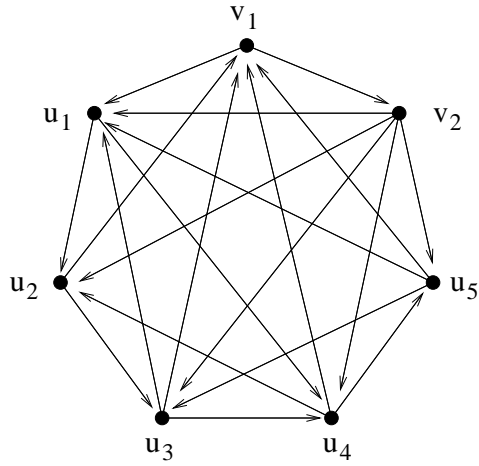


Figure 5.3. The tournament T_B for $m = 5$ and $n = 2$.

| Vertex pair: | Distinguished by: | Paired by: |
|----------------|-------------------|------------|
| $\{v_1, v_2\}$ | u_2 | u_1 |
| $\{v_1, u_1\}$ | u_2 | u_5 |
| $\{v_1, u_2\}$ | v_2 | u_4 |
| $\{v_1, u_3\}$ | v_2 | u_5 |
| $\{v_1, u_4\}$ | v_2 | u_3 |
| $\{v_1, u_5\}$ | v_2 | u_4 |
| $\{v_2, u_1\}$ | u_5 | u_1 |
| $\{v_2, u_2\}$ | v_1 | u_3 |
| $\{v_2, u_3\}$ | v_1 | u_4 |
| $\{v_2, u_4\}$ | v_1 | u_5 |
| $\{v_2, u_5\}$ | v_1 | u_3 |
| $\{u_1, u_2\}$ | u_3 | v_2 |
| $\{u_1, u_3\}$ | u_2 | v_2 |
| $\{u_1, u_4\}$ | u_5 | v_2 |
| $\{u_1, u_5\}$ | u_4 | v_2 |
| $\{u_2, u_3\}$ | u_1 | v_1 |
| $\{u_2, u_4\}$ | u_3 | v_1 |
| $\{u_2, u_5\}$ | u_1 | v_1 |
| $\{u_3, u_4\}$ | u_5 | v_1 |
| $\{u_3, u_5\}$ | u_4 | v_1 |
| $\{u_4, u_5\}$ | u_3 | v_1 |

Table 5.1.

Construction for $m = 4$ or $m \geq 6$. Let $T_O = U_2$ and T_E be any tournament in $\overline{W_{1,m-1}}$ such that u_1 is the vertex with out-degree 0. The remaining arcs in T_B are as follows $\{v_1, v_2\} \rightarrow u_1$, and for $j = 2, 3, \dots, m$, we have $u_j \rightarrow v_1$ and $v_2 \rightarrow u_j$. See Figure 5.4 for an example of the tournament T_B when $m = 4$. Then since every vertex in T_E beats either zero or one of the vertices in T_O , the tournaments T_E , T_O , and T_B satisfy Properties 1 through 4 of Theorem 5.10.

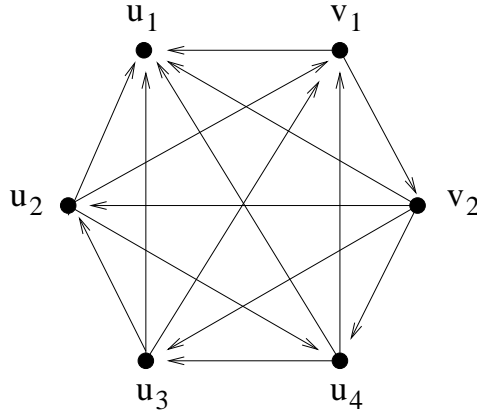


Figure 5.4. The tournament T_B , for $m = 4$ and $n = 2$.

Next we will show that T_B is well-covered. It is easy to see that v_1 and v_2 are paired and distinguished. By Corollary 5.12, we conclude that T_E is well-covered. Thus, since T_E is a subtournament of T_B , every pair of vertices in $V(T_E)$ are paired and distinguished in T_B . Therefore, to show that T_B is well-covered, it suffices to show that v_j and u_i are paired and distinguished for every choice of $v_j \in V(T_O)$ and $u_i \in V(T_E)$. First we show that v_1 is paired and distinguished with every vertex in $V(T_E)$. Clearly, v_1 and u_1 are

paired since they are both beaten by u_2 . Then since v_1 beats v_2 and v_2 beats u_1 , it follows that v_1 and u_1 are also distinguished. Let $u_i \in V(T_E)$ for some $i \in \{2, 3, \dots, m\}$. Since $T_E - \{u_1\}$ induces a subtournament with every arc in a 3-cycle, there exists a vertex $u_l \in V(T_E) \setminus \{u_1\}$, with $l \neq i$, such that u_l beats u_i . But since u_l beats v_1 , it follows that u_i and v_1 are paired. In addition, since v_1 beats v_2 and v_2 beats u_i , they are also distinguished.

Next, since v_1 beats both v_2 and u_1 , we have that u_1 and v_2 are paired. Also since v_2 beats u_2 and u_2 beats u_1 , it follows that u_1 and v_2 are distinguished. Let $u_i \in V(T_E)$ for some $i \in \{2, 3, \dots, m\}$. Since u_i beats v_1 and v_1 beats v_2 , we conclude that u_i and v_2 are distinguished. In addition, they are also paired since both u_i and v_2 beat u_1 .

Thus, we have shown that every pair of vertices in T_B are paired and distinguished. It follows that T_B is well-covered and so Property 5 of Theorem 5.10 is satisfied. Therefore, we may conclude by Theorem 5.10 that for every graph $G \in \wp^*(m, 2)$ with $m = 4$ or $m \geq 6$, there exists a near-regular tournament T with $\text{dom}(T) = G$. ■

Lemma 5.15 *Let $G \in \wp^*(m, 4)$ with $m \geq 2$. Then G is the domination graph of a near-regular tournament.*

Proof: For $m \geq 2$, we will construct tournaments T_E , T_O , and T_B that satisfy Theorem 5.10. Then it will follow by the same theorem that for every graph $G \in \wp^*(m, 4)$, there exists a near-regular tournament T with $\text{dom}(T) = G$.

Construction for $m = 2$. Let $T_E = U_2$ and T_O be the near-regular tournament in Figure 5.5. The remaining arcs in T_B are $u_1 \rightarrow v_2$, $\{v_1, v_3, v_4\} \rightarrow u_1$, $u_2 \rightarrow \{v_2, v_3\}$, $\{v_1, v_4\} \rightarrow u_2$. See Figure 5.6. Then since every vertex in T_E beats either one or two vertices of T_O , the tournaments T_E , T_O , and T_B satisfy Properties 1 through 4 of Theorem 5.10. In addition, Table 5.2 verifies that T_B is well-covered and so Property 5 of Theorem 5.10 is also satisfied. Therefore, by Theorem 5.10 we may conclude that for every graph $G \in \wp^*(2, 4)$, there exists a near-regular tournament T with $\text{dom}(T) = G$.

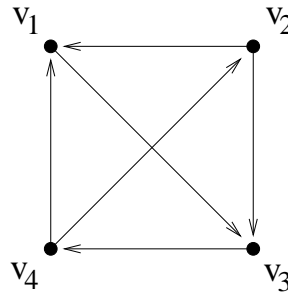


Figure 5.5. The unique near-regular tournament on four vertices.

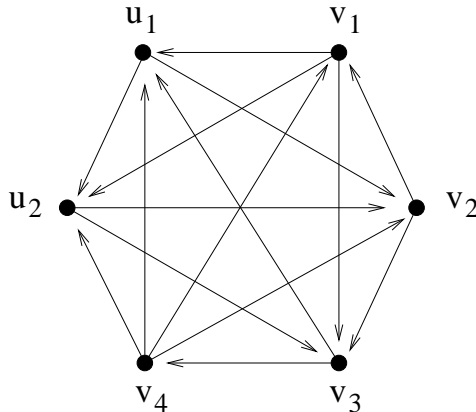


Figure 5.6. The tournament T_B for $m = 2$ and $n = 4$.

| Vertex pair: | Distinguished by: | Paired by: |
|----------------|-------------------|------------|
| $\{u_1, u_2\}$ | v_3 | v_4 |
| $\{u_1, v_1\}$ | v_2 | u_2 |
| $\{u_1, v_2\}$ | v_1 | v_4 |
| $\{u_1, v_3\}$ | v_2 | v_1 |
| $\{u_1, v_4\}$ | v_1 | v_2 |
| $\{u_2, v_1\}$ | u_1 | v_4 |
| $\{u_2, v_2\}$ | v_1 | u_1 |
| $\{u_2, v_3\}$ | u_1 | v_1 |
| $\{u_2, v_4\}$ | u_1 | v_2 |
| $\{v_1, v_2\}$ | u_1 | v_3 |
| $\{v_1, v_3\}$ | u_2 | v_2 |
| $\{v_1, v_4\}$ | v_3 | u_1 |
| $\{v_2, v_3\}$ | u_1 | u_2 |
| $\{v_2, v_4\}$ | v_3 | v_1 |
| $\{v_3, v_4\}$ | u_2 | u_1 |

Table 5.2.

Construction for $m = 3$. Let $T_E = U_3$ and let T_O be the tournament in Figure 5.5. The remaining arcs in T_B are $u_1 \rightarrow v_2$, $\{v_1, v_3, v_4\} \rightarrow u_1$, $\{u_2, u_3\} \rightarrow \{v_2, v_3\}$ and $\{v_1, v_4\} \rightarrow \{u_2, u_3\}$. See Figure 5.7. Then since every vertex in T_E beats either one or two vertices in T_O , the tournaments T_E , T_O , and T_B satisfy Properties 1 through 4 of Theorem 5.10. In addition, Table 5.3 verifies that T_B is a well-covered tournament and so Property 5 of Theorem 5.10 is also satisfied. Thus, we may conclude by Theorem 5.10 that for every graph $G \in \wp^*(3, 4)$, there exists a near-regular tournament T with $\text{dom}(T) = G$.

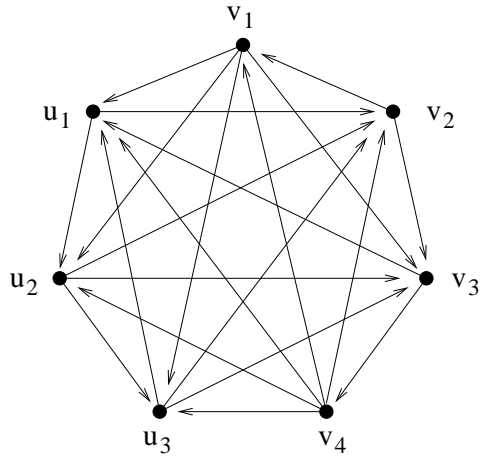


Figure 5.7. The tournament T_B for $m = 3$ and $n = 4$.

| Vertex pair: | Distinguished by: | Paired by: |
|----------------|-------------------|------------|
| $\{u_1, u_2\}$ | u_3 | v_2 |
| $\{u_1, u_3\}$ | u_2 | v_2 |
| $\{u_1, v_1\}$ | v_2 | u_2 |
| $\{u_1, v_2\}$ | v_1 | u_3 |
| $\{u_1, v_3\}$ | v_2 | u_3 |
| $\{u_1, v_4\}$ | v_1 | u_2 |
| $\{u_2, u_3\}$ | u_1 | v_2 |
| $\{u_2, v_1\}$ | v_2 | v_4 |
| $\{u_2, v_2\}$ | v_1 | u_1 |
| $\{u_2, v_3\}$ | v_4 | v_1 |
| $\{u_2, v_4\}$ | v_3 | v_2 |
| $\{u_3, v_1\}$ | v_2 | v_4 |
| $\{u_3, v_2\}$ | v_1 | v_3 |
| $\{u_3, v_3\}$ | v_4 | u_2 |
| $\{u_3, v_4\}$ | v_3 | v_2 |
| $\{v_1, v_2\}$ | u_1 | v_3 |
| $\{v_1, v_3\}$ | v_4 | u_1 |
| $\{v_1, v_4\}$ | v_2 | u_1 |
| $\{v_2, v_3\}$ | u_1 | u_2 |
| $\{v_2, v_4\}$ | u_1 | v_1 |
| $\{v_3, v_4\}$ | v_1 | u_1 |

Table 5.3.

Construction for $m = 5$. Let $T_E = U_5$ and T_O be the near-regular tournament in Figure 5.5. The remaining arcs in T_B are $u_1 \rightarrow v_2$, $\{v_1, v_3, v_4\} \rightarrow u_1$, $\{u_2, u_3, u_4, u_5\} \rightarrow \{v_2, v_3\}$ and $\{v_1, v_4\} \rightarrow \{u_2, u_3, u_4, u_5\}$. See Figure 5.8. Then since every vertex in $V(T_E)$ beats either one or two vertices in $V(T_O)$, the tournaments T_E , T_O , and T_B satisfy Properties 1 through 4 of Theorem 5.10. In addition, Table 5.4 verifies that T_B is well-covered and so Property 5 of Theorem 5.10 is also satisfied. Therefore, it follows by Theorem 5.10 that for every graph $G \in \wp^*(5, 4)$, there exists a near-regular tournament T with $\text{dom}(T) = G$.

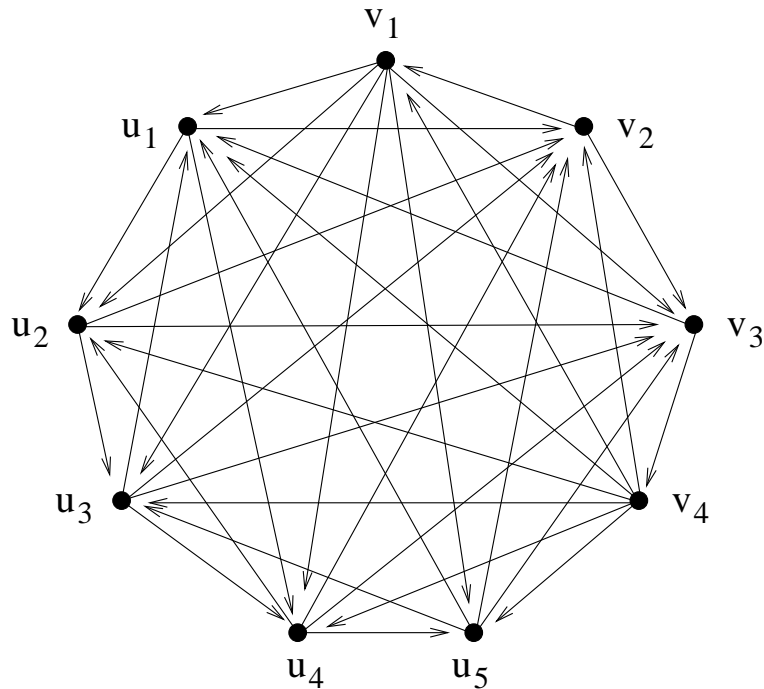


Figure 5.8. The tournament T_B for $m = 5$ and $n = 4$.

| Vertex pair: | Distinguished by: | Paired by: |
|----------------|-------------------|------------|
| $\{v_1, v_2\}$ | u_1 | v_3 |
| $\{v_1, v_3\}$ | v_4 | u_1 |
| $\{v_1, v_4\}$ | v_3 | u_1 |
| $\{v_1, u_1\}$ | v_2 | u_2 |
| $\{v_1, u_2\}$ | u_1 | u_3 |
| $\{v_1, u_3\}$ | v_2 | u_4 |
| $\{v_1, u_4\}$ | u_1 | u_5 |
| $\{v_1, u_5\}$ | v_2 | u_3 |
| $\{v_2, v_3\}$ | u_1 | u_2 |
| $\{v_2, v_4\}$ | u_1 | v_1 |
| $\{v_2, u_1\}$ | v_3 | u_3 |
| $\{v_2, u_2\}$ | u_3 | u_1 |
| $\{v_2, u_3\}$ | u_4 | v_3 |
| $\{v_2, u_4\}$ | u_2 | u_1 |
| $\{v_2, u_5\}$ | u_1 | v_3 |
| $\{v_3, v_4\}$ | v_1 | u_1 |
| $\{v_3, u_1\}$ | v_2 | u_3 |
| $\{v_3, u_2\}$ | u_1 | u_4 |
| $\{v_3, u_3\}$ | v_4 | u_2 |
| $\{v_3, u_4\}$ | u_1 | u_3 |
| $\{v_3, u_5\}$ | v_4 | u_4 |
| $\{v_4, u_1\}$ | u_3 | v_2 |
| $\{v_4, u_2\}$ | u_1 | u_3 |
| $\{v_4, u_3\}$ | u_4 | u_2 |
| $\{v_4, u_4\}$ | u_1 | u_5 |
| $\{v_4, u_5\}$ | u_4 | u_3 |

Table 5.4.

Construction for $m = 4$ or $m \geq 6$. Let T_O be the tournament in Figure 5.5 and let $T_E \in W_{1,m-1}$ such that u_1 is the transmitter in T_E . Then the remaining arcs in T_B are $u_1 \rightarrow v_1$, $\{v_2, v_3, v_4\} \rightarrow u_1$, $\{u_2, u_3, \dots, u_m\} \rightarrow \{v_2, v_3\}$ and $\{v_1, v_4\} \rightarrow \{u_2, u_3, \dots, u_m\}$. See Figure 5.9 for an example of T_B when $m = 4$. Then since every vertex in T_E beats either one or two vertices in T_O ,

the tournaments T_E , T_O , and T_B satisfy Properties 1 through 4 of Theorem 5.10. It remains to be shown that T_B is a well-covered tournament.

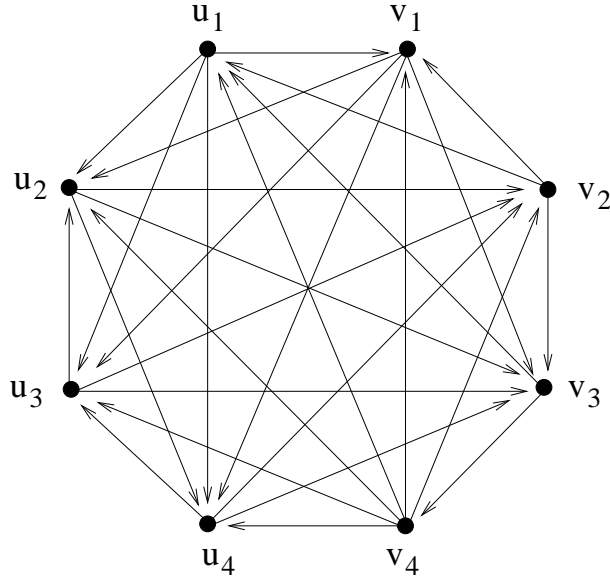


Figure 5.9. The tournament T_B for $m = 4$ and $n = 4$.

By Lemma 5.11, we conclude that T_E is well-covered. Thus, since T_E is a subtournament of T_B , every pair of vertices in $V(T_E)$ is paired and distinguished in T_B . Note that in T_O the pairs $\{v_1, v_4\}$, $\{v_2, v_3\}$, and $\{v_3, v_4\}$ are not paired and the pair $\{v_1, v_2\}$ is not distinguished. But by the way in which T_B is constructed, every pair of vertices in $V(T_O)$ is paired and distinguished in T_B . Thus, to show that T_B is well-covered, it suffices to show that v_j and u_i are paired and distinguished for any choice of $v_j \in V(T_O)$ and $u_i \in V(T_E)$.

Let $u_i \in V(T_E)$ with $i \neq 1$. Since u_i is contained in a 3-cycle with v_2 and v_1 , it follows that u_i is distinguished with both v_1 and v_2 . Also since u_i

is contained in a 3-cycle with v_3 and v_4 , we conclude that u_i is distinguished with every vertex in $V(T_O)$. Next, since every arc in $T_E - \{u_1\}$ is contained in a 3-cycle, there exist distinct vertices $u_j, u_l \in V(T_E) - \{u_1\}$, with $i \neq j, l$, such that u_i beats u_j and u_l beats u_i . Then since both v_1 and v_4 beat u_j , it follows that u_i is paired with both v_1 and v_4 . Also since u_l beats both v_2 and v_3 , we may conclude that u_i is paired with every vertex in $V(T_O)$.

Next since u_1 beats u_2 and u_2 beats both v_2 and v_3 , we can see that u_1 is distinguished with both v_2 and v_3 . Then since v_4 beats v_2 and v_2 beats u_1 , it follows that v_4 and u_1 are distinguished. But then since v_1 beats v_3 and v_3 beats u_1 , we conclude that u_1 is distinguished with every vertex in $V(T_O)$. Next, since v_1, v_4 and u_1 all beat u_2 , it follows that u_1 is paired with v_1 and v_4 . Then since both v_2 and u_1 beat v_1 , we have that u_1 is paired with v_2 . Finally, since v_1 beats both v_3 and u_1 , we may conclude that u_1 is paired with every vertex in $V(T_O)$.

Therefore, it follows that T_B is well-covered and so Property 5 of Theorem 5.10 is also satisfied. Hence, it follows by Theorem 5.10 that for every graph $G \in \wp^*(m, 4)$ with $m = 4$ or $m \geq 6$, there exists a near-regular tournament T with $\text{dom}(T) = G$. ■

For n odd, let \overline{U}_n denote the tournament obtained from U_n by reversing the arcs on the cycle x_1, x_2, \dots, x_n .

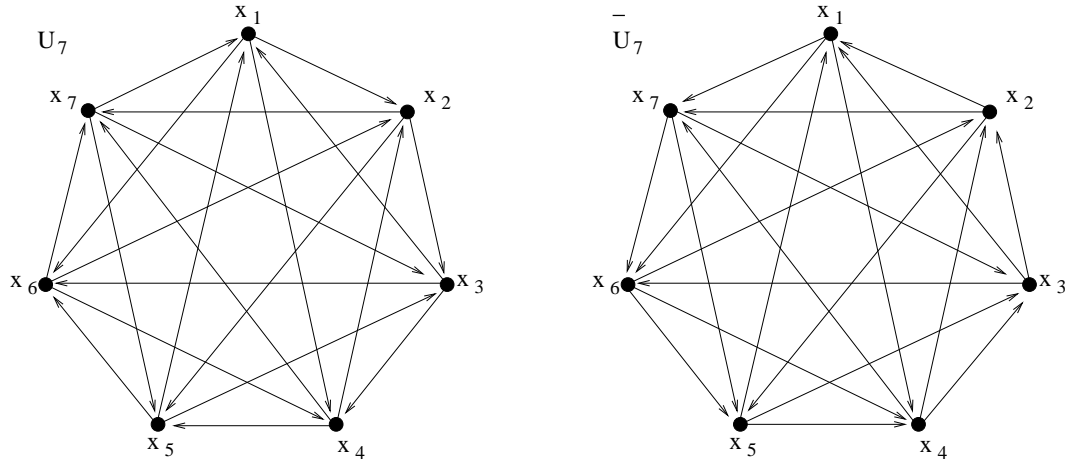


Figure 5.10. The tournaments U_7 and $\overline{U_7}$.

