Name: ____________________________________________________________

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1. __________ 5. __________
2. __________ 6. __________
3. __________ 7. __________
4. __________ 8. __________

Total __________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
Steve Billups, Julien Langou (Chair), Weldon Lodwick.
1. Find the least squares solution of $Ax = b$ where
\[
A = \begin{pmatrix}
1 & -2 \\
-1 & 2 \\
0 & 3 \\
2 & 5
\end{pmatrix}
\text{ and } b = \begin{pmatrix}
3 \\
1 \\
-4 \\
2
\end{pmatrix}.
\]

Solution

The linear least squares solution $x$ is given by $x = (A^T A)^{-1} A^T b$.

\[
A^T b = \begin{pmatrix}
1 & -1 & 0 & 2 \\
-2 & 2 & 3 & 5
\end{pmatrix}
\begin{pmatrix}
3 \\
1 \\
-4 \\
2
\end{pmatrix}
= \begin{pmatrix}
6 \\
-6
\end{pmatrix}
\]

\[
A^T A = \begin{pmatrix}
1 & -1 & 0 & 2 \\
-2 & 2 & 3 & 5
\end{pmatrix}
\begin{pmatrix}
1 & -2 \\
-1 & 2 \\
0 & 3 \\
2 & 5
\end{pmatrix}
= \begin{pmatrix}
6 & 6 \\
6 & 42
\end{pmatrix}
= 6 \begin{pmatrix}
1 & 1 \\
1 & 7
\end{pmatrix}
\]

\[
(A^T A)^{-1} = \frac{1}{36} \begin{pmatrix}
7 & -1 \\
-1 & 1
\end{pmatrix}
\]

\[
x = (A^T A)^{-1} A^T b = \frac{1}{36} \begin{pmatrix}
7 & -1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
6 \\
-6
\end{pmatrix}
= \frac{1}{6} \begin{pmatrix}
7 & -1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
-1
\end{pmatrix}
= \frac{1}{6} \begin{pmatrix}
8 \\
-2
\end{pmatrix}
= \begin{pmatrix}
4/3 \\
-1/3
\end{pmatrix}.
\]
2. Let \( F \) be a field. Let \( P_1 \) denote the standard vector space of polynomials \( f(t) \) with coefficients in the field \( F \) and having degree at most 1. Let \( S = \{1, t\} \) be the standard ordered basis of \( P_1 \).

(a) Define \( T \in \mathcal{L}(P_1) \) by

\[
T : p(t) = a + bt \mapsto q(t) = 5a - 2b + (4a - b)t.
\]

Construct the matrix \( A = [T]_S \) that represents \( T \) with respect to the basis \( S \). Is there an ordered basis \( B \) for \( P_1 \) such that \( [T]_B \) is diagonal? If so, give such a basis and the corresponding matrix representation. If not, explain why not.

(b) Replace \( T \) of part (a) by \( S \in \mathcal{L}(P_1) \) defined by

\[
S : p(t) = a + bt \mapsto q(t) = -a + b - bt,
\]

and repeat question (a).

Solution

(a) Since \( T(1) = 5 + 4t \), the first column of \([T]_S\) is \( \begin{pmatrix} 5 \\ 4 \end{pmatrix} \). Similarly, \( T(t) = -2 - t \) implies the second column is \( \begin{pmatrix} -2 \\ -1 \end{pmatrix} \). So \( A = [T]_S = \begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix} \).

\( A \) has eigenvalues 3 and 1 with corresponding eigenvectors \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 2 \end{pmatrix} \), respectively. Since \( T \) has \( 2 = \dim(P_1) \) distinct eigenvalues, \( T \) is diagonalizable with diagonalization

\[
S^{-1}AS = D, \text{ with } S = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \text{ and } D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Thus, the desired basis is \( B = \{1 + t, 1 + 2t\} \), for which \( [T]_B = D \).

(b) \( A = [T]_S = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \). This matrix is in Jordan form and is an elementary Jordan block that is not diagonal. Hence \( A \) is not diagonalizable. Therefore, there is no basis for which the corresponding matrix representation is diagonal.
3. Let $A$ be a real matrix. A *generalized inverse* of a matrix $A$ is any matrix $G$ such that $AGA = A$. Prove each of the following:

(a) If $A$ is invertible, the unique generalized inverse of $A$ is $A^{-1}$.
(b) If $G$ is a generalized inverse of $(X^TX)$, then

$$XGX^TX = X.$$ 

(c) For any real symmetric matrix $A$, there exists a generalized inverse of $A$.

