Name: ________________________________

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1. ____________ 5. ____________
2. ____________ 6. ____________
3. ____________ 7. ____________
4. ____________ 8. ____________

Total ____________

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Applied Linear Algebra Preliminary Exam Committee:
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1. Let $V$ be a real inner product space with inner product $\langle ., . \rangle$, and suppose that $T \in \mathcal{L}(V)$ is a linear operator $T : V \to V$. Define what an adjoint of $T$ is and show that if $T$ has an adjoint, then this adjoint is unique.

**Solution**

Let $T \in \mathcal{L}(V)$. An adjoint of $T$ is a linear operator $T^* \in \mathcal{L}(V)$ such that

$$\forall x \in V, \forall y \in V, \quad \langle Tx, y \rangle = \langle x, T^* y \rangle.$$ 

**Claim:** The adjoint is unique, if it exists.

**Proof:** Let $A \in \mathcal{L}(V)$ be an adjoint of $T$ and let $B \in \mathcal{L}(V)$ be an adjoint of $T$, then

$$\forall y \in V, \forall x \in V, \quad \langle Tx, y \rangle = \langle x, Ay \rangle \text{ and } \langle Tx, y \rangle = \langle x, By \rangle,$$

so that

$$\forall y \in V, \forall x \in V, \quad \langle x, Ay \rangle = \langle x, By \rangle,$$

using the bilinearity of the inner product,

$$\forall y \in V, \forall x \in V, \quad \langle x, Ay - By \rangle = 0,$$

so that

$$\forall y \in V, \quad (Ay - By) \perp V,$$

but the only vector in $V \perp$ is $0$, so

$$\forall y \in V, \quad Ay - By = 0,$$

so that

$$\forall y \in V, \quad Ay = By,$$

so that

$$A = B.$$

**Note:** The adjoint always exists in finite dimensional inner product spaces. The existence is not necessarily true in infinite dimensional inner product spaces.
2. We consider $\mathcal{M}_n(\mathbb{R})$ the vector space of all $n$–by–$n$ matrices with real coefficients and supplement it with the inner product $(X,Y) \mapsto \text{trace}(X^T Y)$. Let $A \in \mathcal{M}_n(\mathbb{R})$, and

$$
\varphi_A : \mathcal{M}_n(\mathbb{R}) \longrightarrow \mathcal{M}_n(\mathbb{R})
$$

$$
X \mapsto A^T X A
$$

Show that $\varphi_A \in \mathcal{L}(\mathcal{M}_n(\mathbb{R}))$ and compute the adjoint of $\varphi_A$.

**Solution**

We have, $\forall \lambda \in \mathbb{R}, \forall \mu \in \mathbb{R}, \forall X \in \mathcal{M}_n(\mathbb{R}), \forall Y \in \mathcal{M}_n(\mathbb{R})$,

$$
\varphi_A(\lambda X + \mu Y) = A^T (\lambda X + \mu Y) A = \lambda (A^T X A) + \mu (A^T Y A) = \lambda \varphi_A(X) + \mu \varphi_A(Y).
$$

So $\varphi_A \in \mathcal{L}(\mathcal{M}_n(\mathbb{R}))$.

Let $X \in \mathcal{M}_n(\mathbb{R})$ and let $Y \in \mathcal{M}_n(\mathbb{R})$,

$$
\langle \varphi_A(X), Y \rangle = \text{trace} ((\varphi_A(X))^T Y) = \text{trace} ((A^T X A)^T Y) = \text{trace} (A^T X^T YA),
$$

we now use the fact that, $\{ \forall A \in \mathcal{M}_n(\mathbb{R}), \forall B \in \mathcal{M}_n(\mathbb{R}), \text{trace}(AB) = \text{trace}(BA)\}$

$$
= \text{trace} (X^T AYA^T) = \text{trace} (X^T (AYA^T)) = \text{trace} (X^T (\varphi_A(Y))) = \langle X, \varphi_A(Y) \rangle
$$

So

$$
(\varphi_A)^* = \varphi_{A^T}.
$$
3. (a) Let $A$ be a real symmetric $n$–by–$n$ matrix. Prove that $A$ is positive definite, i.e., $x^T A x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, if and only if all the eigenvalues of $A$ are positive.

(b) Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$. Put $V = \mathbb{R}^3$. Define the map $*: V \times V \to \mathbb{R}$ by $u * v = u^T A v$ for all $u, v \in V$. Prove that $*$ is an inner product on $V$.

(c) Use the inner product from above and the Gram-Schmidt orthogonalization process to find an orthonormal basis for $V$.