Lemma 5.16 (Cho, et al., [10]) *If n is odd with $n \geq 7$, then $\overline{U_n}$ is a regular tournament and $\text{dom}(\overline{U_n}) = I_n$.*

Lemma 5.17 *Let $G \in \wp^*(m, 6)$, with $m \geq 0$. Then G is the domination graph of a near-regular tournament.*

Proof: For $m \geq 0$, we will construct tournaments T_E , T_O , and T_B that satisfy Theorem 5.10. It will then follow by the same theorem that for every graph $G \in \wp^*(m, 6)$, there exists a near-regular tournament T with $\text{dom}(T) = G$.

Construction for $m = 0$. Let T_O be the near-regular tournament in Figure 5.11. Then $T_O = T_B$ satisfies Properties 1 through 4 of Theorem 5.10. In addition, Table 5.5 verifies that $T_O = T_B$ is a well-covered tournament and so Property 5 of Theorem 5.10 is also satisfied. Therefore, it follows by Theorem 5.10 that for every graph $G \in \wp^*(0, 6)$, there exists a near-regular tournament

| Vertex pair: | Distinguished by: | Paired by: |
|----------------|-------------------|------------|
| $\{v_1, v_2\}$ | v_3 | v_6 |
| $\{v_1, v_3\}$ | v_2 | v_4 |
| $\{v_1, v_4\}$ | v_2 | v_5 |
| $\{v_1, v_5\}$ | v_6 | v_4 |
| $\{v_1, v_6\}$ | v_5 | v_3 |
| $\{v_2, v_3\}$ | v_1 | v_5 |
| $\{v_2, v_4\}$ | v_1 | v_6 |
| $\{v_2, v_5\}$ | v_4 | v_1 |
| $\{v_2, v_6\}$ | v_3 | v_5 |
| $\{v_3, v_4\}$ | v_6 | v_2 |
| $\{v_3, v_5\}$ | v_2 | v_6 |
| $\{v_3, v_6\}$ | v_2 | v_5 |
| $\{v_4, v_5\}$ | v_6 | v_3 |
| $\{v_4, v_6\}$ | v_5 | v_1 |
| $\{v_5, v_6\}$ | v_4 | v_2 |

Table 5.5.

T with $dom(T) = G$.

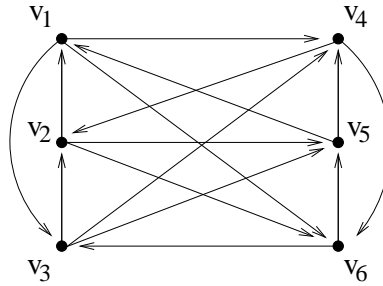


Figure 5.11. A well-covered near-regular tournament on six vertices.

Construction for $m = 1$. Let $T_B = \overline{U_7}$ with vertex set $\{x_1, x_2, \dots, x_7\}$ such that $V(T_O) = \{x_1, x_2, \dots, x_6\}$ and $V(T_E) = \{x_7\}$. See Figure 5.10. Note that T_O is a near-regular tournament and that x_7 beats half of the vertices in $V(T_O)$. Thus, the tournaments T_O , T_E , and T_B satisfy Properties 1 through 4 of Theorem 5.10. By Lemma 5.16, we conclude that $dom(\overline{U_7}) = I_7$ and so every pair of vertices in T_B is paired. In addition, since $\overline{U_7}$ is a regular tournament, every

arc in $T_B = \overline{U_7}$ is contained in a 3-cycle. Thus, every pair of vertices in T_B is distinguished and so we may conclude that T_B is well-covered. Therefore, it follows by Theorem 5.10 that for every graph $G \in \wp^*(1, 6)$, there exists a near-regular tournament T with $\text{dom}(T) = G$.

Construction for $m = 2$. Let T_O be the tournament in Figure 5.11 and $T_E = U_2$. The remaining arcs in T_B are $u_2 \rightarrow \{v_4, v_5, v_6\}$, $u_1 \rightarrow \{v_4, v_6\}$, $v_5 \rightarrow u_1$, and $\{v_1, v_2, v_3\} \rightarrow \{u_1, u_2\}$. See Figure 5.12. Then since every vertex in T_E beats either two or three vertices in T_O , the tournaments T_E , T_O , and T_B satisfy Properties 1 through 4 of Theorem 5.10. We have shown above (in the construction for $m = 0$) that T_O is a well-covered tournament. Thus, to show that T_B is well-covered, it suffices to show that every vertex in $V(T_E)$ is paired and distinguished with every vertex in $V(T_O)$. Table 5.6 verifies that this is in fact true, so Property 5 of Theorem 5.10 is also satisfied. Therefore, we may conclude by Theorem 5.10 that for every graph $G \in \wp^*(2, 6)$, there exists a near-regular tournament T with $\text{dom}(T) = G$.

Construction for $m \geq 3$. Let T_O be the tournament in Figure 5.11 and T_E be any tournament such that every arc is in a 3-cycle if $m = 3$ or 5 or any well-covered tournament, otherwise. The remaining arcs in T_B are $\{u_1, u_2, \dots, u_m\} \rightarrow \{v_4, v_5, v_6\}$ and $\{v_1, v_2, v_3\} \rightarrow \{u_1, u_2, \dots, u_m\}$. See Figures 5.13 and 5.14 for examples of T_B when $m = 3$ and $m = 4$, respectively. Therefore, since every vertex in T_E beats half of the vertices in T_O , it follows that

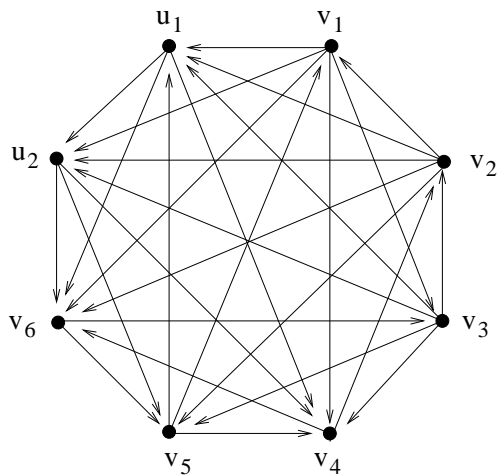


Figure 5.12. The tournament T_B for $m = 2$ and $n = 6$.

| Vertex pair: | Distinguished by: | Paired by: |
|----------------------|-------------------|------------|
| $\{u_{11}, u_{21}\}$ | v_{51} | v_{11} |
| $\{u_{11}, v_{11}\}$ | v_{31} | u_{21} |
| $\{u_{11}, v_{21}\}$ | v_{11} | u_{21} |
| $\{u_{11}, v_{31}\}$ | v_{21} | u_{21} |
| $\{u_{11}, v_{41}\}$ | v_{21} | v_{51} |
| $\{u_{11}, v_{51}\}$ | v_{11} | v_{41} |
| $\{u_{11}, v_{61}\}$ | v_{41} | v_{11} |
| $\{u_{21}, v_{11}\}$ | v_{51} | v_{41} |
| $\{u_{21}, v_{21}\}$ | v_{41} | v_{51} |
| $\{u_{21}, v_{31}\}$ | v_{61} | v_{51} |
| $\{u_{21}, v_{41}\}$ | v_{21} | v_{61} |
| $\{u_{21}, v_{51}\}$ | v_{11} | v_{41} |
| $\{u_{21}, v_{61}\}$ | v_{31} | v_{51} |

Table 5.6.

Properties 1 through 4 of Theorem 5.10 are satisfied. It remains to be shown that T_B is well-covered.

Next we will show that T_B is well-covered. If $m = 3$ or 5 , then by the choice of T_E , every pair of vertices in T_E is distinguished. Thus, since T_E is contained in T_B , every pair of vertices in T_E is distinguished in T_B . In addition, since every vertex in T_E beats v_4 , every pair of vertices in T_E is paired in T_B . On the other hand, if $m = 4$ or $m \geq 6$, then by the choice of T_E every pair of vertices in T_E is paired and distinguished in T_E . Thus, since T_E is contained in T_B , every pair of vertices in T_E is paired and distinguished in T_B . In either case, every pair of vertices in T_E is paired and distinguished in T_B . Then since T_O is well covered (see Table 5.6) and T_O is contained in T_B , every pair of vertices in T_O is paired and distinguished in T_B . Therefore, to show that T_B is well covered, it suffices to show that every pair of vertices v_j and u_i is paired and distinguished in T_B for any choice of $v_j \in V(T_O)$ and $u_i \in V(T_E)$.

Let $u_i \in V(T_E)$. Since u_i is contained in the 3-cycles $u_i \rightarrow v_4 \rightarrow v_2 \rightarrow u_i$, $u_i \rightarrow v_5 \rightarrow v_1 \rightarrow u_i$, and $u_i \rightarrow v_6 \rightarrow v_3 \rightarrow u_i$, we can see that u_i is distinguished with every vertex in $V(T_O)$. Next note that u_i beats v_4 , v_5 and v_6 . Then since v_1 beats v_4 , v_2 beats v_5 , and v_3 beats v_5 , it follows that u_i is paired with the vertices v_1 , v_2 and v_3 . Also since v_4 beats v_6 , v_6 beats v_5 , and v_5 beats v_4 , we have that u_i is paired with the vertices v_4 , v_5 and v_6 . Therefore, we may conclude that u_i is paired and distinguished with every vertex in T_O .

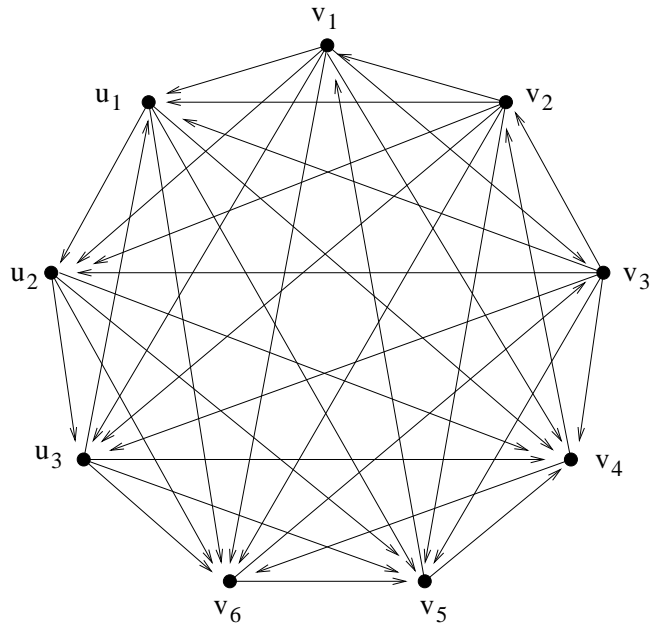


Figure 5.13. The tournament T_B for $m = 3$ and $n = 6$.

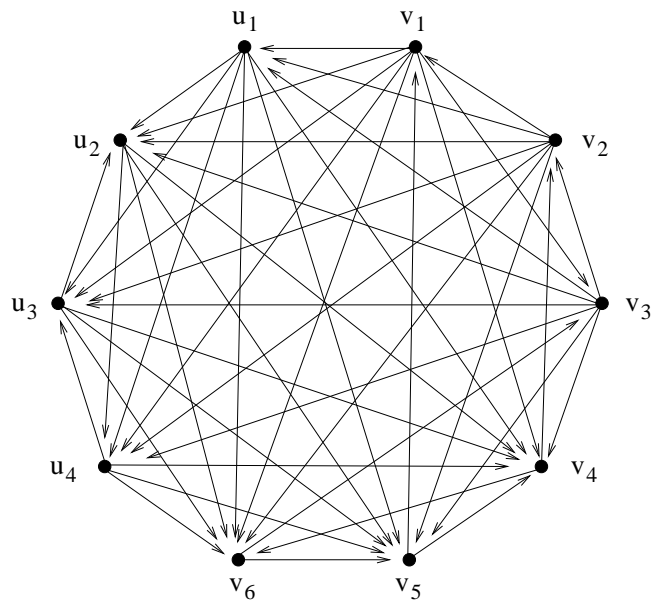


Figure 5.14. The tournament T_B for $m = 4$ and $n = 6$.

Since u_i is an arbitrary vertex in $V(T_E)$, this follows for all vertices in $V(T_E)$.

Hence, we may conclude that T_B is well-covered.

Therefore, we may conclude by Theorem 5.10 that for every graph $G \in \wp^*(m,6)$ with $m \geq 3$, there exists a near-regular tournament T with $\text{dom}(T) = G$. ■

For m even, let \widehat{U}_m denote the tournament obtained from $\overline{U_{m-1}}$ by adding a vertex x_m with arcs (x_i, x_m) , for all i odd with $1 \leq i \leq m-1$, and (x_m, x_j) for all j even with $2 \leq j \leq m-2$.

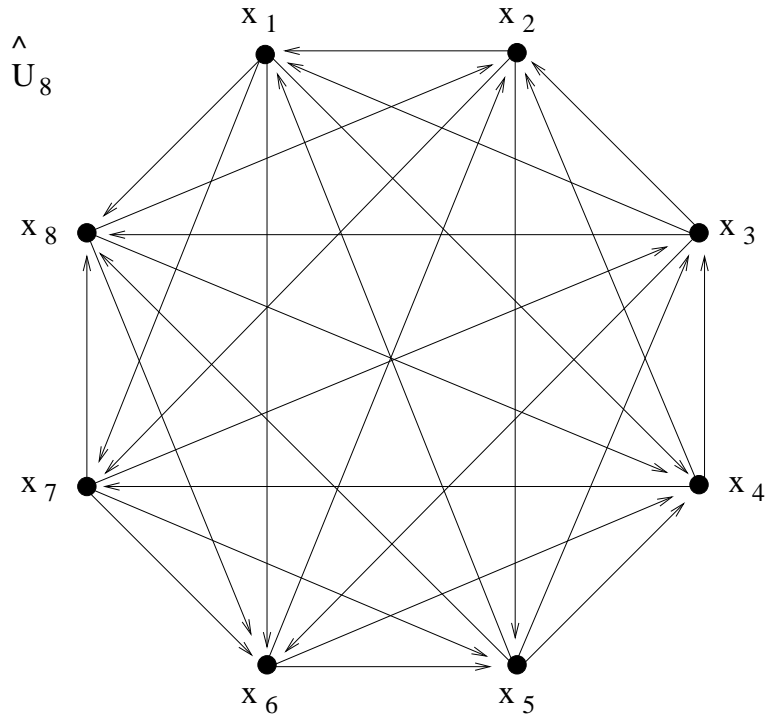


Figure 5.15. The tournament \widehat{U}_8 .

Lemma 5.18 *If m is even with $m \geq 8$, then \widehat{U}_m is a near-regular tournament*

and $\text{dom}(\widehat{U}_m) = I_m$.

Proof: By construction, \widehat{U}_m is a near-regular tournament. To show that $\text{dom}(\widehat{U}_m) = I_m$, we need to show that every pair of vertices is dominated by at least one vertex. Since $T(\{x_1, x_2, \dots, x_{m-1}\}) = \overline{U_{m-1}}$, by Lemma 5.12 we can see that $\text{dom}(T(\{x_1, x_2, \dots, x_{m-1}\})) = I_n$ and so every pair of vertices in $\{x_1, x_2, \dots, x_{m-1}\}$ is dominated by at least one vertex. Next, for i even with $2 \leq i \leq m-2$, the vertex x_{i+1} beats x_i and x_m . Thus, $[x_i, x_m] \notin \text{dom}(T)$ for $2 \leq i \leq m-2$ and i even. Finally since $x_{m-1} \rightarrow x_{m-3} \rightarrow \dots \rightarrow x_1 \rightarrow x_{m-1}$ and every vertex in $\{x_1, x_3, \dots, x_{m-1}\}$ beats x_m , then $[x_m, x_j] \notin \text{dom}(T)$, for j odd with $1 \leq j \leq m-1$. Therefore, we may conclude that $\text{dom}(\widehat{U}_m) = I_m$. ■

Lemma 5.19 *If m is even and $m \geq 8$, then \widehat{U}_m is well-covered.*

Proof: By Lemma 5.18, we conclude that $\text{dom}(\widehat{U}_m) = I_m$. Thus, every pair of vertices in \widehat{U}_m is paired. By construction, every arc in \widehat{U}_m is contained in a 3-cycle. Thus, every pair of vertices in \widehat{U}_m is distinguished. Therefore, \widehat{U}_m is well-covered. ■

Lemma 5.20 *Assume m and n are integers such that $m \geq 0$ and n is even with $n \geq 8$. Let $G \in \wp^*(m, n)$. Then G is the domination graph of a near-regular tournament.*

Proof: Assume m and n are integers such that $m \geq 0$ and n is even with $n \geq 8$. We will construct tournaments T_E , T_O , and T_B that satisfy Theorem 5.10. It

will then follow that for every graph $G \in \wp^*(m, n)$, there is a near-regular tournament T with $\text{dom}(T) = G$.

Construction for $m = 0$. For n even and $n \geq 8$, let $T_O = \widehat{U}_n$. See Figure 5.15. Since $T_O = T_B$ is a near-regular tournament, $T_O = T_B$ satisfies Properties 1 through 4 of Theorem 5.10. Also by Lemma 5.19, it follows that $T_O = T_B$ is well-covered. Thus, we may conclude by Theorem 5.10 that for every graph $G \in \wp^*(0, n)$ with n even and $n \geq 8$, there exists a near-regular tournament T with $\text{dom}(T) = G$.

Construction for $m = 1$. For n even and $n \geq 8$, let $T_O = \widehat{U}_n$. Also let $T_E = \{u_1\}$. The remaining arcs in T_B are $u_1 \rightarrow \{v_2, v_4, \dots, v_{n-2}, v_{n-1}\}$ and $\{v_1, v_3, \dots, v_{n-3}, v_n\} \rightarrow u_1$. See Figure 5.16. Then since u_1 beats half of the vertices in T_O , the tournaments T_O , T_E , and T_B satisfy Properties 1 through 4 of Theorem 5.10. To show that T_B is well-covered, it suffices to show that u_1 is paired and distinguished with every vertex in $V(T_O)$.

Since u_1 is contained in the 3-cycles $u_1 \rightarrow v_i \rightarrow v_{i-1} \rightarrow u_1$ for i even with $2 \leq i \leq n - 2$, we can see that u_1 is distinguished with every vertex in $\{v_1, v_2, \dots, v_{n-2}\}$. In addition, since u_1 is contained in a 3-cycle with v_{n-1} and v_n , it follows that u_1 is distinguished with every vertex in $V(T_O)$. Next note that for every vertex in $\{v_1, v_3, \dots, v_{n-3}, v_n\}$ there exists a vertex in $\{v_2, v_4, \dots, v_{n-2}, v_{n-1}\}$ that it beats. In addition, for every vertex in $\{v_2, v_4, \dots, v_{n-2}, v_{n-1}\}$ there exists a vertex in $\{v_1, v_3, \dots, v_{n-3}, v_n\}$ that beats it. Thus, since u_1 beats

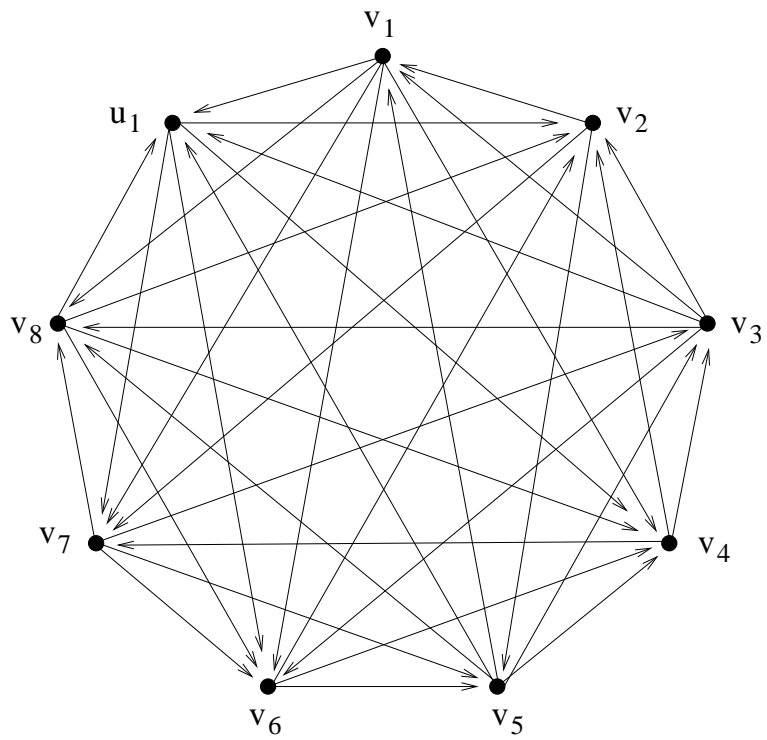


Figure 5.16. The tournament T_B for $m = 1$ and $n = 8$.

every vertex in $\{v_2, v_4, \dots, v_{n-2}, v_{n-1}\}$ and every vertex in $\{v_1, v_3, \dots, v_{n-3}, v_n\}$ beats u_1 , we can see that u_1 is paired with every vertex in T_O . Thus, it follows that T_B is well-covered. Therefore, we may conclude by Theorem 5.10 that for every graph $G \in \wp^*(1, n)$ with n even and at least eight, there exists a near-regular tournament T with $\text{dom}(T) = G$.

Construction for $m = 2$. For n even and $n \geq 8$, let $T_O = \widehat{U}_n$. Also let $T_E = U_2$. The remaining arcs in T_B are $u_1 \rightarrow \{v_2, v_4, \dots, v_{n-2}\}$, $\{v_1, v_3, \dots, v_{n-1}, v_n\} \rightarrow u_1$, $\{v_2, v_4, \dots, v_{n-2}, v_{n-1}\} \rightarrow u_2$ and $u_2 \rightarrow \{v_1, v_3, \dots, v_{n-3}, v_n\}$. See Figure 5.17 for an example of T_B when $n = 8$. Then since every vertex in T_B beats either $\frac{n}{2}$ or $\frac{n}{2} - 1$ of the vertices in T_O , the tournaments T_O , T_E , and T_B satisfy Properties 1 through 4 of Theorem 5.10. It remains to be shown that T_B is well-covered.

First note that since v_{n-1} beats both u_1 and u_2 , we can see that u_1 and u_2 are paired. Also since T_B contains the 3-cycle, $u_1 \rightarrow u_2 \rightarrow v_n \rightarrow u_1$, it follows that u_1 and u_2 are also distinguished. Next by Lemma 5.19, the tournament T_O is well-covered. Thus, since T_O is a subtournament of T_B , every pair of vertices in T_O is paired and distinguished in T_B . Therefore, to show that T_B is well-covered, it suffices to show that u_i and v_j are paired and distinguished for any choice of $u_i \in V(T_E)$ and $v_j \in V(T_O)$.

