The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in.

Exam conditions:

- Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $C$ denotes the field of complex numbers, $R$ denotes the field of real numbers, and $F$ denotes a field which may be either $C$ or $R$. $C^n$ and $R^n$ denote the vector spaces of $n$-tuples of complex and real scalars, respectively. $T^*$ is the adjoint of the operator $T$ and $\lambda^*$ is the complex conjugate of the scalar $\lambda$. $v^T$ and $A^T$ denote vector and matrix transposes, respectively.
- Ask the proctor if you have any questions.

Good luck!

1. __________
2. __________
3. __________
4. __________
5. __________
6. __________
7. __________
8. __________

Total __________
On this exam $V$ is a finite dimensional vector space over the field $F$, where either $F = \mathbb{C}$, the field of complex numbers, or $F = \mathbb{R}$, the field of real numbers. Also, $F^n$ denotes the vector space of column vectors with $n$ entries from $F$, as usual. For $T \in \mathcal{L}(V)$, the image (sometimes called the range) of $T$ is denoted $\text{Im}(T)$.

1. Suppose that $P \in \mathcal{L}(V)$ (the vector space of linear maps from $V$ to itself) and that $P^2 = P$.

   (a) (6 points) Determine all possible eigenvalues of $P$.
   
   (b) (10 points) Prove that $V = \text{null}(P) \oplus \text{Im}(P)$.
   
   (c) (4 points) Is it necessary that all possible eigenvalues found in part (a) actually must occur? Prove that your answer is correct.

**Solution:** $P^2 - P = 0$ implies that the minimal polynomial $p(x)$ of $P$ divides $x^2 - x = x(x - 1)$. Hence $p(x) = x$, or $(x - 1)$, or $x(x - 1)$. So in general the eigenvalues are each equal to either 0 or 1. But $p(x) = x$ if and only if $P = 0$, in which case $V = \text{null}(P)$ and $\{0\} = \text{Im}(P)$. And $p(x) = x - 1$ if and only if $P = I$. In this case $V = \text{Im}(P)$ and $\text{null}(P) = \{0\}$. In these two cases the condition in part (b) clearly holds, and we see that part (c) is also answered.

Finally, suppose $p(x) = x(x - 1)$, so that both 0 and 1 are eigenvalues of $P$. If $v \in \text{null}(P) \cap \text{Im}(P)$, then $P(v) = 0$ on the one hand, and on the other hand there is some $w \in V$ for which $v = P(w) = P^2(w) = P(v) = 0$. Hence $\text{null}(P) \cap \text{Im}(P) = \{0\}$. But also for any $v \in V$ we have $v = (v - P(v)) + P(v)$, where $P(v - P(v)) = P(v) - P(v) = 0$. So $v - P(v) \in \text{null}(P)$ and clearly $P(v) \in \text{Im}(P)$. Hence $V = \text{null}(P) \oplus \text{Im}(P)$. This finishes part (b).

2. Define $T \in \mathcal{L}(F^n)$ by $T: (w_1, w_2, w_3, w_4)^T \mapsto (0, w_2 + w_4, w_3, w_4)^T$.

   (a) (8 points) Determine the minimal polynomial of $T$.
   
   (b) (6 points) Determine the characteristic polynomial of $T$.
   
   (c) (6 points) Determine the Jordan form of $T$.

**Solution:** Let $p(x)$ be the minimal polynomial of $T$. It is easy to see that $T(1, 0, 0, 0) = 0$, so 0 is an eigenvalue of $T$ and hence $x$ is a divisor of $p(x)$. Also, $T(0, 1, 0, 0) = (0, 1, 0, 0)$, so 1 is an eigenvalue of $T$ and $x - 1$ divides $p(x)$. Since $T^2(x_1, x_2, x_3, x_4) = (0, x_2 + 2x_4, x_3, x_4)$, it is clear that $\text{null}(T) = \text{null}(T^2) = \{a, 0, 0, 0\} : a \in F$, hence the dimension of the space of generalized eigenvectors of $T$ associated with 0 is 1. This says that the multiplicity of 0 as a root of the characteristic polynomial $f(x)$ of $T$ is 1. So we check for eigenvalue 1. $(T - I)(x_1, x_2, x_3, x_4) = (-x_1, x_4, 0, 0)$. Repeating this we see $(T - I)^2(x_1, x_2, x_3, x_4) = (x_1, 0, 0, 0)$, which is in the null space of $T$. Hence $T(T - I)^2 = 0$. Since $T(T - I)(x_1, x_2, x_3, x_4) = (0, x_4, 0, 0)$, clearly $T(T - I)$ is not the zero operator, hence $p(x) = x(x - 1)^2$. This finishes part (a).

