1. Let \((x_n)\) and \((y_n)\) be Cauchy sequences in a metric space \((X,d)\). Prove that the sequence 
\((d(x_n, y_n))\) converges regardless of whether or not \((x_n)\) or \((y_n)\) converges.

Observe that \(d(x_n, y_n)\) is in the complete space \(\mathbb{R}\), so all we need to show is that \((d(x_n, y_n))\)
is Cauchy. Let \(\varepsilon > 0\) and \(n_0\) such that \(d(x_n, x_m) + d(y_n, y_m) < \varepsilon\) whenever \(n, m > n_0\).

Then
\[
d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_n, y_m) - d(x_m, y_m) = d(x_n, x_m) + d(y_n, y_m) < \varepsilon.
\]

As the statement is symmetric in \(n\) and \(m\), this implies that \(|d(x_n, y_n) - d(x_m, y_m)| < \varepsilon\).
2. Let \( f : [0, 1] \to \mathbb{R} \) satisfy

(a) \( f(0) = f(1) = 0 \)
(b) \( f(x) > 0, x \in (0, 1), \) and
(c) \( f \) is continuous.

Prove that there exists \( x \in (0, 1) \) satisfying

\[
\int_0^x f(u) \, du = x f(x).
\]

[Hint: use Intermediate Value Theorem]

Consider the continuous function

\[
g(x) = x f(x) - \int_0^x f(u) \, du.
\]

As continuous functions attain a maximum on a closed interval, there is an \( x^* \in (0, 1) \) for which \( f(x) \) is maximized. Then

\[
g(x^*) = x^* f(x^*) - \int_0^{x^*} f(u) \, du > x^* f(x^*) - \int_0^{x^*} f(x^*) \, du = 0,
\]

where the inequality follows from the fact that by continuity there exists an \( \varepsilon > 0 \) such that \( f(u) < f(x^*)/2 \) for all \( u \in [0, \varepsilon] \). Similarly, we have

\[
g(1) = -\int_0^1 f(u) \, du < 0,
\]

as there exists an \( \varepsilon > 0 \) such that \( f(u) > f(x^*)/2 \) for all \( u \in [x^* - \varepsilon, x^*] \). By the intermediate value theorem, there exists \( x \in (x^*, 1) \) with \( g(x) = 0 \).
3. Consider the following proposition:

\textit{Every bounded continuous real-valued function on } \mathbb{R} \textit{attains its maximum.}

The following argument has an error. Find the error and provide a counterexample that the argument indeed fails at that point:

Let \( f(x) \leq M \), where \( M \) is some constant, and let \( f^* = \sup \{ f(x) : x \in \mathbb{R} \} \). Clearly, \( f^* \leq M \). Now let \( x_n \to x^* \) such that \( f(x_n) \to f^* \). Then, since \( f \) is continuous, \( f(x_n) \to f(x^*) \), so \( f(x^*) = f^* \). Hence, \( x^* \) is where \( f \) attains its maximum.

A sequence with \( f(x_n) \to f^* \) does not necessarily have a convergent subsequence. Consider \( f(x) = -\frac{1}{1 + x^2} \) (clearly bounded and continuous). Then \( f^* = 0 \), but no subsequence of any sequence with \( f(x_n) \to 0 \) converges.
4. Let \( f \) and \( g \) be continuous maps of a metric space \((X, d_X)\) into a metric space \((Y, d_Y)\) and let \( E \) be a dense subset of \( X \). If \( g(x) = f(x) \) for all \( x \in E \), prove that \( g(x) = f(x) \) for all \( x \in X \).

Consider the continuous function \( h(x) = f(x) - g(x) \) on \( X \). Let \( x \in X \). As \( E \) is dense in \( X \), there exists a sequence \((x_n) \in E\) with \( x_n \to x \). Notice that \( h(x_n) = 0 \) for all \( n \). By continuity, \( h(x) = \lim h(x_n) = 0 \), so \( g(x) = f(x) \).
5. Prove the following theorem from Rudin:

Suppose $f$ maps a convex open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, $f$ is differentiable in $E$, and there is a real number $M$ such that

$$\|f'(x)\| \leq M$$

for every $x \in E$. Then

$$|f(b) - f(a)| \leq M|b - a|$$

for all $a \in E$, $b \in E$.

See Rudin.
6. Let $F$ be an equicontinuous set of functions from a metric space $(X, d_X)$ to metric space $(Y, d_Y)$. Let $\bar{F}$ be the set of functions defined as pointwise limits of sequences of functions in $F$. Show that $\bar{F}$ is equicontinuous.

Let $x \in X$, $\varepsilon > 0$, and $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ whenever $f \in F$ and $d_X(x, y) < \delta$. Let $g \in \bar{F}$, and let $(f_n) \in F$ converging pointwise to $g$.

Let $y \in X$ with $d_X(x, y) < \delta$. Let $n$ be large enough such that $d_Y(f_n(x), g(x)) + d_Y(f_n(y), g(y)) < \varepsilon$. Then

$$d_Y(g(x), g(y)) < d_Y(f_n(x), g(x)) + d_Y(f_n(y), f_n(y)) + d_Y(f_n(y), g(y)) < 2\varepsilon.$$
7. Let $\ell^1$ be the metric space of all real sequences, $x = (\xi_j)$, such that $\sum_{j=1}^{\infty} \xi_j$ converges absolutely and where the distance between two sequences, $x = (\xi_j)$ and $y = (\eta_j)$, is given by

$$d(x, y) = \sum_{j=1}^{\infty} |\xi_j - \eta_j|.$$ 

Let $\ell^\infty$ be the metric space of all bounded sequences and where the distance between two sequences $x$ and $y$ is given by $d(x, y) = \sup_j |\xi_j - \eta_j|$.

We know that $\ell^1$ and $\ell^\infty$ are metric spaces and that $\ell^1 \subset \ell^\infty$.

Is $\ell^1$ closed in $\ell^\infty$? If yes, prove it. If not, provide a counterexample.

No. Consider $x_n = (\xi^n_i)$, where

$$\xi^n_i = \begin{cases} \frac{1}{i}, & \text{if } i \leq n, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, each $x_n$ is in $\ell^1$, and $x_n \to (\frac{1}{i})$ in $\ell^\infty$. But $\sum \frac{1}{i} = \infty$, so $(\frac{1}{i}) \notin \ell^1$. 