Note that u_i is contained in the following 3-cycles $v_j \rightarrow v_{j-1} \rightarrow u_1 \rightarrow v_j$, for all j even with $2 \leq j \leq n - 2$, $v_2 \rightarrow v_{n-1} \rightarrow u_1 \rightarrow v_2$,

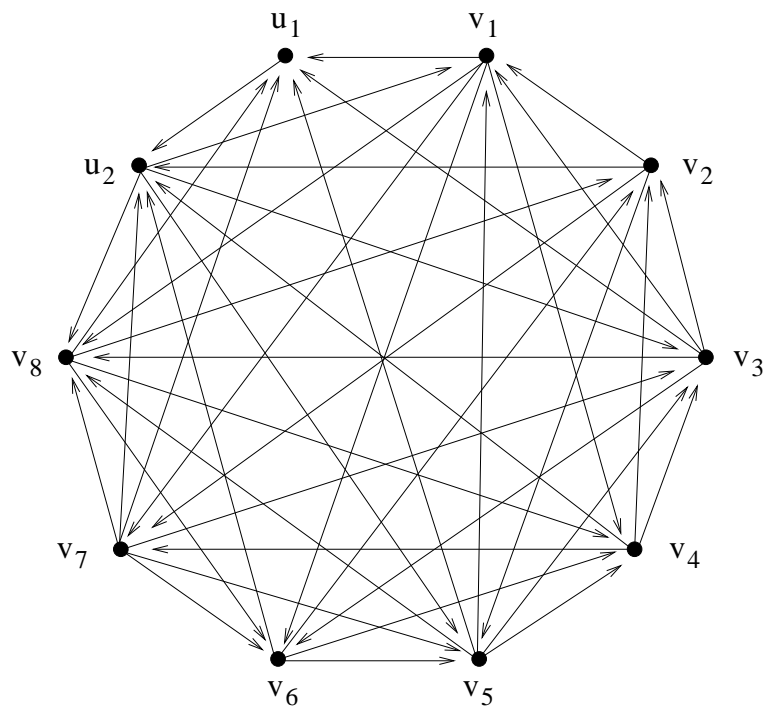


Figure 5.17. The tournament T_B for $m = 2$ and $n = 8$.

and $u_1 \rightarrow u_2 \rightarrow v_n \rightarrow u_1$. Thus, u_1 is distinguished with every vertex in $V(T_O)$. Next notice that for every vertex in $\{v_1, v_3, \dots, v_{n-1}, v_n\}$ there exists a vertex in $\{v_2, v_4, \dots, v_{n-2}\}$ that it beats. In addition, for every vertex in $\{v_2, v_4, \dots, v_{n-2}\}$, there exists a vertex in $\{v_1, v_3, \dots, v_{n-1}, v_n\}$ that beats it. Thus, since u_1 beats every vertex in $\{v_2, v_4, \dots, v_{n-2}\}$ and every vertex in $\{v_1, v_3, \dots, v_{n-1}, v_n\}$ beats u_1 , it follows that u_1 is paired with every vertex in T_O .

Finally, we show that u_2 is paired and distinguished with every vertex in $V(T_O)$. Note that the subtournament of T_B induced by the set of vertices $V(T_O) \cup \{u_2\}$ is isomorphic to the tournament constructed when $m = 1$. In addition, it was shown that the tournament is well-covered. Thus, the subtournament of T_B induced by the set of vertices $V(T_O) \cup \{u_2\}$ is well-covered and so u_2 is paired and distinguished with every vertex in T_O . Therefore, T_B is well-covered.

Hence, we may conclude by Theorem 5.10 that for every graph $G \in \wp^*(2, n)$ with n even and at least eight, there exists a near-regular tournament T with $\text{dom}(T) = G$.

Construction for $m \geq 3$. For n even and $n \geq 8$, let $T_O = \widehat{U}_n$. If $m = 3, 5$, let $T_E = U_m$. Otherwise, let T_E be a well-covered tournament. The remaining arcs in T_B are $\{u_1, u_2, \dots, u_m\} \rightarrow \{v_2, v_4, \dots, v_{n-2}, v_{n-1}\}$ and $\{v_1, v_3, \dots, v_{n-3}, v_n\} \rightarrow \{u_1, u_2, \dots, u_m\}$. See Figure 5.18 for an example of T_B when $m = 3$ and

$n = 8$. Then since every vertex in T_E beats exactly half of the vertices in T_O , the tournaments T_O , T_E , and T_B satisfy Properties 1 through 4 of Theorem 5.10. Thus, it remains to be shown that T_B is well-covered.

Next we will show that T_B is well-covered. If $m = 3$ or 5 , then by the choice of T_E , every pair of vertices in T_E is distinguished. Thus, since T_E is contained in T_B , every pair of vertices in T_E is distinguished in T_B . In addition, since every vertex in T_E beats v_2 , every pair of vertices in T_E is paired in T_B . On the other hand, if $m = 4$ or $m \geq 6$, then by the choice of T_E every pair of vertices in T_E is paired and distinguished in T_E . Thus, since T_E is contained in T_B , every pair of vertices in T_E is paired and distinguished in T_B . In either case, every pair of vertices in T_E is paired and distinguished in T_B . Then by Lemma 5.19, since T_O is well covered and T_O is contained in T_B , every pair of vertices in T_O is paired and distinguished in T_B . Therefore, to show that T_B is well covered, it suffices to show that every pair of vertices v_j and u_i is paired and distinguished in T_B for any choice of $v_j \in V(T_O)$ and $u_i \in V(T_E)$.

Let u_i be an arbitrary vertex in T_E . Note that the subtournament of T_B induced by the set of vertices $V(T_O) \cup \{u_i\}$ is isomorphic to the tournament constructed when $m = 1$. In addition, it was shown that the tournament is well-covered. Thus, the subtournament of T_B induced by the set of vertices $V(T_O) \cup \{u_i\}$ is well-covered and so u_i is paired and distinguished with every

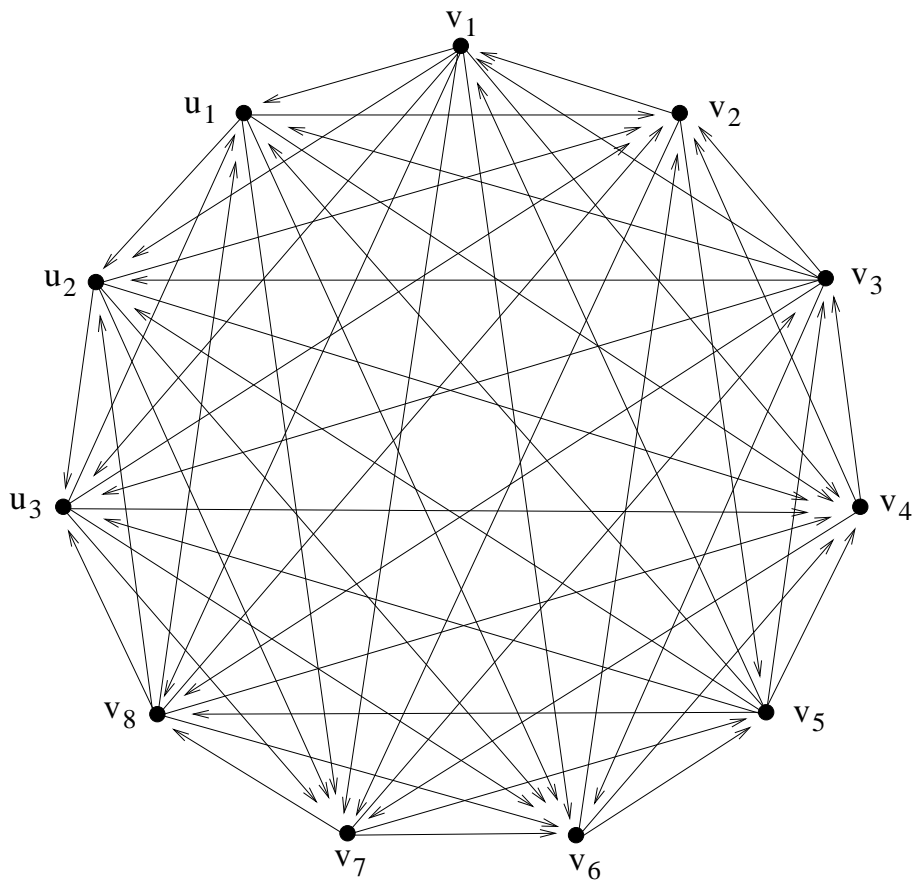


Figure 5.18. The tournament T_B for $m = 3$ and $n = 8$.

vertex in T_O . Thus, since u_i was an arbitrary vertex in T_E , it follows that v_j and u_i are paired and distinguished in T_B for any choice of $v_j \in V(T_O)$ and $u_i \in V(T_E)$. Therefore, T_B is well-covered.

Hence, we may conclude by Theorem 5.10 that for every graph $G \in \wp^*(m, n)$ with n even and at least eight and $m \geq 3$, there exists a near-regular tournament T with $\text{dom}(T) = G$. ■

Theorem 5.21 *If $G \in \wp^*(m, n)$, then G is the domination graph of a near-regular tournament if and only if n is even with $n \neq 4$ and $m + n = 4$ or $m + n \geq 6$.*

Proof: If G is the domination graph of a near-regular tournament T , then it follows from Lemma 5.6 that n must be even. In addition, we may conclude by Theorem 5.1 that $m + n = 4$ or $m + n \geq 6$. Thus, it suffices to show that G cannot be composed solely of four odd paths.

Suppose it were the case that G is the union of four nontrivial odd paths. Then it follows by Lemma 5.6 that $T(O) = T(B)$ is a near-regular tournament. In addition, we have by Theorem 5.21 that $T(B)$ must be well-covered. But there is only one near-regular tournament on four vertices and it is not well-covered. See Figure 5.5. Therefore, we may conclude that n is even with $n \neq 4$ and $m + n = 4$ or $m + n \geq 6$.

The converse follows by Theorem 5.10 and Lemmas 5.13, 5.14, 5.15, 5.17, and 5.20. ■

In this chapter we have determined all forests of nontrivial paths that are the domination graphs of near-regular tournaments. Note that the main result of this chapter, Theorem 5.21, had a close resemblance to Theorem 5.1. The only difference was that four odd paths cannot be the domination graph of a near-regular tournament. If the condition that all paths must be nontrivial is removed, it would be interesting to see how this would change Theorem 5.21. Thus, a possible way in which this research can be continued would be to remove the restriction that the paths must be nontrivial and proceed to determine all forests of paths (including trivial paths) that are the domination graphs of near-regular tournaments.

6. Odd Gapped Pendant Restricted Caterpillars

6.1 Introduction

In this and the next chapter, we continue our work on the domination graphs of near-regular tournaments. In particular, we will focus on forests of nontrivial caterpillars that are the domination graphs of near-regular tournaments. Recall Theorem 4.6, which states that if a caterpillar H is a component of the domination graph of a near-regular tournament, then H must be an odd gapped pendant restricted (OGPR) caterpillar. Therefore, the only forests of nontrivial caterpillars that can be the domination graphs of near-regular tournaments are those that are composed of OGPR caterpillars.

Chapter 5 examined a special class of forests of OGPR caterpillars, namely, forests of paths. Although we have solved the problem for all forests of paths, we have yet to determine all forests of nontrivial OGPR caterpillars that are the domination graphs of near-regular tournaments. However, in Chapter 7, we will exhibit some special classes of forests of nontrivial OGPR caterpillars that are the domination graphs of near-regular tournaments. This will be done by presenting two constructions for near-regular tournaments that have these special classes of forests of nontrivial OGPR caterpillars as their domination graphs. To facilitate the constructions in Chapter 7, in this chapter we will derive properties of the subtournaments induced by OGPR caterpillars.

We will proceed in this chapter as follows. In Section 6.2, we will introduce twelve families of OGPR caterpillars with respect to their structure in $\mathcal{D}(T)$. Then, in Section 6.3 we will determine the subtournament an OGPR caterpillar will induce given that it is a component of the domination of a near-regular tournament.

6.2 Families of Odd Gapped Pendant Restricted Caterpillars

We begin by restating the definition of an OGPR caterpillar. Let H be a caterpillar with k vertices y_1, y_2, \dots, y_k on its spine. Also let W_i represent the set of pendant vertices at y_i . Then H is an odd gapped pendant restricted caterpillar if it has all the following properties:

- $|V_i| \leq 1$ for all i with $2 \leq i \leq k - 1$.
- The set V_1 can have at most two vertices and the set V_k can have at most three vertices or vice versa.
- Consecutive vertices on the spine with degree 3 or more must be separated by an odd number of edges.

Now assume a caterpillar H is a component of the domination graph of a near-regular tournament T . Then it follows by Theorem 4.6 that H must be an odd gapped pendant restricted caterpillar. Let $[\mathcal{D}(T)]_H$ denote the subdigraph of $\mathcal{D}(T)$ on $V(H)$. Recall that the directed spine of $[\mathcal{D}(T)]_H$ is the path (possibly trivial) that results when all vertices with out-degree 0 are removed

from $[\mathcal{D}(T)]_H$. Since Theorem 2.8 provides the existence of the directed spine, we can assume without loss of generality that $[\mathcal{D}(T)]_H$ is as depicted in Figure 6.1. In Figure 6.1, the vertices x_1, x_2, \dots, x_k are the vertices on the directed spine, and the set V_i is the set of pendant vertices at x_i . Note that the spine of H consists of the vertices x_2, x_3, \dots, x_k if $V_1 = \emptyset$ and consists of the vertices x_1, x_2, \dots, x_k , otherwise.

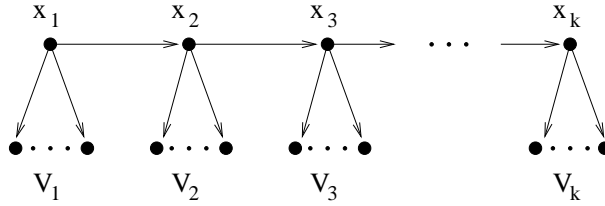


Figure 6.1. The subdigraph of $\mathcal{D}(T)$ on $V(H)$.

To prove Theorem 4.6 we used $[\mathcal{D}(T)]_H$ to show that H must be an OGPR caterpillar. In the proof of Theorem 4.6 it was shown that if $[\mathcal{D}(T)]_H$ is as depicted in Figure 6.1, then for $1 \leq i \leq k-1$, the vertex x_i can have at most one pendant neighbor, i.e., $|V_i| \leq 1$. Most importantly, it was shown that x_k is the only vertex on the spine of H that could have up to three pendant neighbors.

Now in $[\mathcal{D}(T)]_H$ the vertex x_k is always a vertex at one end of the spine of H , and so V_k must have at least one vertex. See Figure 6.1. Therefore, we can always extend the path in $[\mathcal{D}(T)]_H$ that composes the directed spine by adding a vertex in V_k to it, in order to get a longer path in $[\mathcal{D}(T)]_H$. We will define the **extended directed spine** of $[\mathcal{D}(T)]_H$ to be this longer path, i.e.,

it is the directed spine of $[\mathcal{D}(T)]_H$ with a vertex in V_k appended to it. If there is more than one vertex in V_k , then pick a vertex arbitrarily. Note that the extended directed spine is a path with greatest length contained in $[\mathcal{D}(T)]_H$. We proceed by defining the following subsets of vertices on the extended directed spine of $[\mathcal{D}(T)]_H$.

Assume that an OGPR caterpillar H is a component of the domination graph of a near-regular tournament. In addition, assume that $[\mathcal{D}(T)]_H$ is as depicted in Figure 6.1. Starting from the beginning of the extended directed spine (i.e., starting from the vertex that has in-degree 0 in $[\mathcal{D}(T)]_H$), let A denote the set of vertices on the extended directed spine preceding the first vertex on the extended directed spine that has out-degree 2 in $[\mathcal{D}(T)]_H$. Let B denote the set of vertices on the extended directed spine beginning at the first vertex on the extended directed spine that has out-degree 2 in $[\mathcal{D}(T)]_H$ and terminating at the last vertex on the extended directed spine that has out-degree exactly 2 in $[\mathcal{D}(T)]_H$. Let C denote the set of vertices on the extended spine, beginning at the first vertex on the extended directed spine immediately following the final vertex in B and terminating at the last non-end vertex on the extended directed spine, with the property that the last vertex in C is an even distance from the last vertex in B . Thus, C will always have an even number of vertices. Let D denote the single vertex on the extended directed spine immediately following the final vertex in C .

The following observations should be made about the sets defined. First, the set D will always be nonempty and so the vertex in D will correspond to either a vertex that is pendant to the right end of the spine of H or to the right end of the spine of H . In addition, if the vertex in D corresponds to the right end of the spine of H , then it can only have one or three pendant neighbors. One should also observe that the sets A , B and C may be empty. Figure 6.2 illustrates these sets.

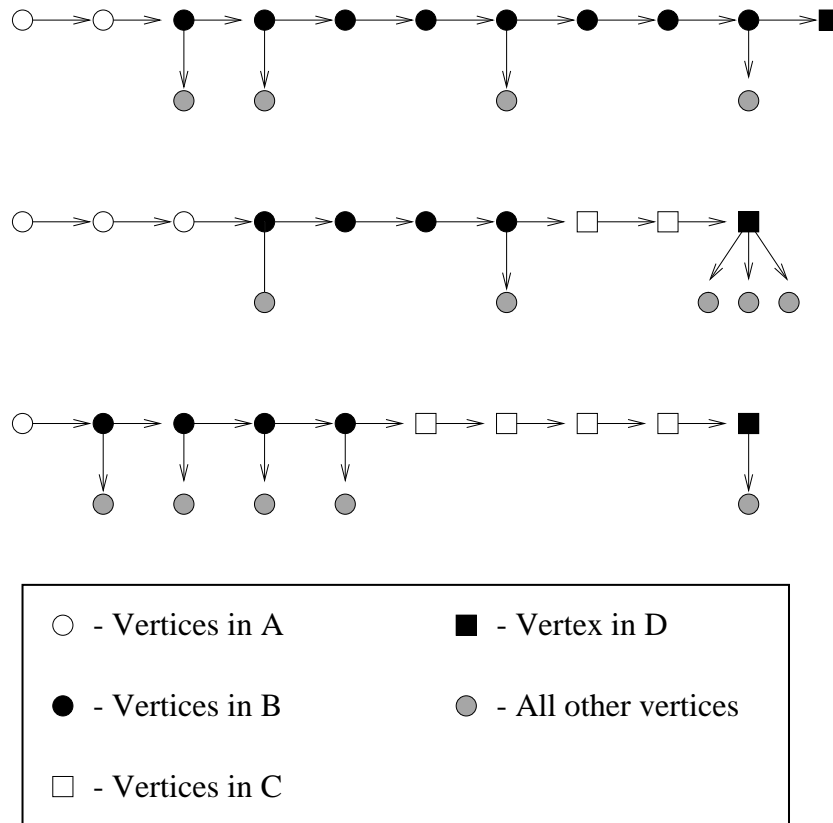


Figure 6.2. Three OGPR caterpillars.

Using the sets A , B , C , and D , define the following families of OGPR caterpillars as follows with respect to the orientation they have in $[\mathcal{D}(T)]_H$. Let H be an OGPR caterpillar. Assume H is a component of the domination graph of a near-regular tournament T and $[\mathcal{D}(T)]_H$ is as depicted in Figure 6.1. Then H is an element of

- EE2** if $|A|$ is even, $|B|$ is even, and the vertex in D has out-degree 1 in $[\mathcal{D}(T)]_H$ (i.e., the vertex in D has exactly one pendant neighbor in H),
- EO2** if $|A|$ is even, $|B|$ is odd, and the vertex in D has out-degree 1 in $[\mathcal{D}(T)]_H$ (i.e., the vertex in D has exactly one pendant neighbor in H),
- OE2** if $|A|$ is odd, $|B|$ is even, and the vertex in D has out-degree 1 in $[\mathcal{D}(T)]_H$ (i.e., the vertex in D has exactly one pendant neighbor in H),
- OO2** if $|A|$ is odd, $|B|$ is odd, and the vertex in D has out-degree 1 in $[\mathcal{D}(T)]_H$ (i.e., the vertex in D has exactly one pendant neighbor in H),
- EE1** if $|A|$ is even, $|B|$ is even, and the vertex in D has out-degree 0 in $[\mathcal{D}(T)]_H$ (i.e., the vertex in D is a pendant vertex in H),
- EO1** if $|A|$ is even, $|B|$ is odd, and the vertex in D has out-degree 0 in $[\mathcal{D}(T)]_H$ (i.e., the vertex in D is a pendant vertex in H),
- OE1** if $|A|$ is odd, $|B|$ is even, and the vertex in D has out-degree 0 in $[\mathcal{D}(T)]_H$ (i.e., the vertex in D is a pendant vertex in H),
- OO1** if $|A|$ is odd, $|B|$ is odd, and the vertex in D has out-degree 0 in $[\mathcal{D}(T)]_H$ (i.e., the vertex in D is a pendant vertex in H),

EE3 if $|A|$ is even, $|B|$ is even, and the vertex in D has out-degree 3 in $[\mathcal{D}(T)]_H$ (i.e., the vertex in D has exactly three pendant neighbors in H),

EO3 if $|A|$ is even, $|B|$ is odd, and the vertex in D has out-degree 3 in $[\mathcal{D}(T)]_H$ (i.e., the vertex in D has exactly three pendant neighbors in H),

OE3 if $|A|$ is odd, $|B|$ is even, and the vertex in D has out-degree 3 in $[\mathcal{D}(T)]_H$ (i.e., the vertex in D has exactly three pendant neighbors in H), and

OO3 if $|A|$ is odd, $|B|$ is odd, and the vertex in D has out-degree 3 in $[\mathcal{D}(T)]_H$ (i.e., the vertex in D has exactly three pendant neighbors in H).

Observe that every OGPR caterpillar belongs to at least one of these families, i.e., every OGPR caterpillar must have an orientation for $[\mathcal{D}(T)]_H$ that places it into one of the families above. For example, consider the OGPR caterpillar in Figure 6.3 and assume it is a component of the domination graph of a near-regular tournament T . Then, by Lemma 2.2 there are three possibilities for the orientation of $[\mathcal{D}(T)]_H$. These are shown in Figure 6.4. If the caterpillar in Figure 6.3 has Orientation 1 or 2 of Figure 6.4, then it belongs to the family $OE1$. Similarly, if the caterpillar in Figure 6.3 has Orientation 3, then it belongs to the family $EE1$. It is important to point out that if we specify that the OGPR caterpillar of Figure 6.3 belongs to $OE1$, then $[\mathcal{D}(T)]_H$ must have

either Orientation 1 or 2.

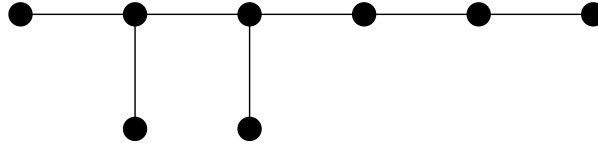


Figure 6.3. An OGPR caterpillar.

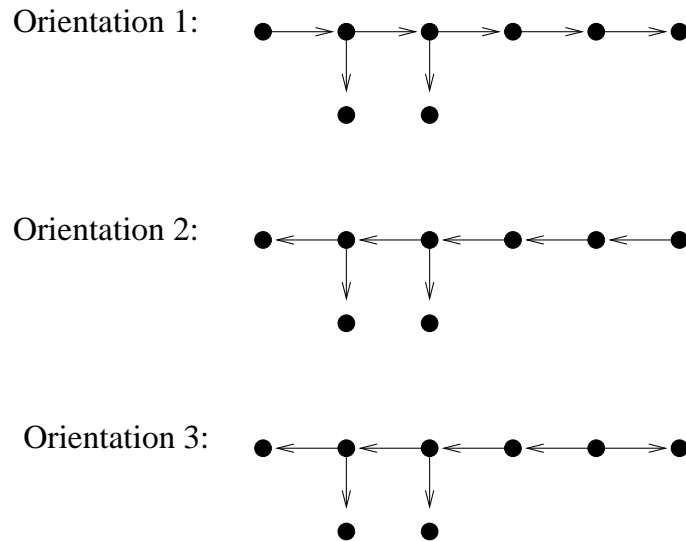


Figure 6.4. Three possible orientations for $[\mathcal{D}(T)]_H$.