**Solution**

(a) $AA^{-1}A = IA = A$, so $A^{-1}$ is a generalized inverse. If $AA^+A = A$, then $AA^+ = AA^+AA^{-1} = AA^{-1} = I$, so $A^+$ is the inverse of $A$.

(b) For arbitrary vector $v$, we can write $v = u + w$, where $u \in \text{null}X^T$ and $w = X\lambda$. Then

$$v^TXGX^TX = (u^T + \lambda^T X^T)XGX^TX = \lambda^T X^TXGX^TX = \lambda^T X^TX = w^TX = v^TX.$$ 

Since $v$ is arbitrary, $XGX^TX = X$.

(c) Since $A$ is real symmetric, it is diagonalizable; so $A = P\Lambda P^T$, where $P$ is orthogonal and $\Lambda$ is diagonal real, with the eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n)$ on the diagonal. Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ where

$$\gamma_i = \begin{cases} 
\frac{1}{\lambda_i} & \text{if } \lambda_i \neq 0 \\
0 & \text{if } \lambda_i = 0.
\end{cases}$$

Let $\Gamma$ be the diagonal matrix with $\gamma$ along the diagonal. Let $G = P\Gamma P^T$.

Since $P$ is orthogonal, $P^TP = I$. Thus,

$$AGA = P\Lambda P^TP\Gamma P^TP\Lambda P^T = P\Lambda\Gamma\Lambda P^T = P\Lambda P^T = A.$$ 

Thus $G$ is a generalized inverse of $A$. 
4. Let $A$ be a real symmetric $n$-by-$n$ matrix which is not just a scalar multiple of the identity matrix. Let $f(x) = (x - 1)(x + 6)^3$ and suppose that $f(A) = 0$ and the trace of $A$ is 0.

(a) Determine the minimal polynomial of $A$.
(b) Determine the trace of $A^2$ as a function of $n$.
(c) Show that $n$ is a multiple of 7.
(d) Determine the characteristic polynomial of $A$ as a function of $n$.

Solution

Since $A$ is real symmetric, its minimal polynomial has no repeated factors, and since $f(A) = 0$ the minimal polynomial divides $f(x)$. Since $A$ is not a scalar times the identity, the minimal polynomial of $A$ has to be exactly $p(x) = (x - 1)(x + 6) = x^2 + 5x - 6$.

Since $p(A) = 0$, we have that $A^2 = -5A + 6I$. So the trace of $A^2$ is $-5(\text{trace}(A)) + 6n = 6n$.

As eigenvalues of $A$, suppose 1 has multiplicity $u$ and $-6$ has multiplicity $v$. (Since $A$ is real symmetric, algebraic and geometric multiplicities are the same.)

On the one hand, we have $u + v = n$. (I.e., for any matrix, the sum of the algebraic multiplicities is always $n$ or, since $A$ is real symmetric, $A$ is diagonalizable, and so the sum of the geometric multiplicities is $n$.) On the other hand, we know that $\text{trace}(A) = 0$ and we know that $\text{trace}(A)$ is the sum of the eigenvalues counting (algebraic – in the general case) multiplicities, therefore $u - 6v = 0$.

Solving $u + v = n$ and $u - 6v = 0$, a system of two linear equations in the two unknowns $u$ and $v$, we find $u = \frac{6n}{7}$ and $v = \frac{n}{7}$, both of which are positive integers. So there is some positive integer $k$ for which $n = 7k$, $u = 6k$, $v = k$. $n$ is a multiple of 7.

The characteristic polynomial is

\[
c_A(x) = (x - 1)^6n(x + 6)^1n.
\]

\[
c_A(x) = (x^7 - 21x^5 + 70x^4 - 105x^3 + 84x^2 - 35x + 6)^\frac{n}{7}.
\]
5. Let $U$ and $W$ be subspaces of the finite-dimensional inner product space $V$.

(a) Prove that $U^\perp \cap W^\perp = (U + W)^\perp$.

(b) Prove that

$$\dim(W) - \dim(U \cap W) = \dim(U^\perp) - \dim(U^\perp \cap W^\perp).$$

**Solution**

Let $x \in U^\perp \cap W^\perp$. Then for any $u \in U$ and $w \in W$, $\langle x, u + w \rangle = \langle x, u \rangle + \langle x, w \rangle = 0$. Thus, $x \in (U + W)^\perp$, so $U^\perp \cap W^\perp \subset (U + W)^\perp$.

