1. Let \( x_1 = 1 \) and \( x_n = \sqrt{3 + \sqrt{x_{n-1}}} \), \( n > 1 \). Prove that \( \{x_n\} \) converges.

**Solution:** First we show that \( \{x_n\} \) is an increasing sequence. \( x_2 = 2 > x_1 \). Suppose \( x_k > x_{k-1} \), \( k < n \). Then
\[
x_n^2 - x_{n-1}^2 = (3 + \sqrt{x_{n-1}}) - (3 + \sqrt{x_{n-2}}) = \sqrt{x_{n-1}} - \sqrt{x_{n-2}} > 0.
\]
To show that the sequence is bounded note that \( x_1 < 3 \), and suppose \( x_k < 3 \), \( k < n \). Then
\[
x_n^2 = 3 + \sqrt{x_{n-1}} \leq 3 + \sqrt{3} < 9,
\]
so \( x_n < 3 \) as well. Since the sequence is increasing and bounded, it must converge, e.g., Rudin, Theorem 3.14.

2. Prove that Cauchy sequences converge.

**Solution:** Standard, e.g., Rudin, Theorem 3.11.

3. Prove that if \( \{f_n\} \) is a sequence of Riemann integrable functions, and \( f_n \to f \) uniformly on \([a, b]\) then \( f \) is Riemann integrable on \([a, b]\).

**Solution:** Standard, e.g., Rudin, Theorem 7.16.

4. Let \( f(x) \) be continuously differentiable with \( f(0) < -1 \), \( f(1) > 0 \), and \( f(2) < 0 \). Prove that \( \forall c \in [0, 1], \exists x_c \in (0, 2) \) with \( f'(x_c) = c \).

**Solution:** By the MVT there is \( z_1 \in (0, 1) \) with \( f'(z_1) > 1 \). By the MVT, there is \( z_2 \in (1, 2) \) with \( f'(z_2) < 0 \). Since \( f'(x) \) is continuous, the IVT implies that for any \( c \in (0, 1) \), there is \( x_c \in (z_1, z_2) \) with \( f'(x) = c \).

5. Prove that
\[
\begin{align*}
\text{(a)} \quad & \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) = \infty \\
\text{(b)} \quad & \sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right) < \infty
\end{align*}
\]

**Solution:** Since
\[
\lim_{n \to \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n \to \infty} \frac{\sin\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} = 1
\]
(a) diverges since \( \sum \frac{1}{n} \) diverges, and (b) converges since \( \sum \frac{1}{n^2} \) converges.
(e.g., Corollary 4.3.12, Trench)
6. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be $F(x, y) = (x + y, x^2 + y^2)$.

(a) Find $A = \{(x, y) \in \mathbb{R}^2 : F \text{ is not locally invertable at } (x, y)\}$. Demonstrate that $F$ is not one-to-one in any neighborhood of $A$.

(b) Find a first order approximation of $F$ at $(x, y) \in \mathbb{R}^2$. (Note: A first order approximation of $F$ at $(x, y)$ is an affine function $G$ such that $\|F(u, v) - G(u, v)\| \sim o(\|(x, y) - (u, v)\|)$. Is the approximation valid on $A$?

Solution: (a) The derivative matrix of $F$ is
\[
dF = \begin{bmatrix} 1 & 1 \\ 2x & 2y \end{bmatrix}.
\]
Since $\det(dF) = 2y - 2x$, $F$ is locally invertable except when $y = x$, so $A$ is the line $y = x$. We can see that $F$ is not one-to-one in any neighborhood of $A$ since $F(x + \epsilon, x - \epsilon) = F(x - \epsilon, x + \epsilon)$ for any $\epsilon > 0$.

(b) Since $F$ is differentiable, it follows (e.g., Trench, Theorem 6.22) that since $F$ is differentiable everywhere, for any $(x, y)$,
\[
\lim_{(u, v) \to (x, y)} \frac{F(u, v) - (F(x, y) + dF \left[ \begin{array}{c} u - x \\ v - y \end{array} \right])}{\|(u, v) - (x, y)\|} = 0.
\]
But the numerator in the expression is exactly $F(u, v) - G(u, v)$, where $G(u, v)$ is the first order approximation.

7. Let $\mathbb{R}^\infty$ be the space of sequences, \{ $x_1, x_2, \ldots$ \}, $x_n \in \mathbb{R}$, and define
\[
H = \{ (x_1, x_2, \ldots) \in \mathbb{R}^\infty : \sum x_i^2 < \infty \}
\]
\[
G_n = \{ (x_1, x_2, \ldots, x_n, 0, 0, \ldots) \in \mathbb{R}^\infty \}
\]
and
\[
G = \bigcup_{n=1}^{\infty} G_n.
\]

(a) Is $H \subset G$, $G \subset H$, or $G = H$? Explain.

(b) Prove that $G$ is dense in $H$ in the $\ell^2$ metric.

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Solution: $G \subset H$ since sequences with a finite number of nonzero elements are always in $\ell^2$, but there are elements of $\ell^2$ with an infinite number of nonzero elements which are therefore in $H$ but not $G$. To show that $G$ is dense in $H$, let $h = (h_1, h_2, \ldots) \in H$. Since $\sum h_i^2 < \infty$, for any $\epsilon > 0$ there is $N$ so that $\sum_{i=N+1}^{\infty} h_i^2 < \epsilon$. Thus $g = (h_1, h_2, \ldots, h_N, 0, 0, \ldots) \in G$ and $\|h - g\| < \epsilon$.

8. Let $X$ and $Y$ be metric spaces, and let $f_n : X \to Y$ be a sequence of continuous functions that converge uniformly to $f$. Prove that $f$ is continuous.

Solution: Standard, e.g., Rudin, Theorem 7.12.