Now note that the families of OGPR caterpillars presented above are defined using $[\mathcal{D}(T)]_H$, i.e., using the orientation they have in $\mathcal{D}(T)$ if they are components of the domination graph of a near-regular tournament T . Thus, it should be understood that if an OGPR caterpillar H is a component of $\text{dom}(T)$ for a near-regular tournament T and belongs to one of the families above, then $[\mathcal{D}(T)]_H$ is fixed and is prescribed by the definition of the family to which it

belongs. Therefore, since $[\mathcal{D}(T)]_H$ is fixed for a caterpillar that belongs to one of the twelve families above, then by properly 2-coloring the vertices of H Lemma 2.9 determines the remaining arcs in the subtournament induced by the vertices of H except for those arcs within the triple end (should H have one).

6.3 Subtournaments Induced by OGPR Caterpillars

This section examines the subtournaments an OGPR caterpillar will induce given that it is a component of the domination graph of a near-regular tournament T and that it belongs to one of the twelve families presented in Section 6.2. We begin by showing that there is only one possibility for the subtournament induced by an OGPR caterpillar H once a structure for $[\mathcal{D}(T)]_H$ is fixed. If this is the case, we will say that $T(H)$ is uniquely determined.

Lemma 6.1 *Let H be an OGPR caterpillar. Assume H is a component of the domination graph of a near-regular tournament T . If $[\mathcal{D}(T)]_H$ is fixed, then the subtournament of T on $V(H)$ is uniquely determined.*

Proof: Assume $[\mathcal{D}(T)]_H$ is fixed, i.e., the orientation of $[\mathcal{D}(T)]_H$ in $\mathcal{D}(T)$ is fixed. Without loss of generality assume that $[\mathcal{D}(T)]_H$ is as depicted in Figure 6.1. We will proceed by showing that $T(H)$ is uniquely determined, i.e., there is only one possibility for $T(H)$.

Since H is an OGPR caterpillar and $[\mathcal{D}(T)]_H$ is as in Figure 6.1, we

have from the proof of Theorem 4.6 that the sets V_i have at most one vertex, for $1 \leq i \leq k - 1$. In addition, we have that V_k is the only pendant set that can have up to three vertices. Now by properly 2-coloring H , Lemma 2.9 determines the remaining arcs in $T(H)$ except for those in each pendant set V_i . In particular, Lemma 2.9 gives us that vertices beat vertices to the right with the same color and vertices to the left with the opposite color. Thus, since $|V_i| \leq 1$ for $1 \leq i \leq k - 1$, it follows that all the arcs in $T(H)$ are forced except for those within V_k . Thus, to show that $T(H)$ is uniquely determined, it suffices to show that there is only one possibility for the arcs in V_k .

There is nothing to prove if V_k has at most two vertices, since there is a unique tournament on one or two vertices. So assume $|V_k| = 3$ and let $V_k = \{u_1, u_2, u_3\}$. We claim that the arcs within V_k must induce a tournament isomorphic to U_3 (a 3-cycle). Suppose this were not the case. Then there must be a vertex in V_k that beats the other two. Assume u_1 is such a vertex. Then since x_k beats every vertex in V_k , we have that $|O(x_k) \cap O(u_1)| \geq 2$. But this contradicts Lemma 4.2 since $[x_k, u_1] \in \text{dom}(T)$. Thus, it follows that the arcs within V_k must induce a subtournament isomorphic to U_3 . ■

Let H be an OGPR caterpillar. Assume H is a component of the domination graph of a near-regular tournament T and that it belongs to one of the twelve families of OGPR caterpillars defined in Section 6.2. Since H

belongs to one of families of Section 6.2, $[\mathcal{D}(T)]_H$ is fixed. Therefore, it follows by Theorem 6.1 that the subtournament $T(H)$ is uniquely determined. So from this point on it should be understood that once it is stated that H belongs to one of the twelve families presented in Section 6.2, then the subtournament $T(H)$ is forced, i.e., it is uniquely determined. With this in mind we proceed by examining some properties of the subtournaments OGPR caterpillars induce given that they are components of the domination graph of a near-regular tournament, and that they belong to one of the twelve families presented in Section 6.2.

Theorems 6.4 through 6.13, which we present next, will be essential in Chapter 7 for our constructions of near-regular tournaments that have forests of nontrivial OGPR caterpillars as their domination graphs. We begin by presenting the following two useful results.

Lemma 6.2 (Fisher, et al., [19]) *Let T be a tournament and $\{v_1, v_2, \dots, v_k\}$ be a subset of $V(T)$. Assume that the set $\{v_1, v_2, \dots, v_k\}$ induces a directed path in $\mathcal{D}(T)$. Then subtournament $T(\{v_1, v_2, \dots, v_k\})$ is isomorphic to U_k .*

Proof: Assume that the set $\{v_1, v_2, \dots, v_k\}$ induces a directed path in $\mathcal{D}(T)$ such that v_i beats v_{i+1} for $1 \leq i \leq k - 1$. Then for $1 \leq i \leq k - 2$, since v_i beats v_{i+1} and $[v_{i+1}, v_{i+2}] \in \text{dom}(T)$, it follows that v_{i+2} beats v_i . Then for $1 \leq i \leq k - 3$, since v_{i+3} beats v_{i+1} and $[v_i, v_{i+1}] \in \text{dom}(T)$, it follows that v_i

beats v_{i+3} . Thus, for $1 \leq i \leq k-4$, since v_i beats v_{i+3} and $[v_{i+3}, v_{i+4}] \in \text{dom}(T)$, we have that v_{i+4} beats v_i . Continuing in this fashion, we conclude that v_i beats v_j if $i - j$ is either odd and negative or even and positive. Thus by definition the subtournament $T(\{v_1, v_2, \dots, v_k\}) \cong U_k$. ■

Lemma 6.3 *Let H be an OGPR caterpillar and T a near-regular tournament. Assume H is a component of $\text{dom}(T)$. Then the subtournament of T induced by the vertices of H that are in B together with vertices pendant to the vertices in B is a near-regular tournament.*

Proof: First assume that $[\mathcal{D}(T)]_H$ is as depicted in Figure 6.1. Let $\{y_1, y_2, \dots, y_k\}$ denote the set of vertices in B such that (y_i, y_{i+1}) is an arc in $[\mathcal{D}(T)]_H$ for $1 \leq i \leq k-1$. Also let $\{v_1, v_2, \dots, v_l\}$ denote the set of vertices pendant to the vertices in B . Note that in $[\mathcal{D}(T)]_H$, all arcs are oriented from vertices in B to vertices in $\{v_1, v_2, \dots, v_l\}$.

Recall that B is the set of vertices on the extended directed spine of $[\mathcal{D}(T)]_H$ that begins at the first vertex with out-degree 2 in $[\mathcal{D}(T)]_H$ and ends at the last vertex on the extended directed spine that has out-degree exactly 2 in $[\mathcal{D}(T)]_H$. Thus, one should observe that v_1 is the pendant neighbor of y_1 in H and v_l is the pendant neighbor of y_k in H . In addition, since H is an OGPR, every vertex in B can have at most one pendant neighbor and vertices in B with pendant neighbors must be separated by an odd number of edges.

Let T' denote the subtournament of T induced by the vertices in $B \cup$

$\{v_1, v_2, \dots, v_l\}$. Now by properly 2-coloring the vertices in $B \cup \{v_1, v_2, \dots, v_l\}$ with red and blue, Lemma 2.9 determines the remaining arcs in T' that are not in $[\mathcal{D}(T)]_H$. In particular, a vertex beats vertices of the same color to the right and beats vertices of the opposite color to the left. Now by Lemma 2.6, the subgraph of H induced by this set of vertices will be a subgraph of $\text{dom}(T')$. We will proceed by showing that T' is a near-regular tournament. Without loss of generality assume that y_1 is colored red.

Case 1. Assume that k is odd. Note that y_k is assigned the color red. Since y_1 is assigned the color red, it beats all vertices to the right that are assigned the color blue. Also since y_k is assigned the color red, it beats all vertices to the left that are assigned the color red. Thus $[y_1, y_k] \in \text{dom}(T')$. Notice that the subgraph of H induced by the vertices in $B \cup \{v_1, v_2, \dots, v_l\}$ together with the edge $[y_1, y_k]$ results in an odd-spiked cycle. Thus, since all the edges in the subgraph of H induced by the vertices in $B \cup \{v_1, v_2, \dots, v_l\}$ are in $\text{dom}(T')$, it follows by Theorem 1.1 that $\text{dom}(T')$ is an odd-spiked cycle, namely, the odd-spiked cycle that results by adding the edge $[y_1, y_k]$ to the subgraph of H induced by the vertices in $B \cup \{v_1, v_2, \dots, v_l\}$. Notice that since H is an OGPR caterpillar, vertices in B with pendant neighbors must be separated by an odd number of edges. Thus, since y_1 and y_k are separated by a single edge in $\text{dom}(T')$, it follows that $\text{dom}(T')$ is an odd gapped single spiked odd cycle. Now recall that in the proof of Lemma 4.11 it was shown

that the tournament that has an odd gapped single spiked odd cycle as its domination graph must be a near-regular tournament. Thus, it follows that T' is a near-regular tournament.

Case 2. Assume that k is even. Note that v_l is assigned the color red. As in Case 1, y_1 beats all vertices to the right that are assigned the color blue. Also v_l beats all vertices to the left that are assigned the color red. Thus, $[y_1, v_l] \in \text{dom}(T')$. Notice that the subgraph of H induced by the vertices in $B \cup \{v_1, v_2, \dots, v_l\}$ together with the edge $[y_1, v_l]$ results in an odd-spiked cycle. Therefore, since all the edges in the subgraph of H induced by the vertices in $B \cup \{v_1, v_2, \dots, v_l\}$ are in $\text{dom}(T')$, it follows by Theorem 1.1 that $\text{dom}(T')$ is an odd-spiked cycle, namely, the odd-spiked cycle that results by adding the edge $[y_1, v_l]$ to the subgraph of H induced by the vertices in $B \cup \{v_1, v_2, \dots, v_l\}$. We claim that $\text{dom}(T')$ is an odd gapped single spiked odd cycle. Let y_r be the second to last vertex in B that has a pendant neighbor. Now since H is an OGPR caterpillar, we have that the vertices in B with pendant neighbors must be separated by an odd number of edges. Thus, y_r and y_k are separated by an odd number of edges. Now notice that in $\text{dom}(T')$, the edges $[y_r, y_{r+1}]$, $[y_{r+1}, y_{r+2}], \dots, [y_{k-1}, y_k], [y_k, v_l]$, and $[v_l, y_1]$ are all on the cycle of $\text{dom}(T')$ and the only vertices on this portion of the cycle that have pendant neighbors are the vertices y_r and y_1 . In addition, it is not hard to see that there are an odd number of edges separating y_r and y_1 on this portion of the cycle. Thus, we can

see that $\text{dom}(T')$ is an odd gapped single spiked odd cycle. Therefore, it follows from the proof of Lemma 4.11 that T' must be a near-regular tournament. ■

Recall that if caterpillar H is a component of the domination graph of a tournament T , then the base vertex of H is the vertex that has in-degree 0 in $[\mathcal{D}(T)]_H$.

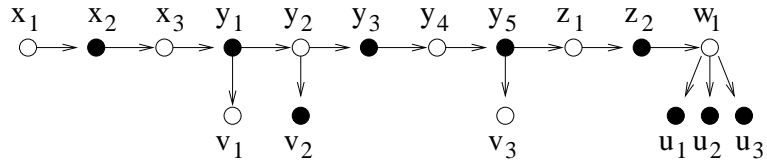


Figure 6.5. A properly colored caterpillar in $OO3$. The non-filled vertices represent red vertices and the filled vertices represent blue vertices.

Theorem 6.4 *Let an OGPR caterpillar H on n vertices be a component of the domination graph of a near-regular tournament T . Assume $H \in OO3$ and is properly 2-colored with red and blue such that the base vertex of H is colored red. Then all red vertices of H have out-degree either $\frac{n+3}{2}$ or $\frac{n+1}{2}$ in $T(H)$, and all blue vertices of H have out-degree either $\frac{n-3}{2}$ or $\frac{n-5}{2}$ in $T(H)$. Furthermore, under this coloring H has $\frac{n-1}{2}$ vertices colored red and $\frac{n+1}{2}$ colored blue.*

Proof: Assume $H \in OO3$ such that $|V(H)| = n$. Let $A = \{x_1, x_2, \dots, x_{s_1}\}$, $B = \{y_1, y_2, \dots, y_{s_2}\}$, $C = \{z_1, z_2, \dots, z_{s_3}\}$ and $D = \{w_1\}$ such that in $[\mathcal{D}(T)]_H$

- (x_i, x_{i+1}) is an arc for $1 \leq i \leq s_1 - 1$,
- (y_i, y_{i+1}) is an arc for $1 \leq i \leq s_2 - 1$, and
- (z_i, z_{i+1}) is an arc for $1 \leq i \leq s_3 - 1$.

Also let $\{v_1, v_2, \dots, v_{s_4}\}$ denote the set of vertices pendant to the vertices in B and $\{u_1, u_2, u_3\}$ denote the set of vertices pendant to w_1 . Thus we have that $s_1 + s_2 + s_3 + s_4 + 4 = n$.

Now properly 2-color H with red and blue such that the base vertex of H is assigned the color red. Note that the base vertex is x_1 . Under this coloring, A has $\frac{s_1 + 1}{2}$ vertices colored red and $\frac{s_1 - 1}{2}$ vertices colored blue, C has half of its vertices colored red and half colored blue, w_1 is colored red and u_1, u_2 , and u_3 are all colored blue. In addition, since consecutive vertices in B that have pendant neighbors are separated by an odd number of edges, it easy to see that $V(B) \cup \{v_1, v_2, \dots, v_{s_4}\}$ has half of its vertices assigned the color blue and half assigned the color red. See Figure 6.5. Thus, H has $\frac{s_1 + s_2 + s_3 + s_4 + 1}{2} + 1 = \frac{n - 1}{2}$ red vertices and $\frac{s_1 + s_2 + s_3 + s_4 - 1}{2} + 3 = \frac{n + 1}{2}$ blue vertices. See Figure 6.5. We proceed by showing that red vertices of H have out-degree either $\frac{n + 3}{2}$ or $\frac{n + 1}{2}$ in $T(H)$ and blue vertices of H have out-degree either $\frac{n - 3}{2}$ or $\frac{n - 5}{2}$ in $T(H)$.

Let $x_i, x_j \in V(A)$ with i even and j odd. Since i is even and j odd, x_i is assigned the color blue and x_j is assigned the color red. Now since the vertices of A induce a directed path in $\mathcal{D}(T)$, we may conclude by Lemma 6.1

that $T(A) = U_{s_1}$. Thus, since s_1 is odd, both x_i and x_j beat $\frac{s_1 - 1}{2}$ vertices in $V(A)$. In addition, it follows by Lemma 2.9 that x_i beats all red vertices in $V(H) \setminus V(A)$ and x_j beats all blue vertices in $V(H) \setminus V(A)$. Thus, we have that x_i has out-degree $\frac{s_1 + s_2 + s_3 + s_4 - 1}{2} + 1 = \frac{n - 3}{2}$ in $T(H)$ and x_j has out-degree $\frac{s_1 + s_2 + s_3 + s_4 - 1}{2} + 3 = \frac{n + 1}{2}$ in $T(H)$.

Let $x \in V(B) \cup \{v_1, v_2, \dots, v_{s_4}\}$. By Lemma 6.2, the subtournament induced by the vertices in $V(B) \cup \{v_1, v_2, \dots, v_{s_4}\}$ is near-regular. Thus, every vertex in $V(B) \cup \{v_1, v_2, \dots, v_{s_4}\}$ beats either $\frac{s_2 + s_4}{2}$ or $\frac{s_2 + s_4}{2} - 1$ vertices of $V(B) \cup \{v_1, v_2, \dots, v_{s_4}\}$. We proceed by showing that x has the appropriate out-degree in $T(H)$ depending on its color. First suppose that x is assigned the color red. Then we have by Lemma 2.9 that x beats red vertices in $V(A)$ and beats blue vertices in $V(C) \cup \{w_1\} \cup \{u_1, u_2, u_3\}$. Thus, x has out-degree $\frac{s_1 + s_2 + s_3 + s_4 + 1}{2} + 3 = \frac{n + 3}{2}$ in $T(H)$ if it beats $\frac{s_2 + s_4}{2}$ vertices of $V(B) \cup \{v_1, v_2, \dots, v_{s_4}\}$ and has out-degree $\frac{s_1 + s_2 + s_3 + s_4 + 1}{2} + 2 = \frac{n + 1}{2}$ if it beats $\frac{s_2 + s_4}{2} - 1$ vertices of $V(B) \cup \{v_1, v_2, \dots, v_{s_4}\}$. Next suppose that x is assigned the color blue. Then it follows by Lemma 2.9 that x beats blue vertices in $V(A)$ and beats red vertices in $V(C) \cup \{w_1\} \cup \{u_1, u_2, u_3\}$. Therefore, x has out-degree $\frac{s_1 + s_2 + s_3 + s_4 - 1}{2} + 1 = \frac{n - 3}{2}$ if it beats $\frac{s_2 + s_4}{2}$ vertices of $V(B) \cup \{v_1, v_2, \dots, v_{s_4}\}$ and has out-degree $\frac{s_1 + s_2 + s_3 + s_4 - 1}{2} = \frac{n - 5}{2}$ if it beats $\frac{s_2 + s_4}{2} - 1$ vertices of $V(B) \cup \{v_1, v_2, \dots, v_{s_4}\}$.

Next let $z_i, z_j \in V(C)$ with i even and j odd. Since $(s_1 + s_2)$ is even,

z_i is assigned the color blue and z_j is assigned the color red. Now since the vertices in C induce a directed path in $\mathcal{D}(T)$, we conclude by Lemma 6.1 that $T(C) = U_{s_3}$. Then since s_3 is even, z_i beats $\frac{s_3}{2} - 1$ vertices of $V(C)$ and z_j beats $\frac{s_3}{2}$ vertices of $V(C)$. But then it follows by Lemma 2.9 that z_i beats all blue vertices of $V(A) \cup V(B) \cup \{v_1, v_2, \dots, v_{s_4}\}$ and also beats w_1 since w_1 is assigned the color red and lies to the right of z_i . Also, by Lemma 2.9, z_j beats all red vertices of $V(A) \cup V(B) \cup \{v_1, v_2, \dots, v_{s_4}\}$ and also beats u_1, u_2 and u_3 since they are all colored blue and lie to the right of z_j . Therefore, z_i has out-degree $\frac{s_1 + s_2 + s_3 + s_4 - 1}{2} = \frac{n - 5}{2}$ in $T(H)$, and z_j has out-degree $\frac{s_1 + s_2 + s_3 + s_4 + 1}{2} + 3 = \frac{n + 3}{2}$ in $T(H)$.

Finally we show that w_1 has out-degree $\frac{n + 3}{2}$ and u_1, u_2 and u_3 all have out-degree $\frac{n - 3}{2}$. Observe that w_1 is colored red and u_1, u_2 and u_3 are all colored blue. Then by Lemma 2.9 we conclude that w_1 beats all other red vertices of $V(H)$ and also beats u_1, u_2 and u_3 . Also u_i beats all blue vertices in $V(H) \setminus \{u_1, u_2, u_3\}$. But from the proof of Theorem 6.1, we have that the subtournament on $\{u_1, u_2, u_3\}$ must be U_3 , and so u_i also beats exactly one vertex in $\{u_1, u_2, u_3\}$. Therefore, w_1 has out-degree $\frac{s_1 + s_2 + s_3 + s_4 + 1}{2} + 3 = \frac{n + 3}{2}$, and u_i has out-degree $\frac{s_1 + s_2 + s_3 + s_4 - 1}{2} + 1 = \frac{n - 3}{2}$ for $i = 1, 2, 3$.

Therefore, we may conclude that all red vertices of H have out-degree either $\frac{n + 3}{2}$ or $\frac{n + 1}{2}$ in $T(H)$, and all blue vertices of H have out-degree either $\frac{n - 3}{2}$ or $\frac{n - 5}{2}$ in $T(H)$. Hence, the result follows. ■

Now by very similar proofs to the proof of Theorem 6.4, we obtain the following results.

Theorem 6.5 *Let an OGPR caterpillar H on n vertices be a component of the domination graph of a near-regular tournament T . Assume $H \in EE2 \cup EO2$ and is properly 2-colored with red and blue such that the base vertex of H is colored red. Then all red vertices in H have out-degree either $\frac{n}{2}$ or $\frac{n}{2} - 1$ in $T(H)$, and all blue vertices have out-degree either $\frac{n}{2}$ or $\frac{n}{2} - 1$ in $T(H)$. Furthermore, under this coloring H has $\frac{n}{2}$ vertices colored red and $\frac{n}{2}$ colored blue. See Figure 6.6.*

Theorem 6.6 *Let an OGPR caterpillar H on n vertices be a component of the domination graph of a near-regular tournament T . Assume $H \in OE2 \cup OO2$ and is properly 2-colored with red and blue such that the base vertex of H is colored red. Then all red vertices of H have out-degree either $\frac{n+1}{2}$ or $\frac{n-1}{2}$ in $T(H)$, and all blue of H have out-degree either $\frac{n-1}{2}$ or $\frac{n-3}{2}$ in $T(H)$. Furthermore, under this coloring H has $\frac{n+1}{2}$ vertices colored red and $\frac{n-1}{2}$ colored blue. See Figure 6.7.*

Theorem 6.7 *Let an OGPR caterpillar H on n vertices be a component of the domination graph of a near-regular tournament T . Assume $H \in EE1$ and is properly 2-colored with red and blue such that the base vertex of H is colored red. Then all blue vertices of H have out-degree either $\frac{n+1}{2}$ or $\frac{n-1}{2}$*

in $T(H)$, and all red vertices of H have out-degree either $\frac{n-1}{2}$ or $\frac{n-3}{2}$ in $T(H)$. Furthermore, under this coloring H has $\frac{n+1}{2}$ vertices colored red and $\frac{n-1}{2}$ colored blue. See Figure 6.8.

Theorem 6.8 *Let an OGPR caterpillar H on n vertices be a component of the domination graph of a near-regular tournament T . Assume $H \in EO1$ and is properly 2-colored with red and blue such that the base vertex of H is colored red. Then all blue vertices of H have out-degree either $\frac{n+1}{2}$ or $\frac{n-1}{2}$ in $T(H)$, and all red vertices of H have out-degree either $\frac{n-1}{2}$ or $\frac{n-3}{2}$ in $T(H)$. Furthermore, under this coloring H has $\frac{n+1}{2}$ vertices colored red and $\frac{n-1}{2}$ colored blue. See Figure 6.9.*

Theorem 6.9 *Let an OGPR caterpillar H on n vertices be a component of the domination graph of a near-regular tournament T . Assume $H \in OE1$ and is properly 2-colored with red and blue such that the base vertex of H is colored red. Then all red vertices of H have out-degree either either $\frac{n+2}{2}$ or $\frac{n}{2}$ in $T(H)$, and all blue vertices have out-degree either $\frac{n-2}{2}$ or $\frac{n-4}{2}$ in $T(H)$. Furthermore, under this coloring H has $\frac{n}{2}$ vertices colored red and $\frac{n}{2}$ colored blue. See Figure 6.10.*

Theorem 6.10 *Let an OGPR caterpillar H on n vertices be a component of the domination graph of a near-regular tournament T . Assume $H \in OO1$ and is properly 2-colored with red and blue such that the base vertex of H is*

colored red. Then all red vertices of H have out-degree either either $\frac{n}{2}$ or $\frac{n-2}{2}$ in $T(H)$, and all blue vertices have out-degree either $\frac{n}{2}$ or $\frac{n-2}{2}$ in $T(H)$. Furthermore, under this coloring H has $\frac{n+2}{2}$ vertices colored red and $\frac{n-2}{2}$ colored blue. See Figure 6.11.