**Solution**

(a) Let $A$ be a real symmetric matrix. We recall that, by definition, $A$ is positive definite if and only $\forall x \in \mathbb{R}^n \setminus \{0\}, x^T A x > 0$. We also recall that a symmetric matrix is diagonalizable in an orthonormal basis with real eigenvalues, so, for our matrix $A$, there exists $\Lambda$ a diagonal $n$–by–$n$ matrix with real coefficients and $V$ a unitary matrix such that $A = V \Lambda V^T$.

Let $A$ be positive definite. Let $\lambda_i$ be an eigenvalue of $A$ and $v_i$ a unit-norm eigenvector associated to $\lambda_i$, (so that $v_i^T v_i = 1$ and $A v_i = \lambda_i v_i$,) then since $A$ is positive definite, we have $v_i^T A v_i > 0$ which means $\lambda_i > 0$. We have proven that if $A$ is positive definite then all eigenvalues of $A$ are positive. (Alternatively, this direction can be proven by contradiction because otherwise $v_i^T A v_i = \lambda_i v_i^T v_i = \lambda_i ||v_i||^2 < 0$ and $A$ was not positive definite.)

Let all eigenvalues of $A$ be positive. Let $x \in \mathbb{R}^n \setminus \{0\}$. We have $x^T A x = x^T V \Lambda V^T x = (V^T x)^T \Lambda (V^T x) = \sum_{i=1}^n \lambda_i (V^T x)_i^2 > 0$. So $A$ is positive definite.

(b) $A$ is symmetric, moreover the eigenvalues of $A$ are 2 and 4 and so are positive, using the previous question, we deduce that $A$ is symmetric positive definite. We check that $*$ satisfies the properties of an inner product on $V$.

i. $x * y = y * x$,

ii. $(\lambda x) * y = \lambda (x * y)$,

iii. $(x + y) * z = (x * z) + (y * z)$,

iv. $x * x \geq 0$ with equality only for $x = 0$.

(i) comes from the symmetry of $A$, (ii) and (iii) comes from the linearity of $A$, (iv) comes from the positive definiteness of $A$.

(c) We take the elementary basis and use the Gram-Schmidt process on it to obtain an orthonormal basis for $V$. We obtain

$$q_1 = \begin{pmatrix} \sqrt{2}/2 \\ 0 \\ 0 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 0 \\ \sqrt{3}/3 \\ 0 \end{pmatrix}, \quad q_3 = \begin{pmatrix} 0 \\ \sqrt{6}/12 \\ \sqrt{6}/4 \end{pmatrix}.$$
4. Let \( M_n(\mathbb{R}) \) be the vector space of all \( n \times n \) matrices with real coefficients, and \( A \in M_n(\mathbb{R}) \) be diagonalizable. We have a nonsingular matrix \( W \) and a diagonal matrix \( \Lambda \), such that \( A = W\Lambda W^{-1} \). Define
\[
B = \begin{pmatrix} 0 & -A \\ 2A & 3A \end{pmatrix}.
\]
Prove that \( B \) is diagonalizable and give the diagonalization of \( B \) (i.e. the \( 2m \) eigencouples of \( B \)).
(Hint: one can first consider the \( m = 1 \) case where \( A = 1 \).)

**Solution**

Let
\[
M = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}.
\]
We have
\[
p_M(x) = \det(\begin{pmatrix} -x & -1 \\ 2 & 3 - x \end{pmatrix}) = x^2 - 3x + 2 = (x - 1)(x - 2).
\]
Since \( M \) has two distinct eigenvalues, \( \lambda_1 = 1 \) and \( \lambda_2 = 2 \), \( M \) is diagonalizable.
Then we look for the eigenvectors of \( M \), we find (for example)
\[
v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.
\]
If we call
\[
V = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix},
\]
we obtain the following diagonalization for \( M \)
\[
M = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix} = VDV^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}.
\]
Extending this relation to blocks, one can check that
\[
B = \begin{pmatrix} 0 & -A \\ 2A & 3A \end{pmatrix} = \begin{pmatrix} I & I \\ -I & -2I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 2A \end{pmatrix} \begin{pmatrix} 2I & I \\ -I & -I \end{pmatrix}.
\]
Using the fact that \( A \) is diagonalizable, there exists a nonsingular matrix \( W \) and a diagonal matrix \( \Lambda \), such that \( A = W\Lambda W^{-1} \). So
\[
B = \begin{pmatrix} 0 & -A \\ 2A & 3A \end{pmatrix} = \begin{pmatrix} I & I \\ -I & -2I \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} \Lambda & 0 \\ 0 & 2A \end{pmatrix} \begin{pmatrix} W^{-1} & 0 \\ 0 & W^{-1} \end{pmatrix} \begin{pmatrix} 2I & I \\ -I & -I \end{pmatrix}.
\]
which gives the diagonalization of $B$

