Analysis Prelim—July 2014

Name:

- All seven answers will be graded, the problem with the lowest point score will be dropped.
- Be sure to show all your work.
- Only write on one side of each sheet.
- Start a new sheet of paper for every problem, and write your name and the problem number on every sheet.
- If you use a statement from Rudin or class, state it. If you are unsure if a statement must be proved or may merely be stated, ask your friendly proctor.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>∑</th>
</tr>
</thead>
</table>
1. Let \( (x_n) \) and \( (y_n) \) be Cauchy sequences in a metric space \( (X, d) \). Prove that the sequence \( (d(x_n, y_n)) \) converges regardless of whether or not \( (x_n) \) or \( (y_n) \) converges.

2. Let \( f : [0, 1] \to \mathbb{R} \) satisfy
   
   (a) \( f(0) = f(1) = 0 \)
   
   (b) \( f(x) > 0, x \in (0, 1) \), and
   
   (c) \( f \) is continuous.

   Prove that there exists \( x \in (0, 1) \) satisfying
   
   \[ \int_0^x f(u) \, du = xf(x). \]

   [Hint: use Intermediate Value Theorem]

3. Consider the following proposition:
   
   Every bounded continuous real-valued function on \( \mathbb{R} \) attains its maximum.

   The following argument has an error. Find the error and provide a counterexample that the argument indeed fails at that point:

   Let \( f(x) \leq M \), where \( M \) is some constant, and let \( f^* = \sup \{ f(x) : x \in \mathbb{R} \} \). Clearly, \( f^* \leq M \). Now let \( x_n \to x^* \) such that \( f(x_n) \to f^* \). Then, since \( f \) is continuous, \( f(x_n) \to f(x^*) \), so \( f(x^*) = f^* \). Hence, \( x^* \) is where \( f \) attains its maximum.

4. Let \( f \) and \( g \) be continuous maps of a metric space \( (X, d_X) \) into a metric space \( (Y, d_Y) \) and let \( E \) be a dense subset of \( X \). If \( g(x) = f(x) \) for all \( x \in E \), prove that \( g(x) = f(x) \) for all \( x \in X \).

5. Prove the following theorem from Rudin:
   
   Suppose \( f \) maps a convex open set \( E \subset \mathbb{R}^n \) into \( \mathbb{R}^m \), \( f \) is differentiable in \( E \), and there is a real number \( M \) such that
   
   \[ \|f'(x)\| \leq M \]

   for every \( x \in E \). Then
   
   \[ |f(b) - f(a)| \leq M|b - a| \]

   for all \( a \in E \), \( b \in E \).

6. Let \( F \) be an equicontinuous set of functions from a metric space \( (X, d_X) \) to metric space \( (Y, d_Y) \). Let \( \bar{F} \) be the set of functions defined as pointwise limits of sequences of functions in \( F \). Show that \( \bar{F} \) is equicontinuous.

7. Let \( \ell^1 \) be the metric space of all real sequences, \( x = (\xi_j) \), such that \( \sum_{j=1}^{\infty} \xi_j \) converges absolutely and where the distance between two sequences, \( x = (\xi_j) \) and \( y = (\eta_j) \), is given by
   
   \[ d(x, y) = \sum_{j=1}^{\infty} |\xi_j - \eta_j|. \]

   Let \( \ell^\infty \) be the metric space of all bounded sequences and where the distance between two sequences \( x \) and \( y \) is given by \( d(x, y) = \sup_j |\xi_j - \eta_j| \).

   We know that \( \ell^1 \) and \( \ell^\infty \) are metric spaces and that \( \ell^1 \subset \ell^\infty \).

   Is \( \ell^1 \) closed in \( \ell^\infty \)? If yes, prove it. If not, provide a counterexample.