Theorem 6.11 *Let an OGPR caterpillar H on n vertices be a component of the domination graph of a near-regular tournament T . Assume $H \in EE3$ and is properly 2-colored with red and blue such that the base vertex of H is colored red. Then all red vertices of H have out-degree either either $\frac{n+2}{2}$ or $\frac{n}{2}$ in $T(H)$, and all blue vertices have out-degree either $\frac{n-2}{2}$ or $\frac{n-4}{2}$ in $T(H)$. Furthermore, under this coloring H has $\frac{n-2}{2}$ vertices colored red and $\frac{n+2}{2}$ colored blue. See Figure 6.12.*

Theorem 6.12 *Let an OGPR caterpillar H on n vertices be a component of the domination graph of a near-regular tournament T . Assume $H \in EO3$ and is properly 2-colored with red and blue such that the base vertex of H is colored red. Then all red vertices of H have out-degree either either $\frac{n-2}{2}$ or $\frac{n-4}{2}$ in $T(H)$, and all blue vertices have out-degree either $\frac{n+2}{2}$ or $\frac{n}{2}$ in $T(H)$. Furthermore, under this coloring H has $\frac{n+2}{2}$ vertices colored red and $\frac{n-2}{2}$ colored blue. See Figure 6.13.*

Theorem 6.13 *Let an OGPR caterpillar H on n vertices be a component of the domination graph of a near-regular tournament T . Assume $H \in OE3$ and*

is properly 2-colored with red and blue such that the base vertex of H is colored red. Then all red vertices of H have out-degree either $\frac{n-3}{2}$ or $\frac{n-1}{2}$ in $T(H)$, and all blue vertices have out-degree either $\frac{n+1}{2}$ or $\frac{n-1}{2}$ in $T(H)$. Furthermore, under this coloring H has $\frac{n+3}{2}$ vertices colored red and $\frac{n-3}{2}$ colored blue. See Figure 6.14.

In this chapter we have defined twelve families of OGPR caterpillars, with respect to their structure in $\mathcal{D}(T)$ for a near-regular tournament T . We have also shown that the subtournament induced by an OGPR caterpillar is uniquely determined once we have specified the family it belongs to. But most importantly we have derived a sequence of results (Theorems 6.4, 6.5, 6.6, 6.7, 6.8, 6.9, 6.10, 6.11, 6.12, and 6.13) that give information regarding the out-degrees of the vertices in the subtournament induced by an OGPR caterpillar. In Chapter 7, Theorems 6.4 through 6.13 will be an essential tool in deriving two constructions for near-regular tournaments that have forests of nontrivial OGPR caterpillars as their domination graph.

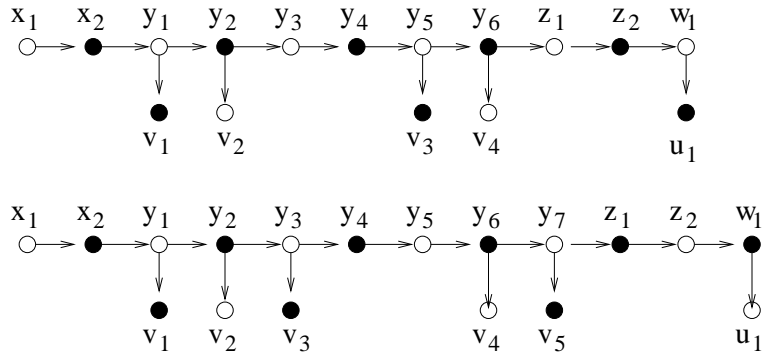


Figure 6.6. Two properly colored caterpillars in $EE2 \cup EO2$. The nonfilled vertices represent red vertices and the filled vertices represent blue vertices.

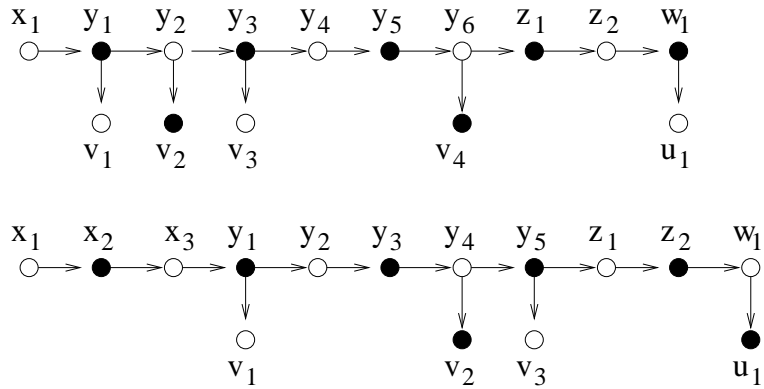


Figure 6.7. Two properly colored caterpillars in $OE2 \cup OO2$. The nonfilled vertices represent red vertices and the filled vertices represent blue vertices.

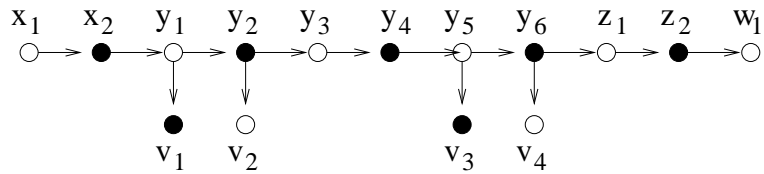


Figure 6.8. A properly colored caterpillar in $EE1$. The nonfilled vertices represent red vertices and the filled vertices represent blue vertices.

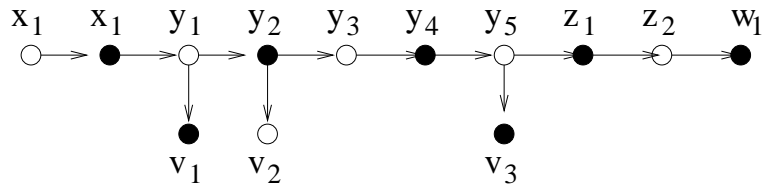


Figure 6.9. A properly colored caterpillar in $EO1$. The non-filled vertices represent red vertices and the filled vertices represent blue vertices.

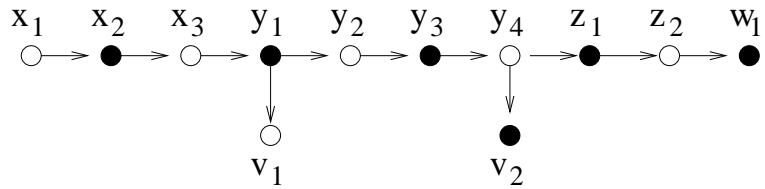


Figure 6.10. A properly colored caterpillar in $OE1$. The non-filled vertices represent red vertices and the filled vertices represent blue vertices.

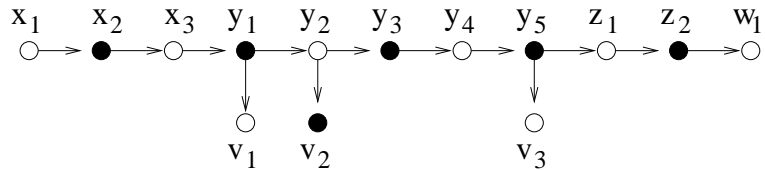


Figure 6.11. A properly colored caterpillar in $OE1$. The non-filled vertices represent red vertices and the filled vertices represent blue vertices.

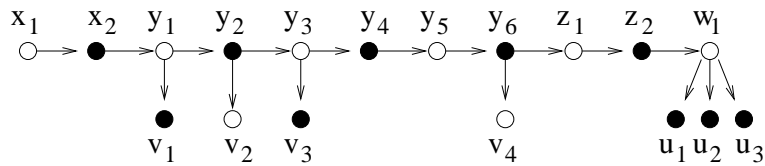


Figure 6.12. A properly colored caterpillar in $OE1$. The non-filled vertices represent red vertices and the filled vertices represent blue vertices.

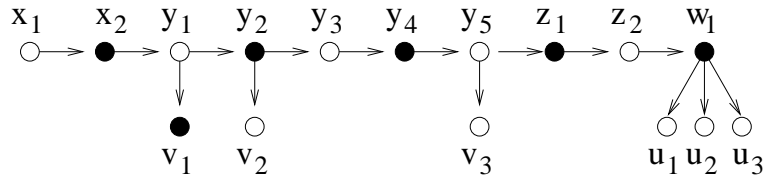


Figure 6.13. A properly colored caterpillar in $OE1$. The non-filled vertices represent red vertices and the filled vertices represent blue vertices.

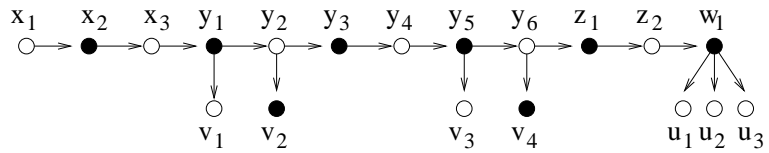


Figure 6.14. A properly colored caterpillar in $OE1$. The non-filled vertices represent red vertices and the filled vertices represent blue vertices.

7. Near-Regular Forest Tournaments

7.1 Introduction

As mentioned in Chapter 6, we have not yet been able to determine all forests of nontrivial OGPR caterpillars that are the domination graphs of near-regular tournaments. But we have been able to determine special forests of OGPR caterpillars that are the domination graphs of near-regular tournaments. We will be presenting these in this chapter. This will be done by exhibiting two constructions for near-regular tournaments that have these special forests of nontrivial OGPR caterpillars as their domination graphs. The results derived in Chapter 6, namely Theorems 6.4 through 6.13, will be essential for the two constructions presented in this setting.

7.2 Two Constructions

In this section, we will exhibit special forests of nontrivial OGPR caterpillars that are the domination graphs of near-regular tournaments. In particular, these forests will consist of OGPR caterpillars in the families $EE2$, $EO2$, $OE1$, $OE2$, $OO2$, $EE1$, and $OO3$. Refer to Chapter 6 for the definition of these families of OGPR caterpillars.

Let $\mathcal{C}(m, n, r)$ denote the set of all graphs that are the union of m caterpillars in $EE2 \cup EO2 \cup OE1$, n nontrivial caterpillars in $OE2 \cup OO2 \cup EE1$,

and r caterpillars in $OO3$. Note that every caterpillar in $EE2 \cup EO2 \cup OE1$ has an even number of vertices and every caterpillar in $OE2 \cup OO2 \cup EE1 \cup OO3$ has an odd number of vertices. We shall refer to those caterpillars in $EE2 \cup EO2 \cup OE1$ as even components, those caterpillars in $OE2 \cup OO2 \cup EE1$ as odd components and those caterpillars in $OO3$ as odd components with a triple end.

Assume $G \in \mathcal{C}(m, n, r)$ is the domination graph of a tournament T . Let $G = E_1 \cup E_2 \cup \dots \cup E_m \cup O_1 \cup O_2 \cup \dots \cup O_n \cup Q_1 \cup Q_2 \cup \dots \cup Q_r$, where E_i is the i th even component, O_j is the j th odd component and Q_l is the l th odd component with a triple end. For all components $R_i \in G$, let the base vertex of R_i be the vertex in $V(R_i)$ that has in-degree 0 in $[\mathcal{D}(T)]_{R_i}$. Let u_i , v_j , and w_l denote the base vertices of E_i , O_j and Q_l , respectively. Also let m_i , n_j and r_l denote the number of vertices in E_i , O_j and Q_l , respectively. It will be useful to define the following subtournaments of T :

$T(B)$ with vertices $\{u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_r\}$,

$T(E)$ with vertices $\{u_1, u_2, \dots, u_m\}$,

$T(O)$ with vertices $\{v_1, v_2, \dots, v_n\}$, and

$T(O3)$ with vertices $\{w_1, w_2, \dots, w_r\}$.

Now assume that $G \in \mathcal{C}(m, n, r)$ is the union of $p = (m + n + r)$ components R_1, R_2, \dots, R_p . Construct a tournament T on $V(G)$ in the following manner. First, note that each component belongs to one of the following

families of OGPR caterpillars: $EE2$, $EO2$, $OE2$, $OO2$, $EE1$, $OE1$, or $OO3$. Thus, by Theorem 6.1 we have that $T(R_i)$ is uniquely determined. So let $T(R_i)$ be as discussed in Chapter 6. Next, properly 2-color each component with red and blue such that its corresponding base vertex is colored red. Note that under this proper coloring, the subtournaments $T(R_i)$ have the properties covered in Theorems 6.4 through 6.13. Then given a structure for $T(E)$, $T(O)$, $T(O3)$, and $T(B)$, Lemma 4.5 determines the remaining arcs in T . We will say that T is a **forest tournament** if it is constructed this way. Furthermore, if T is a near-regular tournament, then we will say that T is a **near-regular forest tournament**. Whether or not $\text{dom}(T) = G$ and T is near-regular will depend on the structure of $T(E)$, $T(O)$, $T(O3)$, and $T(B)$ and the composition of G . The next two results provide sufficient conditions for the structure of tournaments $T(E)$, $T(O)$, $T(O3)$, and $T(B)$ and the composition of G in order to guarantee a near-regular forest tournament.

Lemma 7.1 *Let $G \in \mathcal{C}(m, n, 0)$. Assume T is a forest tournament for G such that the following conditions are satisfied:*

- (1) n is even,
- (2) v_j beats $\frac{n}{2}$ vertices of $T(O)$ in T if $O_j \in EE1$ and beats $\frac{n}{2} - 1$ vertices of $T(O)$ in T if $O_j \in OE2 \cup OO2$, and
- (3) u_i beats $\frac{n}{2}$ vertices of $T(O)$ in T if $E_i \in EE2 \cup EO2$ and beats $\frac{n}{2} - 1$

vertices of $T(O)$ in T if $E_i \in OE1$.

Then T is a near-regular tournament.

Proof: Assume T is a forest tournament for G such that the above conditions are satisfied. Since n is even, T will have an even number of odd components. Thus, it follows that T has an even number of vertices. Now properly 2-color each component of G using red and blue such that the base vertex of each component is assigned the color red. By Theorems 6.6 and 6.7, under this coloring every odd component O_j has $\frac{n_j + 1}{2}$ red vertices and $\frac{n_j - 1}{2}$ blue vertices. In addition, by Theorems 6.5 and 6.9, every even component E_i has $\frac{m_i}{2}$ red vertices and $\frac{m_i}{2}$ blue vertices.

Next let R_i be an arbitrary component in G and x_i its corresponding base vertex. Also let R_j be a component different from R_i with corresponding base vertex x_j . Suppose x_i beats x_j in T . Since x_i and x_j are both colored red, we conclude by Lemma 4.5 that vertices in R_i beat vertices of the same color in R_j . But note that if R_j is an even component, then half of its vertices are colored red and half are colored blue. In addition, if R_j is an odd component, then $\frac{|V(R_j)| + 1}{2}$ of its vertices are colored red, and $\frac{|V(R_j)| - 1}{2}$ of its vertices are colored blue. Thus,

- all vertices in R_i beat $\frac{|V(R_j)|}{2}$ vertices of R_j in T if R_j is an even component and
- red and blue vertices in R_i beat $\frac{|V(R_j)| + 1}{2}$ and $\frac{|V(R_j)| - 1}{2}$ vertices

of R_j in T , respectively, if R_j is an odd component.

By a similar argument, if x_j beats x_i in T , then

- all vertices in R_i beat $\frac{|V(R_j)|}{2}$ vertices of R_j in T if R_j is an even component and
- red and blue vertices in R_i beat $\frac{|V(R_j)| - 1}{2}$ and $\frac{|V(R_j)| + 1}{2}$ vertices of R_j in T , respectively, if R_j is an odd component.

Using these facts we proceed by showing that every vertex in T has degree either $\frac{|V(R_j)|}{2}$ or $\frac{|V(R_j)|}{2} - 1$.

We begin by showing that the vertices of every odd component have the appropriate out-degree in T . Let O_j be an arbitrary odd component. Assume $O_j \in EE1$ and let $x \in V(O_j)$. First assume that x is colored red. By Theorem 6.7, we have that x beats either $\frac{n_j - 1}{2}$ or $\frac{n_j - 3}{2}$ vertices of O_j in T . If x beats $\frac{n_j - 1}{2}$ vertices of O_j in T , then since v_j beats $\frac{n}{2}$ vertices of $T(O)$ in T ,

$$\begin{aligned}
d_T^+(x) &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i: (v_j, v_i) \in A(T)} \frac{n_i + 1}{2} + \sum_{i: (v_i, v_j) \in A(T)} \frac{n_i - 1}{2} \\
&\quad + \frac{n_j - 1}{2} \\
&= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \binom{n}{2} \frac{1}{2} - \left(\frac{n}{2} - 1\right) \frac{1}{2} - \frac{1}{2}
\end{aligned}$$

$$= \frac{|V(T)|}{2}.$$

If x beats $\frac{n_j - 3}{2}$ vertices of O_j in T , then again since v_j beats $\frac{n}{2}$ vertices of $T(O)$ in T ,

$$\begin{aligned} d_T^+(x) &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i: (v_j, v_i) \in A(T)} \frac{n_i + 1}{2} + \sum_{i: (v_i, v_j) \in A(T)} \frac{n_i - 1}{2} \\ &\quad + \frac{n_j - 3}{2} \\ &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \binom{n}{2} \frac{1}{2} - \binom{n}{2} \frac{1}{2} - \frac{3}{2} \\ &= \frac{|V(T)|}{2} - 1. \end{aligned}$$

Next assume that x is assigned the color blue. By Theorem 6.7, it follows that x beats either $\frac{n_j + 1}{2}$ or $\frac{n_j - 1}{2}$ vertices of O_j in T . If x beats $\frac{n_j + 1}{2}$ vertices of O_j in T , then since v_j beats $\frac{n}{2}$ vertices of $T(O)$ in T ,

$$\begin{aligned} d_T^+(x) &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i: (v_j, v_i) \in A(T)} \frac{n_i - 1}{2} + \sum_{i: (v_i, v_j) \in A(T)} \frac{n_i + 1}{2} \\ &\quad + \frac{n_j + 1}{2} \\ &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} - \binom{n}{2} \frac{1}{2} + \binom{n}{2} \frac{1}{2} + \frac{1}{2} \end{aligned}$$

$$= \frac{|V(T)|}{2}.$$

If x beats $\frac{n_j - 1}{2}$ vertices of O_j in T , then since v_j beats $\frac{n}{2}$ vertices of $T(O)$ in T ,

$$\begin{aligned} d_T^+(x) &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i: (v_j, v_i) \in A(T)} \frac{n_i - 1}{2} + \sum_{i: (v_i, v_j) \in A(T)} \frac{n_i + 1}{2} \\ &\quad + \frac{n_j - 1}{2} \\ &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} - \binom{n}{2} \frac{1}{2} + \left(\frac{n}{2} - 1\right) \frac{1}{2} - \frac{1}{2} \\ &= \frac{|V(T)|}{2} - 1. \end{aligned}$$

Next assume $O_j \in OE2 \cup OO2$ and let $x \in V(O_j)$. First assume that x is colored red. By Theorem 6.6, we conclude that x beats either $\frac{n_j + 1}{2}$ or $\frac{n_j - 1}{2}$ vertices of O_j in T . If x beats $\frac{n_j + 1}{2}$ vertices of O_j in T , then since v_j beats $\frac{n}{2} - 1$ vertices of $T(O)$ in T ,

$$\begin{aligned} d_T^+(x) &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i: (v_j, v_i) \in A(T)} \frac{n_i + 1}{2} + \sum_{i: (v_i, v_j) \in A(T)} \frac{n_i - 1}{2} \\ &\quad + \frac{n_j + 1}{2} \\ &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \left(\frac{n}{2} - 1\right) \frac{1}{2} - \binom{n}{2} \frac{1}{2} + \frac{1}{2} \end{aligned}$$

$$= \frac{|V(T)|}{2}.$$

If x beats $\frac{n_j - 1}{2}$ vertices of O_j in T , then again since v_j beats $\frac{n}{2} - 1$ vertices of $T(O)$ in T ,

$$\begin{aligned} d_T^+(x) &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i: (v_j, v_i) \in A(T)} \frac{n_i + 1}{2} + \sum_{i: (v_i, v_j) \in A(T)} \frac{n_i - 1}{2} \\ &\quad + \frac{n_j - 1}{2} \\ &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \left(\frac{n}{2} - 1\right) \frac{1}{2} - \left(\frac{n}{2}\right) \frac{1}{2} - \frac{1}{2} \\ &= \frac{|V(T)|}{2} - 1. \end{aligned}$$

Next assume that x is assigned the color blue. It follows by Theorem 6.6 that x beats either $\frac{n_j - 1}{2}$ or $\frac{n_j - 3}{2}$ vertices of O_j in T . If x beats $\frac{n_j - 1}{2}$ vertices of O_j in T , then since v_j beats $\frac{n}{2} - 1$ vertices of $T(O)$ in T ,

$$\begin{aligned} d_T^+(x) &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i: (v_j, v_i) \in A(T)} \frac{n_i - 1}{2} + \sum_{i: (v_i, v_j) \in A(T)} \frac{n_i + 1}{2} \\ &\quad + \frac{n_j - 1}{2} \\ &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} - \left(\frac{n}{2} - 1\right) \frac{1}{2} + \left(\frac{n}{2}\right) \frac{1}{2} - \frac{1}{2} \end{aligned}$$

$$= \frac{|V(T)|}{2}.$$

If x beats $\frac{n_j - 3}{2}$ vertices of O_j in T , then since v_j beats $\frac{n}{2} - 1$ vertices of $T(O)$ in T ,

$$\begin{aligned} d_T^+(x) &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i: (v_j, v_i) \in A(T)} \frac{n_i - 1}{2} + \sum_{i: (v_i, v_j) \in A(T)} \frac{n_i + 1}{2} \\ &\quad + \frac{n_j - 3}{2} \\ &= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} - \left(\frac{n}{2} - 1\right) \frac{1}{2} + \left(\frac{n}{2}\right) \frac{1}{2} - \frac{3}{2} \\ &= \frac{|V(T)|}{2} - 1. \end{aligned}$$

Thus, it follows that every vertex in $V(O_j)$ has out-degree either $\frac{|V(T)|}{2}$ or $\frac{|V(T)|}{2} - 1$. Since O_j was an arbitrary odd component, we may conclude that for all odd components O_j , every vertex in $V(O_j)$ has out-degree either $\frac{|V(T)|}{2}$ or $\frac{|V(T)|}{2} - 1$ in T .

