University of Colorado at Denver — Mathematics Department
Applied Linear Algebra Preliminary Exam With Solutions
1 June 2009, 10:00 am – 2:00 pm

Name: ________________________________

The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in.

Exam conditions:

• Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
• Each problem is worth 20 points; parts of problems have equal value unless stated otherwise.
• Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
• If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
• Begin each solution on a new page and use additional paper, if necessary.
• Write legibly using a dark pencil or pen.
• Notation: \( \mathbb{C} \) denotes the field of complex numbers, \( \mathbb{R} \) denotes the field of real numbers, and \( F \) denotes a field which may be either \( \mathbb{C} \) or \( \mathbb{R} \). \( \mathbb{C}^n \) and \( \mathbb{R}^n \) denote the vector spaces of \( n \)-tuples of complex and real scalars, respectively, written as column vectors. For \( T \in \mathcal{L}(V) \), the image (sometimes called the range) of \( T \) is denoted \( \text{Im}(T) \). If \( A \) is a matrix over a field, then \( \text{rk}(A) \) is the rank of \( A \). For \( x \in \mathbb{R}^n \), \( \| : \| \) denotes the usual Euclidean norm. If \( A \) is an \( m \times n \) matrix over \( F \), \( T_A \) is the linear map defined by

\[
T_A : F^n \to F^m : x \mapsto Ax. 
\]

\( T^* \) is the adjoint of the operator \( T \) and \( \lambda^* \) is the complex conjugate of the scalar \( \lambda \). \( v^T \) and \( A^T \) denote vector and matrix transposes, respectively.
• Ask the proctor if you have any questions.

Good luck!

1. ____________  5. ____________
2. ____________  6. ____________
3. ____________  7. ____________
4. ____________  8. ____________

Total ____________
1. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Define $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by

$$
T : B \mapsto AB - BA.
$$

(i) (8 points) Fix an ordered basis $\mathcal{B}$ of $M_2(\mathbb{R})$ and compute the matrix $[T]_{\mathcal{B}}$ that represents $T$ with respect to this basis.

(ii) (8 points) Compute a basis for each of the eigenspaces of $T$.

(iii) (4 points) Give the minimal and characteristic polynomials of $T$ and the Jordan form for $T$.

**Solution:** We choose the "standard ordered basis"

$$
\mathcal{B} = (E_{11}, E_{12}, E_{21}, E_{22})
$$

where $E_{ij}$ has a 1 in the $(i, j)$ position and zero elsewhere.

Then routine computations show that

$$
T : E_{11} \mapsto 2(E_{21} - E_{12})
$$

$$
T : E_{12} \mapsto 2(E_{22} - E_{11})
$$

$$
T : E_{21} \mapsto 2(E_{11} - E_{22})
$$

$$
T : E_{22} \mapsto 2(E_{12} - E_{21})
$$

From this it is easy to write down the matrix $[T]_{\mathcal{B}}$, and then write down

$$
\lambda I - [T]_{\mathcal{B}} = \begin{pmatrix}
\lambda & 2 & -2 & 0 \\
2 & \lambda & 0 & -2 \\
-2 & 0 & \lambda & 2 \\
0 & -2 & 2 & \lambda
\end{pmatrix}.
$$

In a few steps we evaluate the determinant of this matrix and find that the characteristic polynomial of $T$ is

$$
f(\lambda) = \lambda^2(\lambda - 4)(\lambda + 4).
$$

Put $\lambda = 0$ in the matrix and row reduce to find that the null space has the basis $(E_{11} + E_{22}, E_{12} + E_{21})$

A basis for the eigenspace with $\lambda = 4$ is

$$
(E_{11} - E_{12} + E_{21} - E_{22}).
$$

A basis for the eigenspace with $\lambda = -4$ is

$$
(E_{11} + E_{12} - E_{21} - E_{22}).
$$
So the Jordan form is a diagonal matrix with diagonal entries 0, 0, 4, −4, and the minimal polynomial is 
\[ p(\lambda) = \lambda(\lambda - 4)(\lambda + 4). \]

2. Let \( A \in M_6(\mathbb{C}) \) be defined by
\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 \\
\end{bmatrix}.
\]

Find all of the eigenvalues, eigenvectors, and generalized eigenvectors of \( A \). Construct the characteristic polynomial, the minimal polynomial, and the Jordan form of \( A \).