$$B = \begin{pmatrix} 0 & -A \\ 2A & 3A \end{pmatrix} = \begin{pmatrix} W & W \\ -W & -2W \end{pmatrix} \begin{pmatrix} \Lambda & 0 \\ 0 & 2\Lambda \end{pmatrix} \begin{pmatrix} 2W^{-1} & W^{-1} \\ -W^{-1} & -W^{-1} \end{pmatrix}. $$

One can check that

$$\left( \begin{pmatrix} W & W \\ -W & -2W \end{pmatrix} \right)^{-1} = \begin{pmatrix} 2W^{-1} & W^{-1} \\ -W^{-1} & -W^{-1} \end{pmatrix}. $$
5. Let $V$ be a vector space over the real numbers $\mathbb{R}$. Let $U_1, U_2, U_3$ be subspaces of $V$.

(a) Prove that $U_1 \subseteq U_3$ implies that $U_1 + (U_2 \cap U_3) = (U_1 + U_2) \cap U_3$ (modular law).

(b) Give examples to show that none of the following distributive laws holds, in general. $U_1 \cap (U_2 + U_3) = (U_1 \cap U_2) + (U_1 \cap U_3)$ and $U_1 + (U_2 \cap U_3) = (U_1 + U_2) \cap (U_1 + U_3)$

Solution

(a) Let $U_1 \subseteq U_3$.
One the one hand, we have that $U_1 + (U_2 \cap U_3) \subseteq U_1 + U_2$, on the other, $U_1 + (U_2 \cap U_3) \subseteq U_3$, so that

$$U_1 + (U_2 \cap U_3) \subseteq (U_1 + U_2) \cap U_3.$$ 

Now let $z \in (U_1 + U_2) \cap U_3$, then there exists $z_1 \in U_1$ and $z_2 \in U_2$ such that $z = z_1 + z_2$ so $z_2 = z - z_1 \in U_3$, so $z_2 \in U_2 \cap U_3$. Therefore $z \in U_1 + (U_2 \cap U_3)$ and so

$$(U_1 + U_2) \cap U_3 \subseteq U_1 + (U_2 \cap U_3).$$

We conclude that

$$(U_1 + U_2) \cap U_3 = U_1 + (U_2 \cap U_3).$$

(b) $U_1 \cap (U_2 + U_3) = (U_1 \cap U_2) + (U_1 \cap U_3)$ does not hold in general. Consider

$$U_1 = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \quad U_2 = \text{Span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \quad \text{and} \quad U_3 = \text{Span}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right).$$

Then

$$(U_2 + U_3) = \mathbb{R}^2, \quad U_1 \cap (U_2 + U_3) = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \quad \text{but}$$

$$(U_1 \cap U_2) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}, \quad (U_1 \cap U_3) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}, \quad (U_1 \cap U_2) + (U_1 \cap U_3) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$ 

$U_1 + (U_2 \cap U_3) = (U_1 + U_2) \cap (U_1 + U_3)$ does not hold in general. Consider again

$$U_1 = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \quad U_2 = \text{Span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \quad \text{and} \quad U_3 = \text{Span}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right).$$

Then

$$(U_2 \cap U_3) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}, \quad U_1 + (U_2 \cap U_3) = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \quad \text{but}$$

$$(U_1 + U_2) = \mathbb{R}^2, \quad (U_1 + U_3) = \mathbb{R}^2, \quad (U_1 + U_2) \cap (U_1 + U_3) = \mathbb{R}^2.$$
6. Let \((u_1, u_2, \ldots, u_m)\) be an orthonormal basis for subspace \(W \neq \{0\}\) of the vector space \(V = \mathbb{R}^n\) (under the standard inner product), let \(U\) be the \(n\)–by–\(m\) matrix defined by \(U = [u_1, u_2, \ldots, u_m]\), and let \(P\) be the \(n\)–by–\(n\) matrix defined by \(P = UU^T\).

(a) Prove that if \(v\) is any given member of \(V\), then among all the vectors \(w\) in \(W\), the one which minimizes \(\|v - w\|\) is given by \(w = (v \cdot u_1)u_1 + (v \cdot u_2)u_2 + \ldots + (v \cdot u_m)u_m\) where \(v \cdot u\) is the standard inner product. (The vector \(w\) is called the projection of \(v\) onto \(W\).)