Next we show that the vertices of every even component have the appropriate out-degree in T . Let E_j is an arbitrary even component. Assume $E_j \in EE2 \cup EO2$ and let $x \in V(E_j)$. First assume that x is colored red. By Theorem 6.5, it follows that x beats either $\frac{m_j}{2}$ or $\frac{m_j}{2} - 1$ vertices of E_j in T .

If x beats $\frac{m_j}{2}$ vertices of E_j in T , then since u_j beats $\frac{n}{2}$ vertices of $T(O)$ in T ,

$$\begin{aligned}
d_T^+(x) &= \sum_{i=1, i \neq j}^m \frac{m_i}{2} + \sum_{i: (u_j, v_i) \in A(T)} \frac{n_i + 1}{2} + \sum_{i: (v_i, u_j) \in A(T)} \frac{n_i - 1}{2} \\
&\quad + \frac{m_j}{2} \\
&= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \binom{n}{2} \frac{1}{2} - \binom{n}{2} \frac{1}{2} \\
&= \frac{|V(T)|}{2}.
\end{aligned}$$

If x beats $\frac{m_j}{2} - 1$ vertices of E_j in T , then again since u_j beats $\frac{n}{2}$ vertices of $T(O)$ in T ,

$$\begin{aligned}
d_T^+(x) &= \sum_{i=1, i \neq j}^m \frac{m_i}{2} + \sum_{i: (u_j, v_i) \in A(T)} \frac{n_i + 1}{2} + \sum_{i: (v_i, u_j) \in A(T)} \frac{n_i - 1}{2} \\
&\quad + \frac{m_j}{2} - 1 \\
&= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \binom{n}{2} \frac{1}{2} - \binom{n}{2} \frac{1}{2} - 1 \\
&= \frac{|V(T)|}{2} - 1.
\end{aligned}$$

Next assume that x is assigned the color blue. Then we conclude by Theorem 6.5 that x beats either $\frac{m_j}{2}$ or $\frac{m_j}{2} - 1$ vertices of E_j in T . If x beats

$\frac{m_j}{2}$ vertices of E_j in T , then since u_j beats $\frac{n}{2}$ vertices of $T(O)$ in T ,

$$\begin{aligned}
d_T^+(x) &= \sum_{i=1, i \neq j}^m \frac{m_i}{2} + \sum_{i: (u_j, v_i) \in A(T)} \frac{n_i - 1}{2} + \sum_{i: (v_i, u_j) \in A(T)} \frac{n_i + 1}{2} \\
&\quad + \frac{m_j}{2} \\
&= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} - \binom{n}{2} \frac{1}{2} + \binom{n}{2} \frac{1}{2} \\
&= \frac{|V(T)|}{2}.
\end{aligned}$$

If x beats $\frac{m_j}{2} - 1$ vertices of E_j in T , then since u_j beats $\frac{n}{2}$ vertices of $T(O)$ in T ,

$$\begin{aligned}
d_T^+(x) &= \sum_{i=1, i \neq j}^m \frac{m_i}{2} + \sum_{i: (u_j, v_i) \in A(T)} \frac{n_i - 1}{2} + \sum_{i: (v_i, u_j) \in A(T)} \frac{n_i + 1}{2} \\
&\quad + \frac{m_j}{2} - 1 \\
&= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} - \binom{n}{2} \frac{1}{2} + \binom{n}{2} \frac{1}{2} - 1 \\
&= \frac{|V(T)|}{2} - 1.
\end{aligned}$$

Next assume $E_j \in OE1$ and let $x \in V(E_j)$. First assume that x is colored red. By Theorem 6.9, we have that x beats either $\frac{m_j + 2}{2}$ or $\frac{m_j}{2}$

vertices of E_j in T . If x beats $\frac{m_j + 2}{2}$ vertices of E_j in T , then since u_j beats $\frac{n}{2} - 1$ vertices of $T(O)$ in T ,

$$\begin{aligned}
d_T^+(x) &= \sum_{i=1, i \neq j}^m \frac{m_i}{2} + \sum_{i: (u_j, v_i) \in A(T)} \frac{n_i + 1}{2} + \sum_{i: (v_i, u_j) \in A(T)} \frac{n_i - 1}{2} \\
&\quad + \frac{m_j + 2}{2} \\
&= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \left(\frac{n}{2} - 1\right) \frac{1}{2} - \left(\frac{n}{2} + 1\right) \frac{1}{2} + 1 \\
&= \frac{|V(T)|}{2}.
\end{aligned}$$

If x beats $\frac{m_j}{2}$ vertices of E_j in T , then again since u_j beats $\frac{n}{2} - 1$ vertices of $T(O)$ in T ,

$$\begin{aligned}
d_T^+(x) &= \sum_{i=1, i \neq j}^m \frac{m_i}{2} + \sum_{i: (u_j, v_i) \in A(T)} \frac{n_i + 1}{2} + \sum_{i: (v_i, u_j) \in A(T)} \frac{n_i - 1}{2} \\
&\quad + \frac{m_j}{2} \\
&= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \left(\frac{n}{2} - 1\right) \frac{1}{2} - \left(\frac{n}{2} + 1\right) \frac{1}{2} \\
&= \frac{|V(T)|}{2} - 1.
\end{aligned}$$

Next assume that x is assigned the color blue. Then it follows by

Theorem 6.9 that x beats either $\frac{m_j - 2}{2}$ or $\frac{m_j - 4}{2}$ vertices of E_j in T . If x beats $\frac{m_j - 2}{2}$ vertices of E_j in T , then since u_j beats $\frac{n}{2} - 1$ vertices of $T(O)$ in T ,

$$\begin{aligned}
d_T^+(x) &= \sum_{i=1, i \neq j}^m \frac{m_i}{2} + \sum_{i: (u_j, v_i) \in A(T)} \frac{n_i - 1}{2} + \sum_{i: (v_i, u_j) \in A(T)} \frac{n_i + 1}{2} \\
&\quad + \frac{m_j - 2}{2} \\
&= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} - \left(\frac{n}{2} - 1\right) \frac{1}{2} + \left(\frac{n}{2} + 1\right) \frac{1}{2} - 1 \\
&= \frac{|V(T)|}{2}.
\end{aligned}$$

If x beats $\frac{m_j - 4}{2}$ vertices of E_j in T , then since u_j beats $\frac{n}{2} - 1$ vertices of $T(O)$ in T we have that

$$\begin{aligned}
d_T^+(x) &= \sum_{i=1, i \neq j}^m \frac{m_i}{2} + \sum_{i: (u_j, v_i) \in A(T)} \frac{n_i - 1}{2} + \sum_{i: (v_i, u_j) \in A(T)} \frac{n_i + 1}{2} \\
&\quad + \frac{m_j - 4}{2} \\
&= \sum_{i=1}^m \frac{m_i}{2} + \sum_{i=1}^n \frac{n_i}{2} - \left(\frac{n}{2} - 1\right) \frac{1}{2} + \left(\frac{n}{2} + 1\right) \frac{1}{2} - 2 \\
&= \frac{|V(T)|}{2} - 1.
\end{aligned}$$

Thus, it follows that every vertex in $V(E_j)$ has out-degree either $\frac{|V(T)|}{2}$ or $\frac{|V(T)|}{2} - 1$. Since E_j was an arbitrary even component, we may conclude that for all even components E_j , every vertex in $V(E_j)$ has out-degree either $\frac{|V(T)|}{2}$ or $\frac{|V(T)|}{2} - 1$ in T .

Therefore, since we have shown that every vertex in $V(T)$ has out-degree either $\frac{|V(T)|}{2}$ or $\frac{|V(T)|}{2} - 1$ and T has an even number of vertices, it follows that T is a near-regular tournament. ■

Lemma 7.2 *Let $G \in \mathcal{C}(0, n, r)$. Assume T is a forest tournament for G such that the following conditions are satisfied:*

- (1) n and r are both odd with $n \geq 3$ and $r \geq 1$,
- (2) every component without a triple end belongs to $OE2 \cup OO2$,
- (3) $T(O)$ and $T(O3)$ are regular tournaments,
- (4) for all i , w_i beats $\frac{n-3}{2}$ vertices of $T(O)$ in T , and
- (5) for all j , v_j beats $\frac{r+1}{2}$ vertices of $T(O3)$ in T .

Then T is a near-regular tournament.

Proof: Let T be a forest tournament for G such that the above conditions are satisfied. Since both n and r are odd, T will have an even number of vertices. Properly 2-color each component of G using red and blue such that the base vertex of each component is colored red. Notice that by Theorem 6.6, every odd component O_j has $\frac{n_j+1}{2}$ red vertices and $\frac{n_j-1}{2}$ blue vertices. Also by

Theorem 6.4, every odd component Q_i with a triple end has $\frac{r_i - 1}{2}$ red vertices and $\frac{r_i + 1}{2}$ blue vertices.

Let R_i be an arbitrary component of G and x_i its corresponding base vertex. Also let R_j be a component different from R_i with corresponding base vertex x_j . Suppose that x_i beats x_j . Since x_i and x_j are both colored red, by Lemma 4.5 vertices in R_i beat vertices of the same color in R_j . But note that if R_j is an odd component, then $\frac{|V(R_j)| + 1}{2}$ of its vertices are colored red, and $\frac{|V(R_j)| - 1}{2}$ of its vertices are colored blue. In addition, if R_j is an odd component with a triple end, then $\frac{|V(R_j)| - 1}{2}$ of its vertices are colored red, and $\frac{|V(R_j)| + 1}{2}$ of its vertices are colored blue. Thus,

- red and blue vertices in R_i beat $\frac{|V(R_j)| + 1}{2}$ and $\frac{|V(R_j)| - 1}{2}$ vertices of R_j in T , respectively, if R_j is an odd component and
- red and blue vertices in R_i beat $\frac{|V(R_j)| - 1}{2}$ and $\frac{|V(R_j)| + 1}{2}$ vertices of R_j in T , respectively, if R_j is an odd component with a triple end.

By a similar argument, if x_j beats x_i in T , then

- red and blue vertices in R_i beat $\frac{|V(R_j)| - 1}{2}$ and $\frac{|V(R_j)| + 1}{2}$ vertices of R_j in T , respectively, if R_j is an odd component and
- red and blue vertices in R_i beat $\frac{|V(R_j)| + 1}{2}$ and $\frac{|V(R_j)| - 1}{2}$ vertices of R_j in T , respectively, if R_j is an odd component with a triple end.

Using these facts we proceed by showing that every vertex in T has out-degree either $\frac{|V(T)|}{2}$ or $\frac{|V(T)|}{2} - 1$ in T .

First we show that the vertices of every odd component have the appropriate out-degree in T . Let O_j be an arbitrary odd component and $x \in V(O_j)$. First assume that x is colored red. Since $O_j \in OE2 \cup OO2$, we have by Theorem 6.6 that x beats either $\frac{n_j + 1}{2}$ or $\frac{n_j - 1}{2}$ vertices of O_j in T . If x beats $\frac{n_j + 1}{2}$ vertices of O_j in T , then since $T(O)$ is regular and v_j beats $\frac{r + 1}{2}$ vertices of $T(O3)$,

$$\begin{aligned}
d_T^+(x) &= \sum_{i: (v_j, w_i) \in A(T)} \frac{r_i - 1}{2} + \sum_{i: (w_i, v_j) \in A(T)} \frac{r_i + 1}{2} + \sum_{i: (v_j, v_i) \in A(T)} \frac{n_i + 1}{2} \\
&\quad + \sum_{i: (v_i, v_j) \in A(T)} \frac{n_i - 1}{2} + \frac{n_j + 1}{2} \\
&= \sum_{i=1}^r \frac{r_i}{2} + \sum_{i=1}^n \frac{n_i}{2} - \left(\frac{r+1}{2}\right) \frac{1}{2} + \left(\frac{r-1}{2}\right) \frac{1}{2} + \left(\frac{n-1}{2}\right) \frac{1}{2} \\
&\quad - \left(\frac{n-1}{2}\right) \frac{1}{2} + \frac{1}{2} \\
&= \frac{|V(T)|}{2}.
\end{aligned}$$

If x beats $\frac{n_j - 1}{2}$ vertices of O_j in T , then since $T(O)$ is regular and v_j beats $\frac{r + 1}{2}$ vertices of $T(O3)$,

$$d_T^+(x) = \sum_{i: (v_j, w_i) \in A(T)} \frac{r_i - 1}{2} + \sum_{i: (w_i, v_j) \in A(T)} \frac{r_i + 1}{2} + \sum_{i: (v_j, v_i) \in A(T)} \frac{n_i + 1}{2}$$

$$\begin{aligned}
& + \sum_{i: (v_i, v_j) \in A(T)} \frac{n_i - 1}{2} + \frac{n_j - 1}{2} \\
= & \sum_{i=1}^r \frac{r_i}{2} + \sum_{i=1}^n \frac{n_i}{2} - \left(\frac{r+1}{2}\right) \frac{1}{2} + \left(\frac{r-1}{2}\right) \frac{1}{2} + \left(\frac{n-1}{2}\right) \frac{1}{2} \\
& - \left(\frac{n-1}{2}\right) \frac{1}{2} - \frac{1}{2} \\
= & \frac{|V(T)|}{2} - 1.
\end{aligned}$$

Next assume that x is colored blue. Since $O_j \in OE2 \cup OO2$, it follows by Theorem 6.6 that x beats either $\frac{n_j - 1}{2}$ or $\frac{n_j - 3}{2}$ vertices of O_j in T . If x beats $\frac{n_j - 1}{2}$ vertices of O_j in T , then since $T(O)$ is regular and v_j beats $\frac{r+1}{2}$ vertices of $T(O3)$,

$$\begin{aligned}
d_T^+(x) & = \sum_{i: (v_j, w_i) \in A(T)} \frac{r_i + 1}{2} + \sum_{i: (w_i, v_j) \in A(T)} \frac{r_i - 1}{2} + \sum_{i: (v_j, v_i) \in A(T)} \frac{n_i - 1}{2} \\
& + \sum_{i: (v_i, v_j) \in A(T)} \frac{n_i + 1}{2} + \frac{n_j - 1}{2} \\
= & \sum_{i=1}^r \frac{r_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \left(\frac{r+1}{2}\right) \frac{1}{2} - \left(\frac{r-1}{2}\right) \frac{1}{2} - \left(\frac{n-1}{2}\right) \frac{1}{2} \\
& + \left(\frac{n-1}{2}\right) \frac{1}{2} - \frac{1}{2}
\end{aligned}$$

$$= \frac{|V(T)|}{2}.$$

If x beats $\frac{n_j - 3}{2}$ vertices of O_j in T , then since $T(O)$ is regular and v_j beats $\frac{r+1}{2}$ vertices of $T(O_3)$,

$$\begin{aligned} d_T^+(x) &= \sum_{i: (v_j, w_i) \in A(T)} \frac{r_i + 1}{2} + \sum_{i: (w_i, v_j) \in A(T)} \frac{r_i - 1}{2} + \sum_{i: (v_j, v_i) \in A(T)} \frac{n_i - 1}{2} \\ &\quad + \sum_{i: (v_i, v_j) \in A(T)} \frac{n_i + 1}{2} + \frac{n_j - 3}{2} \\ &= \sum_{i=1}^r \frac{r_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \left(\frac{r+1}{2}\right) \frac{1}{2} - \left(\frac{r-1}{2}\right) \frac{1}{2} - \left(\frac{n-1}{2}\right) \frac{1}{2} \\ &\quad + \left(\frac{n-1}{2}\right) \frac{1}{2} - \frac{3}{2} \\ &= \frac{|V(T)|}{2} - 1. \end{aligned}$$

Thus, it follows that every vertex in $V(O_j)$ has out-degree either $\frac{|V(T)|}{2}$ or $\frac{|V(T)|}{2} - 1$. Since O_j was an arbitrary odd component, we may conclude that for all odd components O_j , every vertex in $V(O_j)$ has out-degree either $\frac{|V(T)|}{2}$ or $\frac{|V(T)|}{2} - 1$ in T .

Next we show that the vertices in every odd component with a triple end have the appropriate out-degree in T . Let Q_j be an arbitrary odd component with a triple end and $x \in V(Q_j)$. First assume that x is colored red. Since $Q_j \in OO_3$, we may conclude by Theorem 6.4 that x beats either $\frac{r_j + 3}{2}$

or $\frac{r_j + 1}{2}$ vertices of Q_j in T . If x beats $\frac{r_j + 3}{2}$ vertices of Q_j in T , then since $T(O3)$ is regular and w_j beats $\frac{n - 3}{2}$ vertices of $T(O)$,

$$\begin{aligned}
d_T^+(x) &= \sum_{i: (w_j, w_i) \in A(T)} \frac{r_i - 1}{2} + \sum_{i: (w_i, w_j) \in A(T)} \frac{r_i + 1}{2} + \sum_{i: (w_j, v_i) \in A(T)} \frac{n_i + 1}{2} \\
&\quad + \sum_{i: (v_i, w_j) \in A(T)} \frac{n_i - 1}{2} + \frac{r_j + 3}{2} \\
&= \sum_{i=1}^r \frac{r_i}{2} + \sum_{i=1}^n \frac{n_i}{2} - \left(\frac{r-1}{2}\right) \frac{1}{2} + \left(\frac{r-1}{2}\right) \frac{1}{2} + \left(\frac{n-3}{2}\right) \frac{1}{2} \\
&\quad - \left(\frac{n+3}{2}\right) \frac{1}{2} + \frac{3}{2} \\
&= \frac{|V(T)|}{2}.
\end{aligned}$$

If x beats $\frac{r_j + 1}{2}$ vertices of Q_j in T , then since $T(O3)$ is regular and w_j beats $\frac{n - 3}{2}$ vertices of $T(O)$,

$$\begin{aligned}
d_T^+(x) &= \sum_{i: (w_j, w_i) \in A(T)} \frac{r_i - 1}{2} + \sum_{i: (w_i, w_j) \in A(T)} \frac{r_i + 1}{2} + \sum_{i: (w_j, v_i) \in A(T)} \frac{n_i + 1}{2} \\
&\quad + \sum_{i: (w_i, v_j) \in A(T)} \frac{n_i - 1}{2} + \frac{r_j + 1}{2} \\
&= \sum_{i=1}^r \frac{r_i}{2} + \sum_{i=1}^n \frac{n_i}{2} - \left(\frac{r-1}{2}\right) \frac{1}{2} + \left(\frac{r-1}{2}\right) \frac{1}{2} + \left(\frac{n-3}{2}\right) \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
& - \left(\frac{n+3}{2} \right) \frac{1}{2} + \frac{1}{2} \\
& = \frac{|V(T)|}{2} - 1.
\end{aligned}$$

Next assume that x is colored blue. Since $Q_j \in OO3$, by Theorem 6.4 we have that x beats either $\frac{r_j-3}{2}$ or $\frac{r_j-5}{2}$ vertices of Q_j in T . If x beats $\frac{r_j-3}{2}$ vertices of Q_j in T , then since $T(O3)$ is regular and w_j beats $\frac{n-3}{2}$ vertices of $T(O)$,

$$\begin{aligned}
d_T^+(x) &= \sum_{i: (w_j, w_i) \in A(T)} \frac{r_i+1}{2} + \sum_{i: (w_i, w_j) \in A(T)} \frac{r_i-1}{2} + \sum_{i: (w_j, v_i) \in A(T)} \frac{n_i-1}{2} \\
&+ \sum_{i: (v_i, w_j) \in A(T)} \frac{n_i+1}{2} + \frac{r_j-3}{2} \\
&= \sum_{i=1}^r \frac{r_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \left(\frac{r-1}{2} \right) \frac{1}{2} - \left(\frac{r-1}{2} \right) \frac{1}{2} - \left(\frac{n-3}{2} \right) \frac{1}{2} \\
&+ \left(\frac{n+3}{2} \right) \frac{1}{2} - \frac{3}{2} \\
&= \frac{|V(T)|}{2}.
\end{aligned}$$

If x beats $\frac{r_j-5}{2}$ vertices of Q_j in T , then since $T(O3)$ is regular and w_j beats $\frac{n-3}{2}$ vertices of $T(O)$,

$$d_T^+(x) = \sum_{i: (w_j, w_i) \in A(T)} \frac{r_i+1}{2} + \sum_{i: (w_i, w_j) \in A(T)} \frac{r_i-1}{2} + \sum_{i: (w_j, v_i) \in A(T)} \frac{n_i-1}{2}$$

$$\begin{aligned}
& + \sum_{i: (v_i, w_j) \in A(T)} \frac{n_i + 1}{2} + \frac{r_j - 5}{2} \\
& = \sum_{i=1}^r \frac{r_i}{2} + \sum_{i=1}^n \frac{n_i}{2} + \left(\frac{r-1}{2}\right) \frac{1}{2} - \left(\frac{r-1}{2}\right) \frac{1}{2} - \left(\frac{n-3}{2}\right) \frac{1}{2} \\
& \quad + \left(\frac{n+3}{2}\right) \frac{1}{2} - \frac{5}{2} \\
& = \frac{|V(T)|}{2} - 1.
\end{aligned}$$

Thus, it follows that every vertex in $V(Q_j)$ has out-degree either $\frac{|V(T)|}{2}$ or $\frac{|V(T)|}{2} - 1$. Since Q_j was an arbitrary odd component with a triple end, we may conclude that for all odd components with a triple end, every vertex in $V(Q_j)$ has out-degree either $\frac{|V(T)|}{2}$ or $\frac{|V(T)|}{2} - 1$ in T .

Therefore, since we have shown that every vertex has out-degree either $\frac{|V(T)|}{2}$ or $\frac{|V(T)|}{2} - 1$ and T has an even number of vertices, we conclude that T is a near-regular tournament. ■

The next result provides sufficient conditions to guarantee that the domination graph of a forest tournament T is a graph $G \in \mathcal{C}(m, n, r)$ where T is a forest tournament for G .

Theorem 7.3 *Let $G \in \mathcal{C}(m, n, r)$. If there exists a forest tournament T for G such that T_B is well-covered, then $\text{dom}(T) = G$.*

Proof: Suppose there exists a forest tournament T for G such that T_b is well-covered. Then it follows from the proof of Theorem 2.3 of [16] that $\text{dom}(T) = G$. See discussion preceding Theorem 5.8. ■

The next result provides sufficient conditions for a graph in $\mathcal{C}(m, n, 0)$, to be the domination graph of a near-regular tournament.

Theorem 7.4 *Let m and n be integers with n even and $m + n = 4$ or $m + n \geq 6$. Assume that there exist tournaments T_O , T_E , and T_B with the following properties:*

- (1) T_O is a near-regular tournament on n vertices,
- (2) T_E is a tournament on m vertices,
- (3) T_O and T_E are subtournaments of T_B such that $V(T_O)$ and $V(T_E)$ partition the vertex set of T_B ,
- (4) we have in T_B that k vertices in T_E beat $\frac{n}{2}$ vertices of T_O and $m - k$ vertices in T_E beat $\frac{n}{2} - 1$ vertices of T_O , and
- (5) T_B is well-covered.

Then for every graph $G \in \mathcal{C}(m, n, 0)$ with $\frac{n}{2}$ odd caterpillars in $EE1$, $\frac{n}{2}$ odd caterpillars in $OE2 \cup OO2$, k even caterpillars in $EE2 \cup EO2$, and $m - k$ even caterpillars in $OE1$, there exists a near-regular tournament T with $\text{dom}(T) = G$.