**Solution:** The characteristic polynomial is \( x^2(x + 1)^4 \) and the eigenvalues are 0 and -1. The eigenvectors associated with eigenvalue 0 are of the form \( [0, a, 0, b, 0, c]^T \) and the generalized eigenvectors are of the form \( [a, b, 0, 0, 0, 0]^T \). The eigenvectors associated with -1 are of the form \( [0, 0, a, b, -a]^T \), and the generalized eigenvectors are of the form \( [0, 0, a, b, a]^T \). The minimal polynomial is \( x^2(x + 1)^3 \). One Jordan form is
\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}.
\]

3. Norms of Linear Operators

(a) Let \( A \) be an \( m \times n \) real matrix. Prove that there is a real constant \( M_A \) such that \( \|Ax\| \leq M_A \|x\| \) for all \( x \in \mathbb{R}^n \).

**Solution:** We use two basic facts: First, \( \|(x_1, \ldots, x_m)\| \leq \sum_{i=1}^m |x_i| \), by the triangle inequality; Second, recall the Cauchy-Schwartz inequality: \( |x \cdot v| \leq \|x\| \cdot \|v\| \). So we have:
\[
\|Av\| = \|(a_{ij}) \begin{pmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{pmatrix} \| = \left\| \begin{pmatrix}
(a_{i1}, \ldots, a_{in}) \cdot v \\
\vdots \\
(a_{m1}, \ldots, a_{mn}) \cdot v
\end{pmatrix} \right\| \leq \sum_{i=1}^m \|(a_{i1}, \ldots, a_{in}) \cdot v\| \leq \sum_{i=1}^m \|(a_{i1}, \ldots, a_{in})\| \cdot \|v\| = M_A \|v\|,
\]
where \( M_A = \sum_{i=1}^m \sqrt{a_{i1}^2 + a_{i2}^2 + \cdots + a_{in}^2} \).
(b) Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Prove that there is some positive constant $\|T\|$ for which

$$\|T(v)\| \leq \|T\| \cdot \|v\|$$

for all $v \in \mathbb{R}^n$.

**Solution:** If $T = T_A: \mathbb{R}^n \to \mathbb{R}^m$ with $A = (a_{ij})$, then $\|T(v)\| = \|Av\| \leq M_A \|v\|$ (by part (a)) for all $v \in \mathbb{R}^n$, where $M_A = \sum_{i=1}^m \sqrt{a_{i1}^2 + a_{i2}^2 + \cdots + a_{im}^2} \in \mathbb{R}$.

4. Spheres in Finite Dimensional Real Vector Spaces

Let $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ be an ordered basis of the real vector space $V$ with dimension $n$. For each $v \in V$ there are unique scalars $c_1, \ldots, c_n \in \mathbb{R}$ for which $v = \sum_{i=1}^n c_i v_i$. Write the coordinate matrix $[v]_\mathcal{B}$ of $v$ with respect to the ordered basis $\mathcal{B}$ as

$$[v]_\mathcal{B} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \text{ so that } v = (v_1, \ldots, v_n) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathcal{B} \cdot [v]_\mathcal{B}.$$