Part (b): Since the dimension of the space of generalized eigenvectors belonging to 0 is 1, it must be that the dimension of the space of generalized eigenvectors belonging to 1 is 3. Hence the characteristic polynomial of $T$ must be $f(x) = x(x - 1)^3$. 

Part (c) Since the minimal polynomial of $T$ is $x(x - 1)^2$ and the characteristic polynomial is $x(x - 1)^3$, the only possibility (up to the order of the diagonal blocks) for the Jordan form of $T$ is:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

3. Let $T$ be a normal operator on a complex inner product space $V$ of dimension $n$.

(a) (10 points) If $T(v) = \lambda v$ with $0 \neq v \in V$, show that $v$ is an eigenvector of the adjoint $T^*$ with associated eigenvalue $\overline{\lambda}$.

(b) (10 points) Show that $T^*$ is a polynomial in $T$.

Solution to part (a):

$$
T(v) = \lambda v \iff 0 = \| (T - \lambda I)(v) \|^2 \\
= \langle (T - \lambda I)v, (T - \lambda I)v \rangle = \langle v, (T^* - \overline{\lambda} I)(T - \lambda I)v \rangle \\
= \langle v, (T - \lambda I)(T^* - \overline{\lambda} I)v \rangle = \| (T^* - \overline{\lambda} I)v \|^2 \\
\iff T^*(v) = \overline{\lambda} v.
$$

Solution to part (b): Since $T$ is a normal operator on a complex vector space $V$, there is an orthonormal basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ of $V$ consisting of eigenvectors of $T$. Suppose that $T(v_i) = \lambda_i v_i$ for $1 \leq i \leq n$. So by part (a) we know that $T^*(v_i) = \overline{\lambda}_i v_i$, for $1 \leq i \leq n$. WLOG we may assume that the eigenvalues have been ordered so that $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the distinct eigenvalues of $T$. Using Lagrange interpolation (or any method have at hand) construct a polynomial $f(x) \in \mathbb{C}[x]$ (having degree at most $r - 1$, if desired), such that $f(\lambda_i) = \overline{\lambda_i}$, for $1 \leq i \leq r$. Then $f(T)(v_j) = f(\lambda_j)(v_j) = \overline{\lambda_j}(v_j) = T^*(v_j)$, $1 \leq j \leq r$, so that $f(T)$ and $T^*$ have the same effect on each member of the basis $\mathcal{B}$. This implies that $f(T) = T^*$.

4. Let $A$ and $B$ be $n \times n$ Hermitian matrices over $\mathbb{C}$.

(a) (10 points) If $A$ is positive definite, show that there exists an invertible matrix $P$ such that $P^*AP = I$ and $P^*BP$ is diagonal.

(b) (10 points) If $A$ is positive definite and $B$ is positive semidefinite, show that

$$
\det(A + B) \geq \det(A).
$$

Solution:

(a) Since $A$ is positive definite, there exists an invertible matrix $T$ such that $A = T^*T$. $(T^{-1})^*B(T^{-1})$ is Hermitian, so is diagonalizable. That is, there exists a unitary matrix $U$ and a diagonal matrix $D$ such that $U^*(T^{-1})^*B(T^{-1})U = D$. Let $P = T^{-1}U$. Then $P^*BP = D$, and

$$
P^*AP = U^*(T^{-1})^*(T^*T)T^{-1}U = U^*U = I.
$$
(b) Let $P$ and $D$ be as defined above. Then $A = (P^{-1})^* P^{-1}$ and $B = (P^*)^{-1} D P^{-1}$. Since $B$ is positive semidefinite, then the diagonal entries in $D$ are nonnegative. Thus

$$
\det(A + B) = \det \left((P^{-1})^* (I + D) P^{-1}\right) = \det \left((P^*)^{-1} (I - P^{-1}) P^{-1}\right) \det(1 + D) = \det(1 + D) \geq \det A.
$$