Proof: Let $G \in \mathcal{C}(m, n, 0)$ be the union of m even caterpillars E_1, E_2, \dots, E_m

and n odd caterpillars O_1, O_2, \dots, O_n with $\frac{n}{2}$ odd caterpillars in $EE1$, $\frac{n}{2}$ odd caterpillars in $OE2 \cup OO2$, k even caterpillars in $EE2 \cup EO2$, and $m - k$ even caterpillars in $OE1$. First we construct a forest tournament T with vertex set $V(G)$ as follows. Let $T(E_i)$ and $T(O_j)$ be as discussed in Chapter 6 for all components E_i and O_j . Next let $T(O) = T_O$ be such that the vertices that have out-degree $\frac{n}{2}$ in $T(O)$ correspond to the base vertices of the components in $EE1$ and the vertices that have out-degree $\frac{n}{2} - 1$ in $T(O)$ correspond to the base vertices of the components in $OE2 \cup OO2$. Let $T(E) = T_E$ be such that the vertices in $T(E)$ that beat $\frac{n}{2}$ of the vertices in $T(O)$ correspond to the base vertices of the components in $EE2 \cup EO2$ and the vertices in $T(E)$ that beat $\frac{n}{2} - 1$ of the vertices in $T(O)$ correspond to the base vertices of the components in $OE1$. Finally let $T(B) = T_B$. Now by properly 2-coloring G with red and blue such that the base vertices of all the components get colored red, Lemma 4.5 determines the remaining arcs in T .

Now since $T(O)$, $T(E)$ and $T(B)$ satisfy Properties 1 through 4, it follows by Lemma 7.1 that T is a near-regular tournament. In addition, since $T(B)$ is well-covered, by Theorem 7.3 we conclude that $\text{dom}(T) = G$. Therefore, since G was arbitrary, it follows that for every graph $G \in \mathcal{C}(m, n, 0)$ with $\frac{n}{2}$ odd caterpillars in $EE1$, $\frac{n}{2}$ odd caterpillars in $OE2 \cup OO2$, k even caterpillars in $EE2 \cup EO2$, and $m - k$ even caterpillars in $OE1$, there exists a near-regular

tournament T with $\text{dom}(T) = G$. ■

Now using Lemma 7.2 and Theorems 7.3 and 7.4 we proceed by exhibiting two infinite classes of forests of nontrivial OGPR caterpillars that are the domination graphs of near-regular tournaments.

Theorem 7.5 *Suppose $G \in \mathcal{C}(m, n, 0)$ with $m + n = 4$ or $m + n \geq 6$. Then the following combinations of m even caterpillars and n odd caterpillars are the domination graphs of near-regular tournaments:*

| Combination | $n =$ | $m =$ | $EE1$ | $OE2 \cup OO2$ | $EE2 \cup EO2$ | $OE1$ |
|-------------|----------|-------------|---------------|----------------|----------------|-------|
| 1 | 0 | $4, \geq 6$ | 0 | 0 | m | 0 |
| 2 | 2 | 2 | 1 | 1 | 1 | 1 |
| 3 | 2 | 5 | 1 | 1 | 4 | 1 |
| 4 | 2 | $4, \geq 6$ | 1 | 1 | $m - 1$ | 1 |
| 5 | 4 | 2 | 2 | 2 | 1 | 1 |
| 6 | 4 | 3 | 2 | 2 | 2 | 1 |
| 7 | 4 | 5 | 2 | 2 | 4 | 1 |
| 8 | 4 | $4, \geq 6$ | 2 | 2 | $m - 1$ | 1 |
| 9 | 6 | 0 | 3 | 3 | 0 | 0 |
| 10 | 6 | 1 | 3 | 3 | 1 | 0 |
| 11 | 6 | 2 | 3 | 3 | 1 | 1 |
| 12 | 6 | ≥ 3 | 3 | 3 | m | 0 |
| 13 | ≥ 8 | 0 | $\frac{n}{2}$ | $\frac{n}{2}$ | 0 | 0 |
| 14 | ≥ 8 | 2 | $\frac{n}{2}$ | $\frac{n}{2}$ | 1 | 1 |
| 15 | ≥ 8 | $1, \geq 3$ | $\frac{n}{2}$ | $\frac{n}{2}$ | m | 0 |

Proof: The tournaments T_E , T_O and T_B constructed in Lemmas 5.13, 5.14, 5.15, 5.17, and 5.20 together with Theorem 7.4 show that the above combinations of OGPR caterpillars are the domination graphs of near-regular tournaments. To see this, consider the first combination.

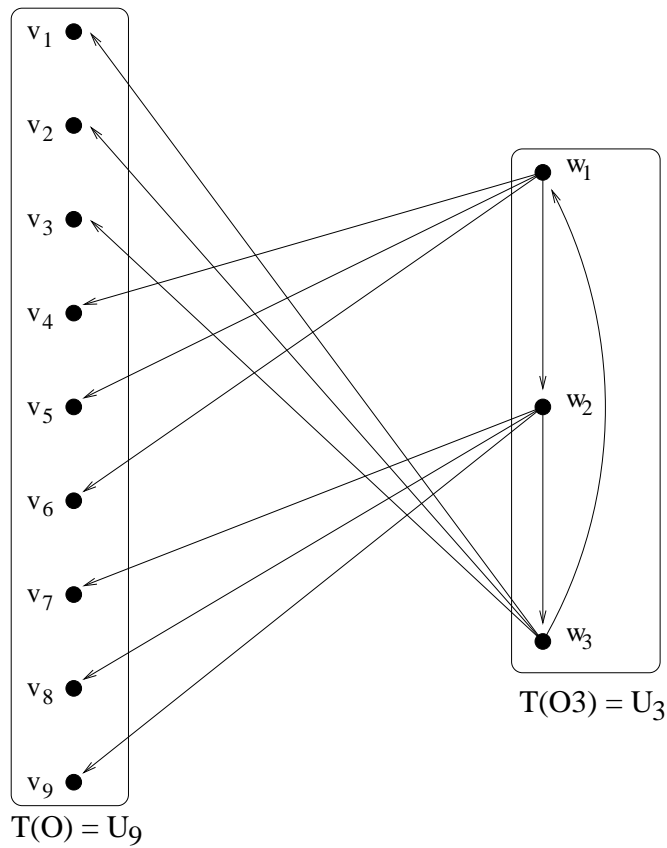
Combination 1. For $m = 4$ or $m \geq 6$, let T_O , T_E , and T_B be the tournaments provided in the proof of Lemma 5.13. Note that T_O , T_E , and T_B satisfy Properties 1, 2, 3 and 5 of Theorem 7.4. In addition, since T_O is the empty tournament, m vertices of T_E beat 0 vertices of T_O . Thus, we may conclude by Theorem 7.4 that for every graph $G \in \mathcal{C}(m, 0, 0)$ with m components in $EE2 \cup EO2$, there exists a near-regular tournament T with $\text{dom}(T) = G$.

The remaining combinations in the above table can be verified using similar arguments. In particular, Lemma 5.14 is used to verify combinations 2 through 4, Lemma 5.15 is used to verify combinations 5 through 8, Lemma 5.17 is used to verify combinations 9 through 12 and Lemma 5.20 is used to verify combinations 13 through 15. Therefore, we may conclude that the above combinations of OGPR caterpillars are the domination graphs of near-regular tournaments. ■

Theorem 7.6 *Assume $G \in \mathcal{C}(0, n, r)$ with r odd, $n = 3r$, and every odd component of G belonging to $OE2 \cup OO2$. Then G is the domination graph of a near-regular tournament.*

Proof: For the following all arithmetic is done modulo $3r$. We start by constructing a forest tournament T with vertex set $V(G)$. Let $T(O_j)$ and $T(Q_i)$ be as discussed in Chapter 6. Next let $T(O) = U_{3r}$ and $T(O3) = U_r$. The remaining arcs in $T(B)$ are $w_i \rightarrow \{v_{(3i-2)+3k}, v_{(3i-1)+3k}, v_{3i+3k}\}$ for k odd with $1 \leq k \leq r-2$, and $\{v_{(3i-2)+3k}, v_{(3i-1)+3k}, v_{3i+3k}\} \rightarrow w_i$ for k even with $0 \leq k \leq r-1$. See Figure 7.1 for an example of $T(B)$ when $r = 3$. Now by properly 2-coloring the vertices of G with red and blue such that the base vertex of each component is colored red, Lemma 4.5 determines the remaining arcs in T . Since all vertices in $T(O3)$ beat $\frac{3r-3}{2}$ vertices of $T(O)$ and all vertices of $T(O)$ beat $\frac{r+1}{2}$ of $T(O3)$, it follows by Lemma 7.2 that T is a near-regular tournament. To show that $\text{dom}(T) = G$, by Theorem 7.3 it suffices to show that $T(B)$ is a well-covered tournament. Thus we proceed by verifying that $T(B)$ is a well-covered tournament.

Notice that since $T(O) = U_{3r}$, we may conclude that $T(O)$ is a regular tournament. Thus, since every arc in $T(O)$ is contained in a 3-cycle, every pair of vertices in $T(O)$ are distinguished in $T(B)$. Similarly, since $T(O3) = U_r$, every pair of vertices in $T(O3)$ are distinguished in $T(B)$. Next let $v_i, v_j \in T(O)$. If $j \not\equiv i+1, i-1 \pmod{3r}$, then there must be a vertex in $T(O)$ that beats both v_i and v_j since they do not form a dominant pair in U_{3r} . Therefore, if $j \not\equiv i+1, i-1 \pmod{3r}$, it follows that v_i and v_j are paired in $T(O)$ and so are paired in $T(B)$. Also, by construction, we have that $\{v_{(3i-2)}, v_{(3i-1)}, v_{3i}\} \rightarrow w_i$



The remaining arcs are oriented from vertices in $T(O)$ to vertices in $T(O3)$.

Figure 7.1. The tournament $T(B)$ when $r = 3$.

for $1 \leq i \leq r$. Thus, every pair of vertices in $V(T(O))$ are paired in $T(B)$.

Next let $w_i, w_j \in V(T(O3))$. If $j \not\equiv i + 1, i - 1 \pmod{3r}$, then there must be a vertex in $T(O3)$ that beats both w_i and w_j since they do not form a dominant pair in U_r . Therefore, if $j \not\equiv i + 1, i - 1 \pmod{3r}$ it follows that w_i and w_j are paired in $T(O3)$ and so are paired in $T(B)$. Also, by construction, $v_{3i} \rightarrow \{w_i, w_{i+1}\}$ for $1 \leq i \leq r$. Thus, we have that every pair of vertices in $V(T(O3))$ are paired in $T(B)$.

It remains to be shown that every pair of vertices v_i and w_j are paired and distinguished for any choice of $v_i \in V(T(O))$ and $w_j \in V(T(O3))$. Let $w_i \in V(T(O3))$. By construction, the subtournament of $T(B)$ induced by $\{w_i, v_{(3i-2)+3k}, v_{(3i-1)+3k}, v_{3i+3k}\}$, for all k with $0 \leq k \leq r - 1$, belongs to $W_{1,3} \cup \overline{W_{1,3}}$. By Lemma 5.11 and Corollary 5.12 this subtournament is well-covered. Thus, w_i is paired and distinguished with every vertex in $\{v_{(3i-2)+3k}, v_{(3i-1)+3k}, v_{3i+3k}\}$ for all k with $0 \leq k \leq r - 1$. Since w_i was an arbitrary vertex in $V(T(O3))$ this holds true for all vertices in $V(T(O3))$. Therefore, v_i and w_j are paired and distinguished for all $w_i \in V(T(O3))$ and $v_j \in V(T(O))$.

Thus, we may conclude that $T(B)$ is a well-covered tournament and $dom(T) = G$. ■

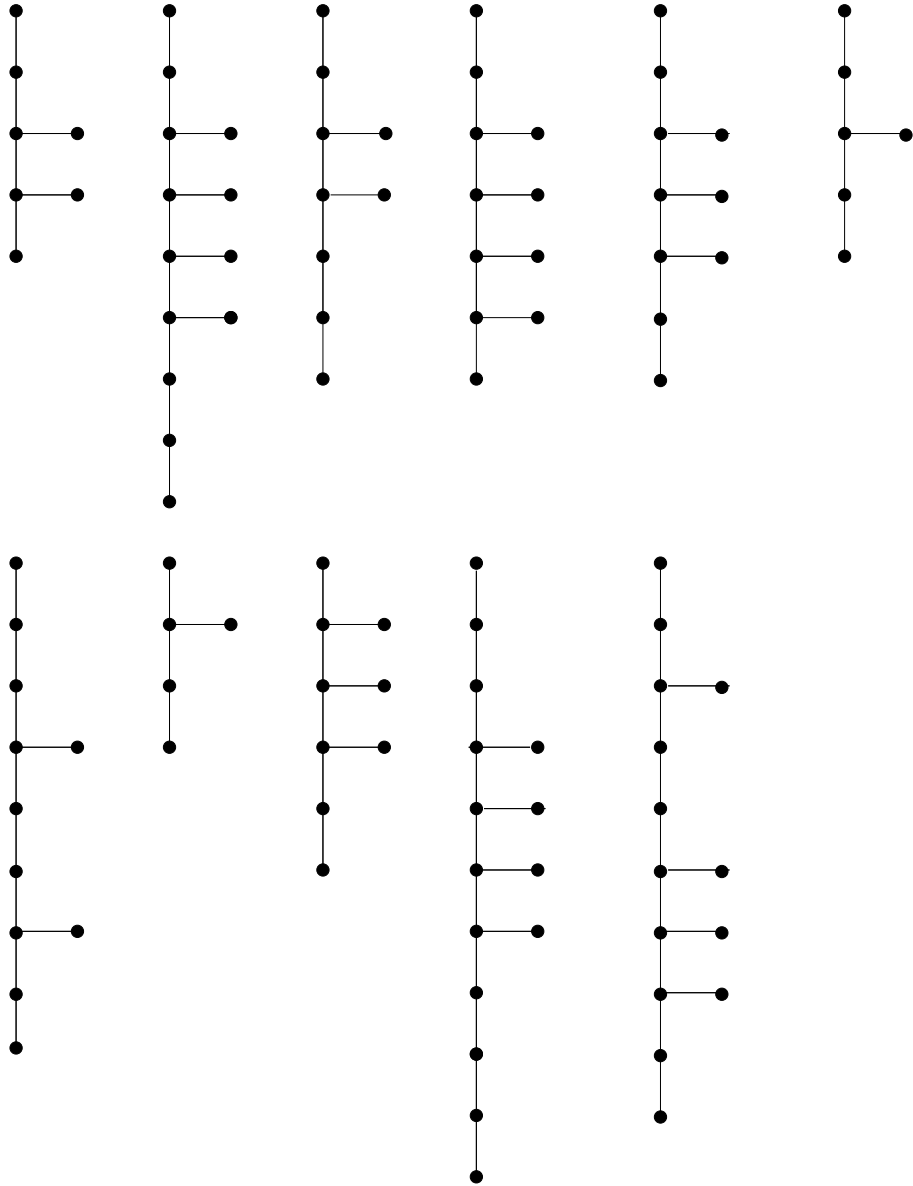


Figure 7.2. Theorem 7.5 verifies that this forest is the domination graph of a near-regular tournament on 110 vertices.

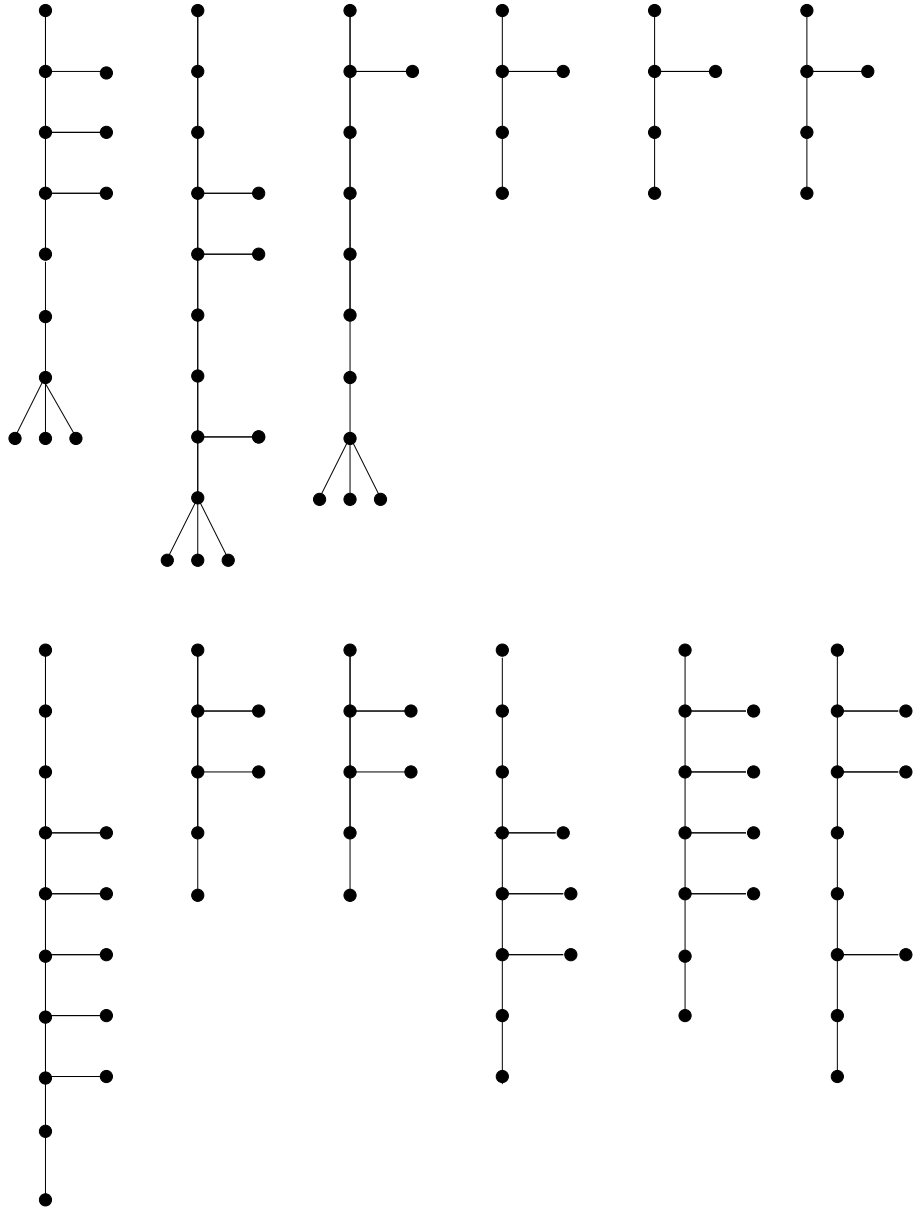


Figure 7.3. Theorem 7.6 verifies that this forest is the domination graph of a near-regular tournament on 116 vertices.

In this chapter together with Chapter 6, initial results were established toward the characterization of forests of nontrivial caterpillars that are the domination graphs of near-regular tournaments. By Theorem 4.6, the only forests of caterpillars that can be the domination graphs of near-regular tournaments are those which are composed of OGPR caterpillars. In addition, it was shown in Chapter 6 that once an orientation is established for an OGPR caterpillar H in $\mathcal{D}(T)$, the subtournament $T(H)$ is uniquely determined. We used this property to derive some results about the subtournaments an OGPR caterpillar induces. In turn, these results were used in this chapter to exhibit two infinite classes of forests of nontrivial OGPR caterpillars that are the domination graphs of near-regular tournaments. In any event, a characterization for all forests of nontrivial OGPR caterpillars that are the domination graphs of near-regular tournaments remains to be established. Therefore, a way in which the research on this topic can be continued is to try to determine all forests of nontrivial OGPR caterpillars that are the domination graphs of near-regular tournaments.

8. Conclusion

In this dissertation we examined three closely related problems which stem from the study of domination in tournaments. There are still many new avenues in which the research on topics discussed in this dissertation can be continued. Therefore, we shall conclude by presenting a list of open problems which have originated from this body of work.

- (1) Characterize all connected graphs that are the domination graph of a unique tournament (up to isomorphism).
- (2) Prove or disprove: If G is the domination-compliance graph of a tournament, then G can be properly 4-colored.
- (3) Prove or disprove: If G is the domination-compliance graph of a tournament, then G is planar.
- (4) Characterize all graphs which are the domination-compliance graph of a strongly connected tournament.
- (5) Characterize all forests of nontrivial caterpillars that are the domination graph of a near-regular tournament.
- (6) Answer: Which odd-spiked cycles with isolated vertices are the domination graph of a near-regular tournament?
- (7) Answer: Which forest of caterpillars with isolated vertices are the domination graph of a near-regular tournament?

A. APPENDIX Proof of Theorem 3.10

Proof: Suppose T is a tournament on six vertices and assume that $K_{3,3}$ is a subgraph of $DC(T)$. Then it follows by Theorem 3.6 that $DC(T)$ can have at most ten edges. Therefore, $DC(T)$ is either $K_{3,3}$ or isomorphic to the graph in Figure A.1.

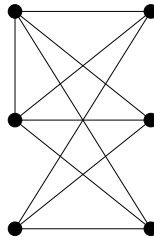


Figure A.1. The only possible domination-compliance graph that has $K_{3,3}$ as a subgraph.

Since $DC(T)$ has at least nine edges, then either $dom(T)$ has at least five edges or $com(T)$ has at least five edges. Since every domination graph is a compliance graph, without loss of generality we may assume that $dom(T)$ has at least five edges. By Theorem 1.1, the domination graph of a tournament must be an odd-spiked cycle with or without isolated vertices or a forest of caterpillars. Thus, the domination graph of a tournament on n vertices can have at most n edges. Therefore, we may conclude that $dom(T)$ must have either five or six edges. We proceed by considering the following cases.

Case 1. Suppose that $dom(T)$ has six edges. Then we have by Theorem 1.1 that $dom(T)$ must be a graph in Figure A.2.

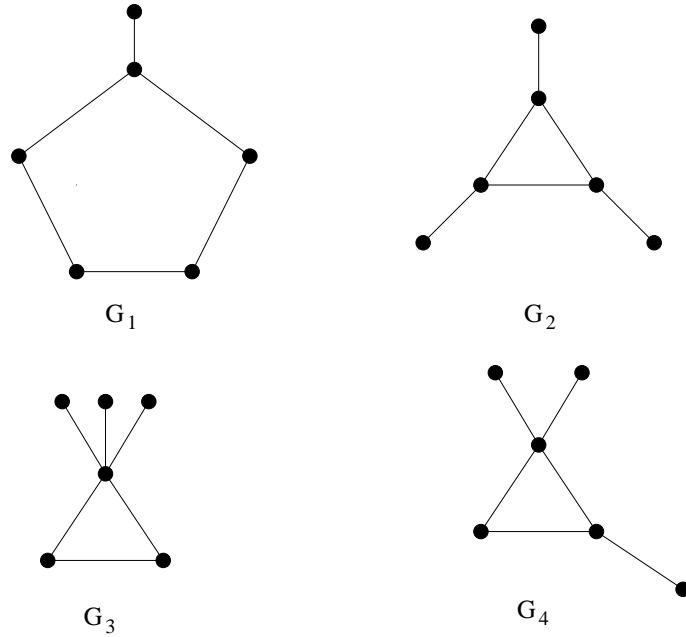


Figure A.2. All connected domination graphs, up to isomorphism, on six vertices and six edges.