For $c = [c_1, \ldots, c_n]^T \in \mathbb{R}^n$, we employ the usual Euclidean norm:

$$\|c\| = \sqrt{\sum_{i=1}^n c_i^2}.$$

For an arbitrary ordered basis $\mathcal{B}$ of $V$, we define the norm with respect to $\mathcal{B}$ as follows:

$$\|v\|_\mathcal{B} := \|[v]_\mathcal{B}\|.$$

Given the basis $\mathcal{B}$, a specific vector $v_0$ and a positive number $r$ we can define the $n$-dimensional sphere with center $v_0$ and radius $r$ (with respect to $\mathcal{B}$) by

$$S_{r,\mathcal{B}}(v_0) = \{w \in V : \|v_0 - w\|_\mathcal{B} \leq r\}.$$

**Problem** Let $r > 0$ and let $\mathcal{B}$, $\mathcal{B}'$ be any two ordered bases of $V$. Show that there is an $r' > 0$ such that

$$S_{r,\mathcal{B}}(0) \subseteq S_{r',\mathcal{B}'}(0).$$

**Solution:** If $\mathcal{B}$ and $\mathcal{B}'$ are two given ordered bases of $V$, there is an invertible, real $n \times n$ matrix $A$ for which $\mathcal{B}' = \mathcal{B}A$, so that $[v]_\mathcal{B} = A \cdot [v]_\mathcal{B}'$. Then there is a constant $\|A\|$ such that $\|AX\| \leq \|A\| \cdot \|X\|$ for any $X \in \mathbb{R}^n$. Hence

$$\|v\|_\mathcal{B} = \|[v]_\mathcal{B}\| = \|A \cdot [v]_\mathcal{B}'\| \leq \|A\| \cdot \|[v]_\mathcal{B}'\| = \|A\| \cdot \|v\|_\mathcal{B}'.$$

From this we see that

$$S_{r,\mathcal{B}A}(0) \subseteq S_{r\|A\|,\mathcal{B}'}(0).$$

Of course the argument is symmetric in $\mathcal{B}$ and $\mathcal{B}'$. 

\[\text{(1)}\]
5. Fredholm Alternative

Let $A$ be an $m \times n$ real matrix and $b \in \mathbb{R}^m$. Show that exactly one of the following systems has a solution:

i) $Ax = b$
ii) $A^Ty = 0$, $y^Tb \neq 0$.

Note: Our notation is $y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$, so $y^T = [y_1, \ldots, y_m]$.

Solution: If $b \in \text{col}A$, then statement i) has a solution, but since $\text{col}A \perp \text{null}A^T$, statement ii) has no solution.

If $b \not\in \text{col}A$, then statement i) does not have a solution. In this case, let $z = \text{proj}_{\text{col}A}b$ (the orthogonal projection of $b$ onto the column space of $A$), and define $y = b - z$.

Note that $y \neq 0$ (since $b \not\in \text{col}A$). Note also that since $z$ is an orthogonal projection, $y \in (\text{col}A)^\perp = \text{null}A^T$. Thus, $A^Ty = 0$ and $y^Tb = y^T(y + z) = y^Ty \neq 0$, so statement ii) has a solution.

6. Upper-triangularization

(a) (12 points) For each of the following, if it is true, merely say so; if it is false, give a counterexample.

(i) If $V$ is a finite-dimensional vector space over $\mathbb{R}$ and $T \in \mathcal{L}(V)$, then $V$ has a basis $B$ with respect to which $[T]_B$ is upper triangular.

Solution: FALSE Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2: (x, y) \mapsto (y, -x)$. Then if $S$ is the standard ordered basis of $\mathbb{R}^2$, the matrix

$$[T]_S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so that the characteristic polynomial of $T$ is $x^2 + 1$. This polynomial has no real roots, so $T$ has no real eigenvalues, which would have to lie along the diagonal of $[T]_B$ if $V$ had such a basis.

(ii) If $V$ is a finite-dimensional vector space over $\mathbb{C}$ and $T \in \mathcal{L}(V)$, then $V$ has a basis $B$ with respect to which $[T]_B$ is upper triangular.

Solution: TRUE (This is usually called the Theorem of Schur.)

(iii) If $V$ is a finite-dimensional vector space over $\mathbb{C}$ and $S, T \in \mathcal{L}(V)$, then $V$ has a basis $B$ for which both $[S]_B$ and $[T]_B$ are upper triangular.