5. Let $\| \cdot \| : \mathbb{C}^n \to \mathbb{R}$ be defined by

$$
\|x\|_{\infty} = \max_{1 \leq i \leq n} |x_i|.
$$

(a) (8 points) Prove that $\| \cdot \|_{\infty}$ is a norm.

(b) (12 points) A norm $\| \cdot \|$ is said to be derived from an inner product if there is an inner product $\langle \cdot , \cdot \rangle$ such that $\|x\| = \langle x, x \rangle^{1/2}$ for all $x \in \mathbb{C}^n$. Show that $\| \cdot \|_{\infty}$ cannot be derived from an inner product.

**Solution:**

(a) We verify the properties of norms:

i. $\|x\|_{\infty} = \max_{1 \leq i \leq n} |x_i| \geq 0$, for all $x \in \mathbb{C}^n$.

ii. $\|x\|_{\infty} = 0 \iff \max_{1 \leq i \leq n} |x_i| = 0 \iff x = 0$.

iii. For any $c \in \mathbb{C}$ and $x \in \mathbb{C}^n$, $\|cx\|_{\infty} = \max_{1 \leq i \leq n} |cx_i| = |c| \max_{1 \leq i \leq n} |x_i| = |c| \|x\|_{\infty}$.

iv. For all $x, y \in \mathbb{C}^n$, $\|x + y\|_{\infty} = \max_{1 \leq i \leq n} |x_i + y_i| \leq \max_{1 \leq i \leq n} |x_i| + \max_{1 \leq i \leq n} |y_i| = \|x\|_{\infty} + \|y\|_{\infty}$.

(b) Assume there exists an inner product $\langle \cdot , \cdot \rangle$ such that $\|x\|_{\infty} = \langle x, x \rangle^{1/2}$ for all $x \in \mathbb{C}^n$. Then for any $x, y \in \mathbb{C}^n$, we have

$$
\|x + y\|_{\infty}^2 + \|x - y\|_{\infty}^2 = 2 \langle x, x \rangle + 2 \langle y, y \rangle = 2 \|x\|_{\infty}^2 + 2 \|y\|_{\infty}^2.
$$

But, choosing $x = (1, 0, \ldots, 0)^T$ and $y = (0, 1, 0, \ldots, 0)^T$, this yields the following contradiction:

$$
2 = \|x + y\|_{\infty}^2 + \|x - y\|_{\infty}^2 = 2 \|x\|_{\infty}^2 + 2 \|y\|_{\infty}^2 = 2 + 2 = 4.
$$

(One of our theorems said that a norm is derived from an inner product if and only if it satisfies the parallelogram equality, so this type of proof should naturally come to mind.)

6. Suppose that $F = \mathcal{C}$ and that $S, T \in \mathcal{L}(V)$ satisfy $ST = TS$. Prove each of the following:

(a) (4 points) If $\lambda$ is an eigenvalue of $S$, then the eigenspace

$$
V_\lambda = \{x \in V | Sx = \lambda x\}
$$

is invariant under $T$.

(b) (4 points) $S$ and $T$ have at least one common eigenvector (not necessarily belonging to the same eigenvalue).
(c) (12 points) There is a basis $B$ of $V$ such that the matrix representations of $S$ and $T$ are both upper triangular.

Solution:

(a) If $x \in V_\lambda$, then $Sx = \lambda x$. Thus,

$$S(Tx) = TSx = T(\lambda x) = \lambda Tx,$$

so $Tx \in V_\lambda$.