For any $y \in (U + W)^\perp$, and any $u \in U$ and $w \in W$, we have $u = u + 0 \in U + W$, so $\langle y, u \rangle = 0$. Similarly, $\langle y, w \rangle = 0$. Thus, $y \in U^\perp \cap W^\perp$. Thus, $(U + W)^\perp \subset U^\perp \cap W^\perp$.

It follows that $(U + W)^\perp = U^\perp \cap W^\perp$, proving part (a).

Keep in mind that for finite-dimensional inner product spaces we know that $\dim(U^\perp) = \dim(V) - \dim(U)$. Then for the proof of (b) consider the following:

$$\dim(U^\perp) - \dim(U^\perp \cap W^\perp) = (\dim(V) - \dim(U)) - \dim((U + W)^\perp)$$

$$= \dim(V) - \dim(U) - (\dim(V) - \dim(U + W))$$

$$= \dim(U) + \dim(W) - \dim(U \cap W) - \dim(U)$$

$$= \dim(W) - \dim(U \cap W), \text{ as desired.}$$
6. Let $B$ be an $n$-by-$n$ Hermitian matrix. Then $B$ has real eigenvalues which we may order as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. For $\overline{0} \neq x \in \mathbb{C}^n$, and using the usual 2-norm $\|x\| = \|x\|_2$, define the Rayleigh Quotient $\rho_B(x)$ for $B$ by

$$\rho_B(x) = \frac{\langle Bx, x \rangle}{\langle x, x \rangle} = \frac{x^*Bx}{\|x\|^2}.$$ 

Prove the following:

(i) If $B$ is an $n$-by-$n$ Hermitian with eigenvalues as above, prove that $\lambda_1 = \max\{\rho_B(x) : x \in \mathbb{C}^n \text{ and } \|x\| = 1\}$.

(ii) Let $A$ be any $n \times n$ complex matrix with largest singular value $\sigma_1$. If $\|A\|_2 = \max\{\|Ax\| : x \in \mathbb{C}^n \text{ and } \|x\| = 1\}$, show that $\|A\|_2 = \sigma_1$.

Solution

First note that if $0 \neq k \in \mathbb{C}$ and $\overline{0} \neq x \in \mathbb{C}^n$, then $\rho_B(kx) = \rho_B(x)$. If we put $O = \{x \in \mathbb{C}^n : \|x\| = 1\}$, then

$$\sup\{\rho_B(x) : \overline{0} \neq x \in \mathbb{C}^n\} = \sup\{\rho_B(x) : x \in O\}.$$ 

Second, since $B$ is hermitian, there is an orthonormal basis $B = (v_1, \ldots, v_n)$ of eigenvectors so that $Bv_j = \lambda_j v_j$, for $j = 1, 2, \ldots, n$. If we put $v_j$ in as the $j$th column of the $n \times n$ matrix $P$, then $P$ is unitary ($P^* = P^{-1}$) and $P^*BP = \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Since $y \mapsto Py = x$ maps $O$ to $O$ in a one-to-one and onto manner, we have

$$\sup\{\rho_B(x) : x \in O\} = \sup\{x^*Bx : x \in O\}$$

$$= \sup\{(Py)^*B(Py) : y \in O\} = \sup\{y^*\Lambda y : y \in O\}$$

$$= \sup\{\sum_{j=1}^n \lambda_j |y_j|^2 : (y_1, \ldots, y_n)^T \in O\}$$

$$\leq \sup\{\lambda_1 \sum_{j=1}^n |y_j|^2 : \sum_{j=1}^n |y_j|^2 = 1\} = \lambda_1.$$ 

So to prove part (i), we just need to find an $x \in O$ for which $\rho_B(x) = \lambda_1$. Clearly $x = v_1$ will work (with $y = P^{-1}x = (1, 0, \ldots, 0)^T$).

For part (ii), we note that $B = A^*A$ is hermitian, and we can adapt the notation of part (i) and use the fact that the largest eigenvalue of $A^*A$ is $\lambda_1 = \sigma_1^2$ to obtain

$$\|A\|_2 = \max\{\|Ax\| : x \in \mathbb{C}^n \text{ and } \|x\| = 1\}$$

$$= \max\{\sqrt{x^*A^*Ax} : x \in O\}$$

$$= \sqrt{\sigma_1^2} = \sigma_1. \text{(By part (i))}$$
7. Let $T$ be a normal operator on a finite-dimensional complex inner product space $V$.