Solution: False Suppose that with respect to some basis $B'$, $S$ and $T$ have the following matrices:

$$[S]_{B'} = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix} \quad \text{and} \quad [T]_{B'} = \begin{pmatrix} 0 & 1 \\ d & 0 \end{pmatrix},$$
with \( c \neq d \). Then a basis \( \mathcal{B} \) of the desired type would exist if and only if there were an invertible matrix \( P = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \) for which

\[
\begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix} \begin{pmatrix} h & -f \\ -g & e \end{pmatrix} = \begin{pmatrix} \ast & \ast \\ ch^2 - g^2 & \ast \end{pmatrix},
\]

with a similar equation holding for the other matrix. It follows that there would have to be an invertible matrix \( P \) as above with \( g^2 = ch^2 \) and \( g^2 = dh^2 \).

If \( g = 0 \), then \( h = 0 \), implying that \( c = 0 \) and \( d = 0 \), contradicting \( c \neq d \).

(b) (8 points) Show that a normal, upper triangular matrix must be diagonal.

**Solution:** We may assume that \( A \) is \( n \times n \) with entries in \( \mathbb{C} \), with \( A_{kj} = 0 \) if \( k > j \). Then

\[
\overline{A}_{11}A_{11} = \sum_{k=1}^{n} \overline{A}_{k1}A_{k1} = \sum_{k=1}^{n} A_{k1}^*A_{k1} = (A^*A)_{11} = \]

\[
= (AA^*)_{11} = \sum_{k=1}^{n} A_{1k}A_{k1}^* = \sum_{k=1}^{n} A_{1k}\overline{A}_{1k}.
\]

It now follows that \( A_{12} = A_{13} = \cdots = A_{1n} = 0 \). Consider the \((2, 2)\) entry.

\[
(AA^*)_{22} = \sum_{k=1}^{n} A_{2k}(A^*)_{k2} = \sum_{k=2}^{n} A_{2k}\overline{A}_{2k}.
\]

This must also equal

\[
(A^*A)_{22} = \sum_{k} (A^*)_{2k}A_{k2} = \sum_{k=1}^{2} (A^*)_{2k}A_{k2} = \overline{A}_{22}A_{22}.
\]

It follows that \( A_{23} = A_{24} = \cdots = A_{2n} \).

Proceed down the rows to show recursively that in fact \( A \) must be diagonal.

7. Tournament Matrices

The matrices of this problem are all \( n \times n \) with real entries.

(a) Show that if the matrix \( A \) is skew-symmetric then \( I + A \) is nonsingular.

(b) Show that for arbitrary matrices \( A \) and \( B \), \( \text{rk}(A + B) \leq \text{rk}(A) + \text{rk}(B) \).

(c) If \( A \) is arbitrary and \( J \) is the matrix of all 1’s, then show that \( \text{rk}(A - J) \geq \text{rk}(A) - \text{rk}(J) \).

(d) If \( M \) is a \((0, 1)\)-matrix with zeros on the main diagonal and with \( M_{ij} = 0 \) if and only if \( M_{ji} = 1 \), show that \( \text{rk}(M) \geq n - 1 \). (Such a matrix is called a tournament matrix.)
Solution: Suppose $A^T = -A$ and that $X$ is a column vector for which $(I + A)X = 0$. Then $AX = -X$ implies that $X = (-A)X = A^T X$, so $X^T A = X^T$. Then $X^T X = (X^T A)X = X^T (AX) = X^T (-X) = -X^T X$, which implies that $X^T X = 0$, and hence $X = 0$. So 0 is not an eigenvalue of $I + A$. For the second part, observe that the union of a maximal independent set of rows of $A$ with a maximal independent set of rows of $B$ will certainly span the row space of $A + B$. For the third part, apply the second part to the matrix $A = (A - J) + J$. For the last part, let $M$ be a tournament matrix of order $n$. Then $M + M^T = J - I$, i.e., $J = I + M + M^T$. Clearly $M - M^T$ is skew-symmetric, so $A = I + M - M^T$ is nonsingular by the first part. Hence $\text{rk}(A) = n$. Then $\text{rk}(A - J) \geq \text{rk}(A) - \text{rk}(J) = n - 1$. But $A - J = -2M^T$, so $\text{rk}(M) = \text{rk}(M^T) = \text{rk}(A - J) \geq n - 1$.