(b) Let $T|_{V_\lambda}$ denote the restriction of $T$ to the subspace $V_\lambda$. $T|_{V_\lambda}$ has at least one eigenvector $v \in V_\lambda$, with eigenvalue $\mu$. It follows that $Tv = T|_{V_\lambda}v = \mu v$, so $v$ is an eigenvector of $V$. And since $v \in V_\lambda$, it is also an eigenvector of $S$.

(c) The matrix of a linear transformation with respect to a basis $\{v_1, \ldots, v_n\}$ is upper triangular if and only if $\text{span}(v_1, \ldots, v_k)$ is invariant for each $k = 1, \ldots, n$.

Using part (b) above, we shall construct a basis $\{v_1, \ldots, v_n\}$ for $V$ such that $\text{span}(v_1, \ldots, v_k)$ is invariant under both $S$ and $T$ for each $k$.

We proceed by induction on $n$, the dimension of $V$, with the result being clearly true if $n = 1$. So suppose that $n > 1$ with the desired result holding for all operators on spaces of positive dimension less than $n$. By part (b) there is a vector $v_1 \in V$ such that $Tv_1 = \lambda_1 v_1$ and $Sv_1 = \mu_1 v_1$ for some scalars $\lambda_1$ and $\mu_1$.

Let $W$ be the subspace spanned by $v_1$. Then the dimension of the quotient space $V/W$ is $n - 1$, and the operators $\bar{T}$ and $\bar{S}$ induced on $V/W$ commute, so by our induction hypothesis there is a basis $B_1 = (v_2 + W, v_3 + W, \ldots, v_n + W)$ of $V/W$ with respect to which both $\bar{T}$ and $\bar{S}$ have upper triangular matrices.

It follows that $B = (v_1, v_2, \ldots, v_n)$ is a basis of $V$ with respect to which both $T$ and $S$ have upper triangular matrices.

7. Let $F = \mathcal{L}$ and suppose that $T \in \mathcal{L}(V)$.

(a) (10 points) Prove that the dimension of $\text{Im}(T)$ equals the number of nonzero singular values of $T$.

(b) (10 points) Suppose that $T \in \mathcal{L}(V)$ is positive semidefinite. Prove that $T$ is invertible if and only if $\langle T(x), x \rangle > 0$ for every $x \in V$ with $x \neq 0$.

Solution:

Let $T \in \mathcal{L}(V)$. Since $T^*T$ is self-adjoint, there is an orthonormal basis $(v_1, \ldots, v_n)$ of $V$ whose members are eigenvectors of $T^*T$, say $T^*Tv_j = \lambda_j v_j$, for $1 \leq j \leq n$. Note $\|Tv_j\|^2 = \langle Tv_j, Tv_j \rangle = \langle T^*Tv_j, v_j \rangle = \lambda_j \|v_j\|^2$, so in particular $\lambda_j \geq 0$.

Then $T^*T$ has real, nonnegative eigenvalues, so we may suppose they are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > \lambda_{r+1} = \cdots = \lambda_n = 0$. Put $s_i = \sqrt{\lambda_i}$, $1 \leq i \leq n$, so that $s_1 \geq s_2 \geq \cdots \geq s_r > 0$ are the nonzero singular values of $T$ and in general $s_i = \|Tv_i\|$, for all $i = 1, 2, \ldots, n$. It follows that $(v_{r+1}, v_{r+2}, \ldots, v_n)$ is a basis of the null space of $T$ and $(Tv_1, Tv_2, \ldots, Tv_r)$ is a basis for the Image of $T$. Clearly $r$ is the number of nonzero singular values and also the dimension of the range of $T$, finishing part (a).

(b) Suppose that $T \in \mathcal{L}(V)$ is positive semidefinite, i.e., $T$ is self-adjoint and $\langle T(v), v \rangle \geq 0$ for all $v \in V$. Since $T$ is self-adjoint we know there is an operator $S$ for which $T = S^*S$. So $\langle T(v), v \rangle = \langle S^*S(v), v \rangle = \langle S(v), S(v) \rangle = 0$ if and only if $S(v) = 0$. So $T$
is invertible if and only if $S$ is invertible iff $S(v) \neq 0$ whenever $v \neq 0$ iff $\langle T(v), v \rangle > 0$ whenever $v \neq 0$.