(b) Prove: For any vector \(v \in V\), the projection \(w\) of \(v\) onto \(W\) is given by \(w = P v\).  

(c) Prove: \(P\) is a projection matrix. (Recall that a matrix \(P \in M_n(\mathbb{R})\) is called a projection matrix if and only if \(P\) is symmetric \((P^T = P)\) and idempotent \((P^2 = P)\).)

(d) If \(V = \mathbb{R}^3\), and \(W = \text{Span}\{(1, 2, 2)^T, (1, 0, 1)^T\}\), find the projection matrix \(P\) described above and use it to find the projection of \((2, 2, 2)^T\) onto \(W\).

**Solution**

(a) First it is clear that \(w \in W\). Note as well that \(v - w \perp W\) since for all \(x \in W\),

\[
(v - w \cdot x) = ((v - (v \cdot u_1)u_1 - \ldots - (v \cdot u_m)u_m), x) = (v, x) - (v \cdot u_1)(u_1, x) - \ldots - (v \cdot u_m)(u_m, x) = 0.
\]

The last equality comes from the fact that since \(x \in W\), \(x = (x \cdot u_1)u_1 + \ldots + (x \cdot u_m)u_m\).

Now consider \(x \in W\). We define

\[
\|v - x\|^2 = \|(v - w) + (w - x)\|^2 = \|v - w\|^2 + 2(v - w) \cdot (w - x) + \|w - x\|^2
\]

Since \(v - w \perp W\) and \(w - x \in W\), we have that \((v - w) \cdot (w - x) = 0\), so that

\[
\|v - x\|^2 = \|v - w\|^2 + \|w - x\|^2
\]

We see that the minimum for \(\|v - x\|\) is \(\|v - w\|^2\) and is realized when \(x = w\).

(b)

\[
w = (v \cdot u_1)u_1 + (v \cdot u_2)u_2 + \ldots + (v \cdot u_m)u_m = u_1(u_1^Tv) + u_2(u_2^Tv) + \ldots + u_m(u_m^Tv) = (u_1^Tu_1 + u_2^Tu_2 + \ldots + u_m^Tu_m)v = UU^Tv = Pv.
\]

(c) First, \(P^T = (UU^T)^T = UU^T = P\), second, \(P^2 = (UU^T)^2 = U(U^TU)U^T = UU^T = P\) where we have used the fact that \(U^TU = I\).
(d) An orthogonal basis for $W$ is for example

$$(u_1, u_2) = \left( \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix} \right).$$

We get


Finally

$$w = Px = \begin{pmatrix} 14/9 \\ 16/9 \\ 22/9 \end{pmatrix}.$$
7. Let $V$ be a real inner product space with inner product $\langle ., . \rangle_V$ and let $W$ be a real inner product space with inner product $\langle ., . \rangle_W$ such that $\dim V = \dim W = n < \infty$. Show that there exists a bijective linear mapping $f : V \rightarrow W$ so that $\langle x, y \rangle_V = \langle f(x), f(y) \rangle_W$ for all $x, y \in V$.

**Solution**

Let \{\(v_1, \ldots, v_n\)\} be an orthonormal basis of $V$ and let \{\(w_1, \ldots, w_n\)\} be an orthonormal basis of $W$. We define the linear mapping $f : V \rightarrow W$ so that

\[\forall i = 1, \ldots, n, \ f(v_i) = w_i.\]

We note that $f$ is correctly and uniquely defined and is bijective.

*Claim: $f$ conserves the scalar product (from $V$ to $W$).*

Let $x \in V$, let $y \in V$, then we can decompose $x$ and $y$ onto the orthonormal basis \{\(v_1, \ldots, v_n\)\} as follows:

\[x = \sum_{i=1}^{n} v_i \langle v_i, x \rangle_V \quad \text{and} \quad y = \sum_{j=1}^{n} v_j \langle v_j, y \rangle_V.\]

(1)

We form the inner product $\langle x, y \rangle_V$ and get

\[\langle x, y \rangle_V = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle v_i, v_j \rangle_V \langle v_i, x \rangle_V \langle v_j, y \rangle_V.\]

Using the bilinearity of the inner product $\langle ., . \rangle_V$

\[= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle v_i, x \rangle_V \langle v_i, v_j \rangle_V \langle v_j, y \rangle_V,\]

Using the orthonormality of \{\(v_1, \ldots, v_n\)\}, we get

\[= \sum_{i=1}^{n} \langle v_i, x \rangle_V \langle v_i, y \rangle_V,\]