Using Theorem 2.7, we may construct the tournaments which have these graphs in Figure A.2 as their domination graph. Figures A.3, A.4, A.5 and A.6 depict these tournaments and their corresponding domination-compliance graphs. But notice that none of these domination-compliance graphs are $K_{3,3}$ nor are any isomorphic to the graph in Figure A.1. Thus we have a contradiction.

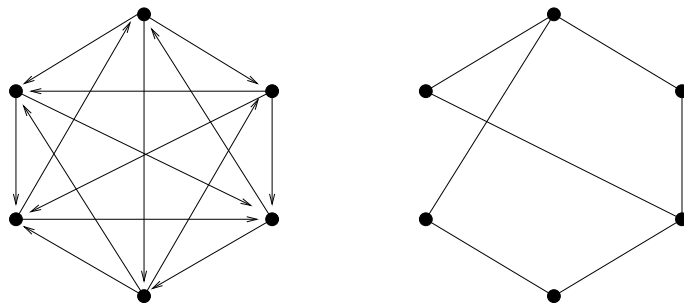


Figure A.3. The only tournament that has G_1 as its domination graph and its domination-compliance graph.

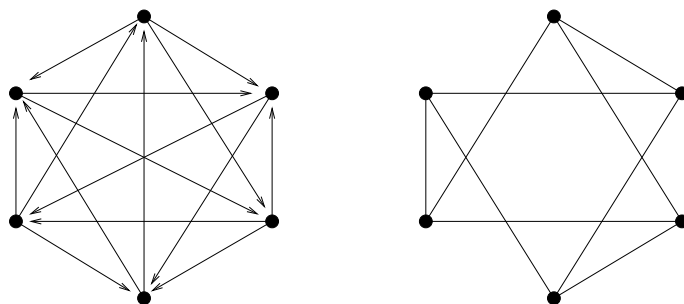


Figure A.4. The only tournament that has G_2 as its domination graph and its domination-compliance graph.

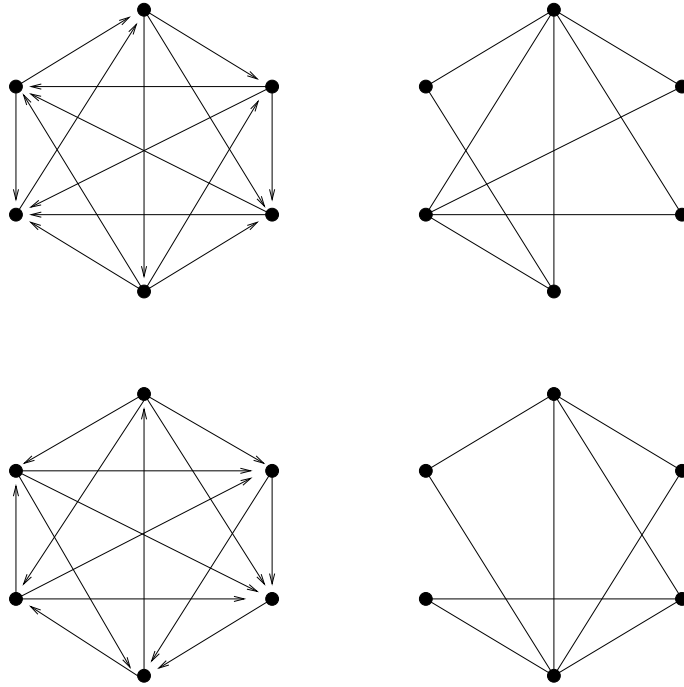


Figure A.5. Two tournaments that have $dom(T) = G_3$ and their corresponding domination-compliance graphs.

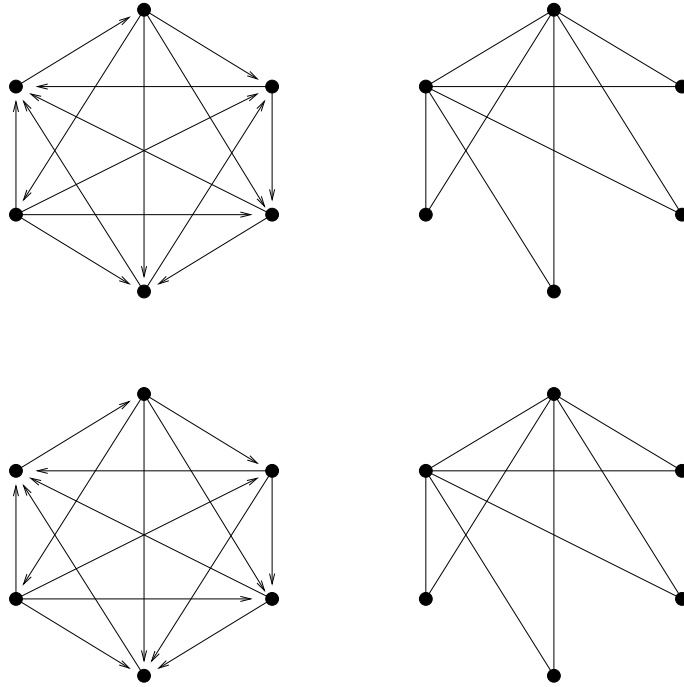


Figure A.6. The two tournaments that have $dom(T) = G_4$ and their corresponding domination-compliance graphs.

Case 2. Suppose that $\text{dom}(T)$ has exactly five edges. Then by Theorem 1.1, $\text{dom}(T)$ must be one of the graphs in figure A.7.

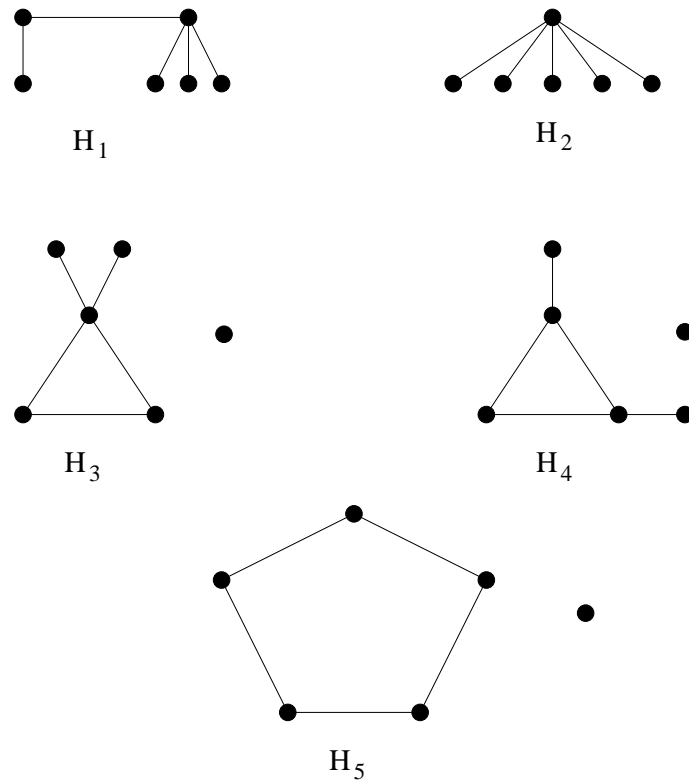


Figure A.7. All domination graphs, up to isomorphism, on six vertices and five edges.

Subcase 2.1. Suppose that $\text{dom}(T) = H_1$. Using Theorem 2.10, we may construct the tournament T that has H_1 as its domination graph. Figure A.8 depicts this tournament along with its domination-compliance graph. It is easy to see that the domination-compliance graph of Figure A.8 does not have

$K_{3,3}$ as a subgraph. Thus we have a contradiction.

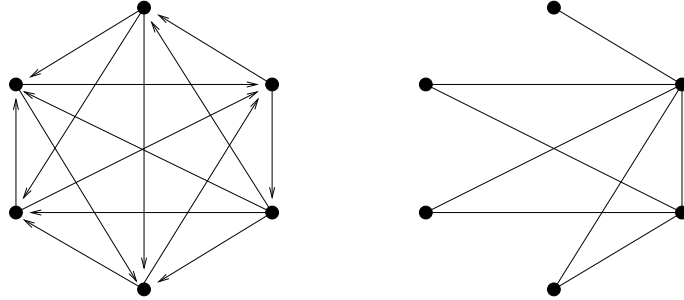
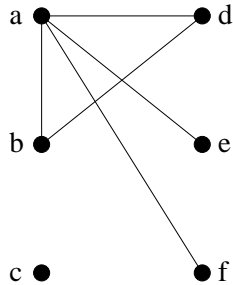


Figure A.8. The tournament that has H_1 as its domination graph and its domination-compliance graph.

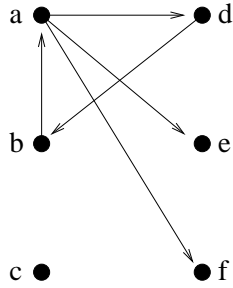
Subcase 2.2. Suppose that $dom(T) = H_2$. Note that $dom(T)$ has a vertex of degree 5. But since $dom(T)$ is a subgraph of $DC(T)$ and $DC(T)$ does not have a vertex of degree 5, it follows that $dom(T)$ cannot be H_2 .

Subcase 2.3. Suppose that $dom(T) = H_3$. Since $dom(T)$ is a subgraph of $DC(T)$ and $dom(T)$ has a 3-cycle, it follows that $DC(T)$ must be the graph in Figure A.1. Without loss of generality, we may assume that $dom(T)$ is contained in $DC(T)$ as depicted in the following figure.

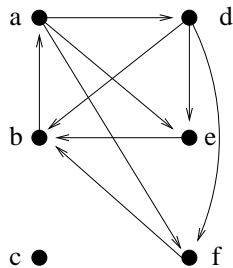


Lemma 2.2 determines that the domination digraph has two possible

orientations; one obtained by orienting the arcs on the 3-cycle clockwise and the other by orienting the arcs on the 3-cycle counterclockwise. First assume that $\mathcal{D}(T)$ has the following orientation.



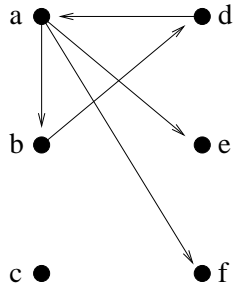
Then by Lemma 2.6, the subgraph of $dom(T)$ induced by the set of vertices $\{a, b, d, e, f\}$ must be a subgraph of $dom(T(\{a, b, d, e, f\}))$. Then by Theorem 1.1, it follows that $dom(T(\{a, b, d, e, f\}))$ must be an odd-spiked cycle. But by Theorem 2.7, we conclude that $T(\{a, b, d, e, f\})$ is a spiked cycle tournament. Therefore, the following arcs must be in $T(\{a, b, d, e, f\})$ and also in T since $T(\{a, b, d, e, f\})$ is contained in T .



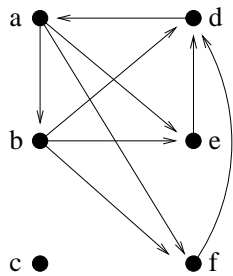
Since $[a, c]$ is not an edge in $dom(T)$, then b must beat c in T . Then since $[b, c]$ is not an edge in $DC(T)$, the edge $[b, c]$ cannot be in $com(T)$ and so c must beat a in T . Then since $[a, e]$ is an edge in $dom(T)$, we conclude that

e must beat c in T . Finally since $[b, e]$ is an edge in $DC(T)$ and $[b, e]$ is not an edge in $dom(T)$, we have that $[b, e]$ must be an edge in $com(T)$. But this is not possible since both b and e beat c in T , a contradiction.

Next suppose that $\mathcal{D}(T)$ has the following orientation.



Then by Lemma 2.6, the subgraph of $dom(T)$ induced by the set of vertices $\{a, b, d, e, f\}$ must be a subgraph of $dom(T(\{a, b, d, e, f\}))$. Then by Theorem 1.1, it follows that $dom(T(\{a, b, d, e, f\}))$ is an odd-spiked cycle. But by Theorem 2.7, we conclude that $T(\{a, b, d, e, f\})$ is a spiked cycle tournament. Therefore, the following arcs must be in $T(\{a, b, d, e, f\})$ and also in T since $T(\{a, b, d, e, f\})$ is contained in T .



Since T is a tournament, either (e, f) is an arc or (f, e) is an arc. Suppose that (e, f) is an arc. Then since $[b, e]$ is an edge in $DC(T)$ and $[b, e]$ is

not an edge in $dom(T)$, it follows that $[b, e]$ must be an edge in $com(T)$. But this is impossible since both b and e beat f in T , a contradiction. Similarly if (e, f) is an arc, then $[b, f]$ is not an edge in $com(T)$, a contradiction since $[b, f]$ is an edge in $DC(T)$ and it is not an edge in $dom(T)$.

Subcase 2.4. Suppose that $dom(T) = H_4$. Since $dom(T)$ is a subgraph of $DC(T)$ and $dom(T)$ has a 3-cycle it follows that $DC(T)$ must be the graph in Figure A.1. The following figure depicts the only two possible ways that $dom(T)$ can be contained in $DC(T)$.

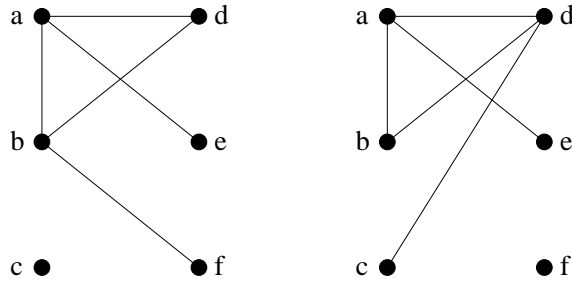
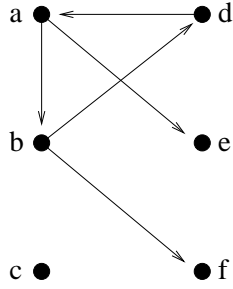
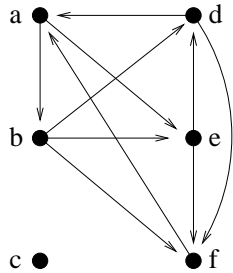


Figure A.9. Two possible ways, up to isomorphism, that H_4 can be contained in $DC(T)$.

First suppose that $dom(T)$ is contained in $DC(T)$ as depicted by the left graph in Figure A.9. Lemma 2.2 determines that the domination digraph has two possible orientations; one obtained by orienting the arcs on the 3-cycle clockwise and the other by orienting the arcs on the 3-cycle counterclockwise. First assume that $\mathcal{D}(T)$ has the following orientation.

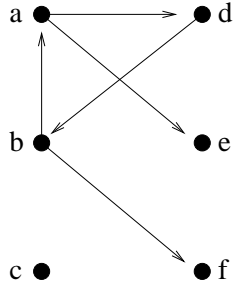


Then by Lemma 2.6, the subgraph of $\text{dom}(T)$ induced by the set of vertices $\{a, b, d, e, f\}$ must be a subgraph of $\text{dom}(T(\{a, b, d, e, f\}))$. Then by Theorem 1.1, it follows that $\text{dom}(T(\{a, b, d, e, f\}))$ is an odd-spiked cycle. But by Theorem 2.7, we conclude that $T(\{a, b, d, e, f\})$ is a spiked cycle tournament. Therefore, the following arcs must be in $T(\{a, b, d, e, f\})$ and also in T since $T(\{a, b, d, e, f\})$ is contained in T .

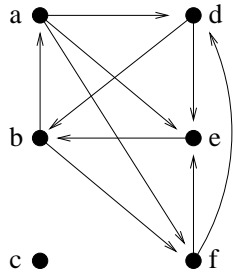


Since $[b, e]$ is an edge in $DC(T)$ and $[b, e]$ is not an edge in $\text{dom}(T)$, it follows that $[b, e]$ must be an edge in $\text{com}(T)$. But since both b and e beat d in T , then $[b, e]$ cannot be an edge in $\text{com}(T)$. This is a contradiction since $[b, e] \in DC(T)$ and $[b, e] \notin \text{dom}(T)$.

Next suppose that $\mathcal{D}(T)$ has the following orientation.



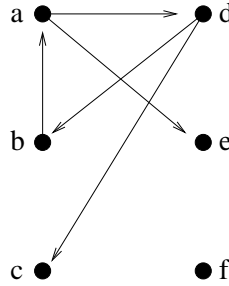
Then by Lemma 2.6, the subgraph of $dom(T)$ induced by the set of vertices $\{a, b, d, e, f\}$ must be a subgraph of $dom(T(\{a, b, d, e, f\}))$. Then by Theorem 1.1, it follows that $dom(T(\{a, b, d, e, f\}))$ is an odd-spiked cycle. But by Theorem 2.7, we conclude that $T(\{a, b, d, e, f\})$ is a spiked cycle tournament. Therefore, the following arcs must be in $T(\{a, b, d, e, f\})$ and also in T since $T(\{a, b, d, e, f\})$ is contained in T .



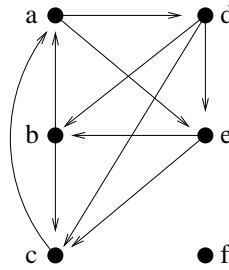
Since $[a, f]$ is an edge in $DC(T)$ and $[a, f]$ is not an edge in $dom(T)$, it follows that $[a, f]$ must be an edge in $com(T)$. But since both a and f beat d in T , then $[a, f]$ cannot be an edge in $com(T)$. This is a contradiction since $[a, f] \in DC(T)$ and $[a, f] \notin dom(T)$.

Next assume that $dom(T)$ is contained in $DC(T)$ as depicted by the right graph in Figure A.9. Again, Lemma 2.2 determines that the domination

digraph has two possible orientations. First assume that $\mathcal{D}(T)$ has the following orientation.

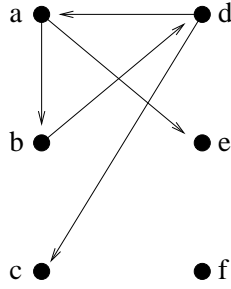


Then by Lemma 2.6, the subgraph of $dom(T)$ induced by set of the vertices $\{a, b, c, d, e\}$ must be a subgraph of $dom(T(\{a, b, c, d, e\}))$. Then by Theorem 1.1, it follows that $dom(T(\{a, b, c, d, e\}))$ is an odd-spiked cycle. But by Theorem 2.7, we conclude that $T(\{a, b, c, d, e\})$ is a spiked cycle tournament. Therefore, the following arcs must be in $T(\{a, b, c, d, e\})$ and also in T since $T(\{a, b, c, d, e\})$ is contained in T .

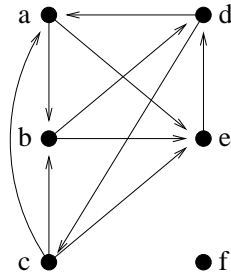


Since $[b, e]$ is an edge in $DC(T)$ and $[b, e]$ is not an edge in $dom(T)$, we must have that $[b, e]$ is an edge in $com(T)$. But this is impossible since both b and e beat c in T , a contradiction.

Next assume that $\mathcal{D}(T)$ has the following orientation.



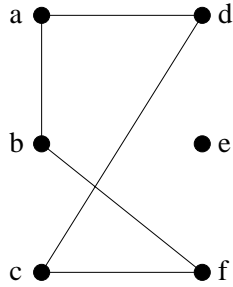
Then by Lemma 2.6, the subgraph of $dom(T)$ induced by the set of vertices $\{a, b, c, d, e\}$ must be a subgraph of $dom(T(\{a, b, c, d, e\}))$. Then by Theorem 1.1, it follows that $dom(T(\{a, b, c, d, e\}))$ is an odd-spiked cycle. But by Theorem 2.7, we have that $T(\{a, b, c, d, e\})$ is a spiked cycle tournament. Therefore, the following arcs must be in $T(\{a, b, c, d, e\})$ and also in T since $T(\{a, b, c, d, e\})$ is contained in T .



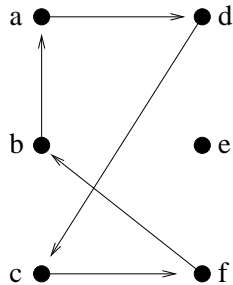
Since $[b, e]$ is an edge in $DC(T)$ and $[b, e]$ is not an edge in $dom(T)$, we must have that $[b, e]$ is an edge in $com(T)$. But this is impossible since both b and e beat d in T , a contradiction.

Subcase 2.5. Suppose that $dom(T) = H_5$. Since $dom(T)$ is contained in $DC(T)$ and $dom(T)$ has a 5-cycle it follows that $DC(T)$ must be the graph in Figure A.1. There is only one possible way, up to isomorphism,

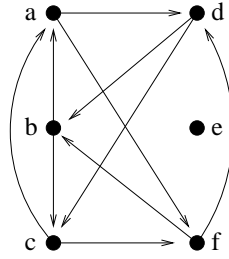
that $dom(T)$ can be contained in $DC(T)$. Therefore, without loss of generality assume that $dom(T)$ is contained in $DC(T)$ as depicted in the following figure.



Lemma 2.2 determines that the domination digraph has two possible orientations; one obtained by orienting the arcs on the 5-cycle clockwise and the other by orienting the arcs on the 5-cycle counterclockwise. First assume that $\mathcal{D}(T)$ has the following orientation.

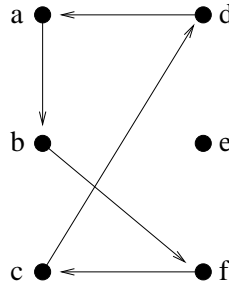


Then by Lemma 2.6, the subgraph of $dom(T)$ induced by the set of vertices $\{a, b, c, d, f\}$ must be a subgraph of $dom(T(\{a, b, c, d, f\}))$. Then by Theorem 1.1, it follows that $dom(T(\{a, b, c, d, f\}))$ is an odd cycle. But by Theorem 2.4, we conclude that $T(\{a, b, c, d, f\})$ is isomorphic to U_5 . Therefore, the following arcs must be in $T(\{a, b, c, d, f\})$ and also in T since $T(\{a, b, c, d, f\})$ is contained in T .

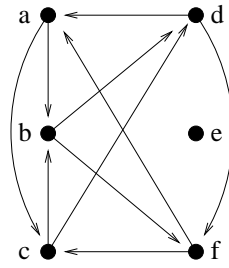


Since $[a, f]$ is an edge in $DC(T)$ and $[a, f]$ is not an edge in $dom(T)$, it follows that $[a, f]$ must be an edge in $com(T)$. This is impossible since a and f beat d in T , a contradiction.

Next assume that $\mathcal{D}(T)$ has the following orientation.



Then by Lemma 2.6, the subgraph of $dom(T)$ induced by the set of vertices $\{a, b, c, d, f\}$ must be a subgraph of $dom(T(\{a, b, c, d, f\}))$. Then by Theorem 1.1, it follows that $dom(T(\{a, b, c, d, f\}))$ is an odd cycle. Thus, by Theorem 2.7, it follows that $T(\{a, b, c, d, f\})$ is isomorphic to U_5 . Therefore, the following arcs must be in $T(\{a, b, c, d, f\})$ and also in T since $T(\{a, b, c, d, f\})$ is contained in T .



Since $[a, f]$ is an edge in $DC(T)$ and $[a, f]$ is not an edge in $dom(T)$, it follows that $[a, f]$ must be an edge in $com(T)$. This is impossible since a and f beat c in T , a contradiction.

Therefore, $K_{3,3}$ is not a subgraph of $DC(T)$. ■

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