8. Let $N$ be a real $n \times n$ matrix of rank $n - m$ and nullity $m$. Let $L$ be an $m \times n$ matrix whose rows form a basis of the left null space of $N$, and let $R$ be an $n \times m$ matrix whose columns form a basis of the right null space of $N$. Put $Z = L^T R^T$. Finally, put $M = N + Z$.

(a) (2 points) For $x \in \mathbb{R}^n$, show that $N^T x = 0$ if and only if $x = L^T y$ for some $y \in \mathbb{R}^n$.

(b) (2 points) For $x \in \mathbb{R}^n$, show that $N x = 0$ if and only if $x = R y$ for some $y \in \mathbb{R}^m$.

(c) (4 points) Show that $Z$ is an $n \times n$ matrix with rank $m$ for which $N^T Z = 0$, $N Z^T = 0$ and $M M^T = N N^T + Z Z^T$.

(d) (12 points) Show that the eigenvalues of $M M^T$ are precisely the positive eigenvalues of $N N^T$ and the positive eigenvalues of $Z Z^T$, and conclude that $M M^T$ is nonsingular.

Solution:

(a) $N^T X = 0$ iff $X^T N = 0$ iff $X^T$ is in the row space of $L$, i.e., iff $X^T = Y^T L$ for some $Y \in \mathbb{R}^m$, iff $X = L^T Y$ for some $Y \in \mathbb{R}^m$.

(b) $N X = 0$ iff $X$ is in the column space of $R$, i.e., iff $X = R Y$ for some $Y \in \mathbb{R}^m$.

(c) $N^T Z = N^T L^T R^T = 0$ by part (a). Similarly, $N Z^T = N R L = 0$ by part (b). The columns of $L^T$ are independent and $m$ in number, so $Z v = 0$ iff $L^T (R^T v) = 0$ iff $R^T v = 0$. Since $R^T$ is $m \times n$ with rank $m$ and right nullity $n - m$, $Z = L^T R^T$ must have nullity $n - m$, and hence rank $m$. It now is easy to compute $M M^T = (N + Z)(N^T + Z^T) = N N^T + N Z^T + Z N^T + Z Z^T = N N^T + Z Z^T$.

(d) $N N^T$ and $Z Z^T$ are real, symmetric commuting matrices (both products are 0), so there must be an orthogonal basis $B = (v_1, v_2, \ldots, v_n)$ of $\mathbb{R}^n$ consisting of eigenvectors of both $N N^T$ and $Z Z^T$. We know that all the eigenvalues of $N N^T$ and $Z Z^T$ are real and nonnegative. Suppose that $v_i$ is a member of $B$ for which $N N^T v_i = \lambda_i v_i \neq 0$. $v_i$ must be orthogonal to all the vectors in the right null space of $N N^T$, i.e., $v_i$ orthogonal to the right null space of $N N^T$. This says $v_i^T Z = 0$, which implies $Z Z^T v_i = 0$. Hence each $v_i$ not in the null space of $N N^T$ must be in the null space of $Z Z^T$. $N$ and $Z$ play symmetric roles, so a similar argument shows that each $v_j$ not in the right null space of $Z Z^T$ must be in the null space of $N N^T$. Hence we may assume that the members of $B$ are ordered so that $v_1, \ldots, v_{n - m}$ are not in the null space of $N N^T$ and are in the null space of $Z Z^T$. Similarly, $v_{n - m + 1}, \ldots, v_n$ are in the null space of $N N^T$ and not in the null space of $Z Z^T$. It follows immediately that $v_1, \ldots, v_{n - m}$ are eigenvectors of $M M^T$ belonging to the positive eigenvalues of $N N^T$ and $v_{n - m + 1}, \ldots, v_n$ are eigenvectors of $M M^T$ belonging to the positive eigenvalues of $Z Z^T$. Finally, since all the eigenvalues of $M M^T$ are positive (i.e., none of them is zero), $M M^T$ is nonsingular.