Therefore we have

\[\langle x, y \rangle_V = \sum_{i=1}^{n} \langle v_i, x \rangle_V \langle v_i, y \rangle_V,\]

(2)

Back to Equation (1), Applying $f$ and using the linearity of $f$, we get:

\[f(x) = \sum_{i=1}^{n} f(v_i) \langle v_i, x \rangle_V \quad \text{and} \quad f(y) = \sum_{j=1}^{n} f(v_j) \langle v_j, y \rangle_V.\]
And using the definition of $f$, we get

$$f(x) = \sum_{i=1}^{n} w_i \langle v_i, x \rangle_V \quad \text{and} \quad f(y) = \sum_{j=1}^{n} w_j \langle v_j, y \rangle_V.$$  

We now form the inner product $\langle f(x), f(y) \rangle_W$ and get

$$\langle f(x), f(y) \rangle_W = \left( \sum_{i=1}^{n} w_i \langle v_i, x \rangle_V \right) \left( \sum_{j=1}^{n} w_j \langle v_j, y \rangle_V \right).$$

Using the bilinearity of the inner product $\langle \cdot, \cdot \rangle_W$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle v_i, x \rangle_V \langle w_i, w_j \rangle_W \langle v_j, y \rangle_V,$$

Using the orthonormality of $\{w_1, \ldots, w_n\}$, we get

$$= \sum_{i=1}^{n} \langle v_i, x \rangle_V \langle v_i, y \rangle_V.$$

Using Equation (2), we conclude that

$$\langle f(x), f(y) \rangle_W = \langle x, y \rangle_V.$$
8. Let \( n \) a natural integer, \( \mathcal{M}_n(\mathbb{C}) \) be the vector space of all \( n \times n \) matrices with complex coefficients, and \( A = (a_{ij})_{ij} \in \mathcal{M}_n(\mathbb{C}) \). Show that

\[
\text{Spectrum}(A) \subset \bigcup_{i=1}^{n} \left\{ B'\left(a_{ii}, \sum_{1 \leq j \leq n, j \neq i} |a_{ij}| \right) \right\},
\]

where we define for any \( a \in \mathbb{C} \) and any \( r \in [0, +\infty) \), \( B'(a, r) \) by

\[
B'(a, r) = \{ z \in \mathbb{C}, |z - a| \leq r \}.
\]

The \( B'\left(a_{ii}, \sum_{1 \leq j \leq n, j \neq i} |a_{ij}| \right) \) are called the Gershgorin circles of \( A \).

Solution

Let \( \lambda \in \text{Spectrum}(A) \) and consider an associated eigenvector \( x \in \mathbb{R}^n \). (So that \( x \neq 0 \) and \( Ax = x\lambda \).) We write the equality \( Ax = x\lambda \) row by row and get

\[
\forall i = 1, \ldots, n, \quad \sum_{j=1}^{n} a_{ij} x_j = x_i \lambda.
\]

Consider \( i_0 \) such that

\[
|x_{i_0}| = \max_{i=1, \ldots, n} |x_i|.
\]

(Note that \( |x_{i_0}| \neq 0 \) since \( x \neq 0 \).) Then we get:

\[
|x_{i_0}(\lambda - a_{i_0i_0})| = \left| \sum_{1 \leq j \leq n, j \neq i_0} a_{i_0j} x_j \right|,
\]

\[
\leq \sum_{1 \leq j \leq n, j \neq i_0} |a_{i_0j}| |x_j|,
\]

\[
\leq \left( \sum_{1 \leq j \leq n, j \neq i_0} |a_{i_0j}| \right) |x_{i_0}|.
\]

Since \( |x_{i_0}| \neq 0 \),

\[
|\lambda - a_{i_0i_0}| \leq \left( \sum_{1 \leq j \leq n, j \neq i_0} |a_{i_0j}| \right).
\]

So

\[
\lambda \in \left\{ B'\left(a_{i_0i_0}, \sum_{1 \leq j \leq n, j \neq i_0} |a_{i_0j}| \right) \right\}.
\]
So

\[ \lambda \in \bigcup_{i=1}^{n} \left\{ B' \left( a_{ii}, \sum_{1 \leq j \leq n} \mid a_{ij} \mid \right) \right\}. \]

Since \( \lambda \) was an arbitrary eigenvalue

\[ \text{Spectrum}(A) \subset \bigcup_{i=1}^{n} \left\{ B' \left( a_{ii}, \sum_{1 \leq j \leq n} \mid a_{ij} \mid \right) \right\}. \]