8. Given an $m \times n$ matrix $A$, the **pseudoinverse of** $A$, denoted $A^+$, can be defined as the matrix such that for all $b \in \mathbb{C}^m$, $x^+ := A^+b$ is the least squares solution to the equation $Ax = b$ that has the smallest norm.

(a) Using the above definition, explain why $AA^+$ and $A^+A$ must be projection matrices (and are therefore Hermitian). Onto what fundamental subspaces do these matrices project?

(b) Prove that $AA^+A = A$ and $A^+AA^+ = A^+$. (Note: these two properties, together with the Hermitian properties in part (a) uniquely determine the pseudoinverse).

(c) If $\Sigma$ is a real diagonal matrix, what is $\Sigma^+$?

(d) Give an explicit formula for $A^+$ in terms of the singular value decomposition $A = V\Sigma W^*$. Justify your answer.

Solution:

(a) For $x^+$ to be a least squares solution to $Ax = b$, $Ax^+$ must be the orthogonal projection of $b$ onto the column space of $A$. Let $p(b)$ be this projection. Then $AA^+b = Ax^+ = p(b)$. It follows that $AA^+$ is the projection matrix onto the column space of $A$.

Since $x^+$ is the least norm solution to $Ax = p(b)$, it must lie in the row space of $A$.

Consider any $y \in \mathbb{C}^n$. Let $b = Ay$ and $x^+ = A^+b = A^+Ay$. Since $b$ is in the column space of $A$, $p(b) = b$. It follows that $Ax^+ = p(b) = Ay$, so $A(x^+ - y) = 0$. Thus $x^+ = A^+Ay$ is the orthogonal projection onto the row space of $A$.

(b) Observe that $A^+b = A^+p(b)$. Thus, for any $b$, $A^+AA^+b = A^+Ax^+ = A^+p(b) = A^+b$. Since this is true for all $b$, $A^+AA^+ = A^+$. Similarly, for any $y$, $AA^+Ay = Ax^+ = Ay$. Thus, $AA^+A = A$.

(c) $\Sigma^+$ is the diagonal matrix with entries

$$
\Sigma^+_{ii} = \begin{cases} 
1/\Sigma_{ii}, & \text{if } \Sigma_{ii} \neq 0; \\
0, & \text{otherwise.}
\end{cases}
$$

It is easy to verify that $\Sigma \Sigma^+ = \Sigma^+ \Sigma$ is diagonal (and hence Hermitian); $\Sigma \Sigma^+ \Sigma = \Sigma$, and $\Sigma^+ \Sigma \Sigma^+ = \Sigma^+$. Thus $\Sigma^+$ is the pseudoinverse.
(d) $A^+ = W\Sigma^+ V^*$. To prove that this is the pseudoinverse, check each of the properties. Let $D := \Sigma \Sigma^+$, and observe that $D$ is diagonal with $D_{ii} = 0$ if $\Sigma_{ii} = 0$, and $D_{ii} = 1$ otherwise. Then

- $AA^+ V \Sigma^+ W^* = V \Sigma \Sigma^+ V^* = VDV^*$, which is clearly Hermitian.
- Similarly, $A^+ A = W \Sigma^+ V^* V \Sigma W^* = WDW^*$, which is Hermitian.
- $AA^+ A = V \Sigma W^* W \Sigma^+ V^* V \Sigma W^* = VD\Sigma W^* = V \Sigma W^* = A$.
- $A^+ AA^+ = W \Sigma^+ V^* V \Sigma W^* V \Sigma^+ V^* = W D \Sigma^+ W^* = W \Sigma^+ W^* = A^+$. 