(a) Prove that $T$ is self-adjoint if and only if its eigenvalues are all real.
(b) Prove that $T$ is positive (i.e., positive semidefinite) if and only if all its eigenvalues are nonnegative.

Solution

Since $T$ is normal, by the complex spectral theorem, there is an orthonormal basis $\{e_1, \ldots, e_n\}$ of $V$ consisting of eigenvectors of $T$, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. The matrix of $T$ with respect to the basis $\{e_1, \ldots, e_n\}$ is the diagonal matrix $D = \text{diag} \{\lambda_1, \ldots, \lambda_n\}$.

(a) $T$ is self-adjoint if and only if $D = D^*$ if and only if $\lambda_j = \bar{\lambda}_j$ (i.e., $\lambda_j$ is real) for all $j$.

(b) First suppose $T$ is positive, so $\langle Tv, v \rangle \geq 0$ for all $v \in V$. Then, for each eigenpair $(\lambda_j, e_j)$, $\langle Te_j, e_j \rangle = \langle \lambda_j e_j, e_j \rangle = \bar{\lambda}_j \langle e_j, e_j \rangle = \lambda_j \geq 0$. So all eigenvalues are nonnegative.

Conversely, suppose all eigenvalues are nonnegative. For any $v \in V$, we can write $v = v_1 e_1 + \cdots + v_n e_n$. Then

$$\langle Tv, v \rangle = \left\langle \sum_{j=1}^n T(v_j e_j), v \right\rangle = \sum_{j=1}^n \lambda_j \langle v_j e_j, v \rangle = \sum_{j=1}^n \lambda_j \langle v_j e_j, v_j e_j \rangle \geq 0,$$

so $T$ is positive.
8. (a) (Frobenius inequality) If $A$, $B$, and $C$ are rectangular matrices such that the product $ABC$ is defined, then

$$\text{rank}(AB) + \text{rank}(BC) \leq \text{rank}(B) + \text{rank}(ABC)$$

(b) In particular, prove that

$$\text{rank}(AB) \leq \min \{\text{rank}(A), \text{rank}(B)\}.$$ 

Solution

(a) Let $A$ be $m$-by-$n$, $B$ be $n$-by-$p$, and $C$ be $p$-by-$q$.

We consider $A_{\text{Range}(B)}$, the restriction of $A$ to the subspace $\text{Range}(B)$. We apply the rank theorem to $A_{\text{Range}(B)}$ and get

$$\text{Rank}(B) = \dim \text{Null} \left( A_{\text{Range}(B)} \right) + \text{Rank} \left( A_{\text{Range}(B)} \right).$$

Note that

$$\text{Range} \left( A_{\text{Range}(B)} \right) = \text{Range}(AB).$$

Therefore

$$\text{Rank}(B) = \dim \text{Null} \left( A_{\text{Range}(B)} \right) + \text{Rank} (AB). \quad (1)$$

We now consider $A_{\text{Range}(BC)}$, the restriction of $A$ to the subspace $\text{Range}(BC)$. We apply the rank theorem and follow the same process as above and get:

$$\text{Rank}(BC) = \dim \text{Null} \left( A_{\text{Range}(BC)} \right) + \text{Rank} (ABC). \quad (2)$$

Note that

$$\text{Range}(BC) \subset \text{Range}(B),$$

therefore

$$\dim \text{Null} \left( A_{\text{Range}(BC)} \right) \leq \dim \text{Null} \left( A_{\text{Range}(B)} \right). \quad (3)$$

Combining Equations 1, 2, and 3 gives the Frobenius inequality.

(b) Let $A$ be $m$-by-$n$, $B$ be $n$-by-$p$. We set $C$ to be the zero $p$-by-$p$ matrix. Then the Frobenius inequality applied to the product $ABC$ gives

$$\text{rank}(AB) \leq \text{rank}(B).$$

Now we set $C$ to be the zero $m$-by-$m$ matrix. Then the Frobenius inequality applied to the product $CAB$ gives

$$\text{rank}(AB) \leq \text{rank}(A).$$

In summary,

$$\text{rank}(AB) \leq \min \{\text{rank}(A), \text{rank}(B)\}.$$