VOTING PROFILES WITH DISJOINT CONDORCET CYCLES

by

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DEDICATION

I dedicate this thesis to my parents and grandparents in gratitude for what they have given me.

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ABSTRACT

Condorcet voting can be used to order all the candidates or alternatives in an election; however, Condorcet voting is not always successful in producing such an order but may instead result in the voting paradox which is also called Condorcet's paradox. This thesis discusses the interrelationships amongst Condorcet voting, tournaments of graph theory, and the algebraic structure of latin squares. The notion of p-groupoids is developed and it is shown how these algebraic structures decompose complete graphs of odd order into disjoint cycles. The thesis then shows the relationship between idempotent symmetric quasigroups and p-groupoids. This relationship is then exploited to construct voting profiles which, under Condorcet voting, produce the Condorcet paradox.
This abstract accurately represents the content of the candidate's thesis. I recommend its publication.

Signed

William E. Cherowitzo
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1. Introduction

This thesis investigates the interrelationships among discrete structures connected with Condorcet voting such as latin squares and quasigroups, and tournaments, a topic in graph theory. In particular, this paper describes a method to construct square voting profiles of odd order that yield an extreme form of the paradox that can result under Condorcet voting. Condorcet voting stems from the idea that a candidate that wins all pairwise elections against the other candidates should be elected.

The Condorcet voting procedure was first described by John Charles Borda, a French scientist who lived during the end of the 1700's. In the literature of voting theory, Borda is best known for the social choice function which bears his name, the Borda count, see chapters 2 and 3. However, Borda had a wide and varied adult life. He was an officer in the French navy and also served as a soldier, even serving for a time with French forces aiding the American revolution. He is mostly known for his contributions to fluid dynamics, navigation, and the development of the metric system. Toward the end of his life he was a member of the French Revolutionary Committee on Weights and Measures from which sprang the metric system.
He also developed trigonometric tables during this time as an aid to navigation. Although born to the nobility, Borda did not seem to suffer any tragic consequences due to the French Revolution.

Borda was not satisfied with the Condorcet voting scheme since it did not always work, that is, it sometimes does not give an answer or even a tie result. He later developed his (Borda count) system, which always yields a result which could be a tie. It is interesting that the Borda count is not necessarily compatible with Condorcet voting, see chapter 3. [9]

Borda is contrasted with his contemporary Marie Jean Antoine Nicolas Caritat Marquis de Condorcet, a mathematician and philosopher of the enlightenment. Born to an important aristocratic family, Condorcet was known for his progressive views. He supported enfranchisement of women, advocated abolition of slavery, was in favor of free speech, and supported the American revolution. He was widely known to be sympathetic to the suffering of the French people and called for the end of compulsory labor. Because of these attitudes he did not suffer because of his noble birth during the revolution, at least at first. He became heavily involved in politics after the revolution. He represented Paris in the Legislative Assembly after the revolution and was presiding officer of that body for a time. He was instrumental in the establishment of a republic and the educational systems
which came about after the revolution (and after Condorcet's death).
Condorcet drafted a constitution for the French Republic, but this was
rejected in favor of a constitution drafted by the radical Jacobins whose
leader was Maximilien Robespierre. The ascendency of the Jacobins saw
Condorcet and many of his fellow and more moderate Girondins outlawed
and arrested. Condorcet was arrested after an extended period in hiding
during which he wrote some of his most important philosophical works. He
died in prison probably by his own hand.

Condorcet accepted the principle of the voting system which carries
his name. He proposed that any voting system which did not select a
Condorcet winner when one existed should not be adopted. He was critical
of the Borda count for this very reason.

Condorcet was critical of Borda in general as an experimentalist and
even denounced him in the French Academy of Sciences as having
abandoned the philosophical purity of mathematics for the pettiness of the
applied sciences [1].

Since this era, the problem with Condorcet voting, usually referred to
as the Condorcet paradox was rediscovered every few years and many
solutions have been offered to fix the problem.
This thesis does not offer a solution but examines aspects of a restricted domain of the Condorcet social choice function. Condorcet voting is applied to finding an acceptable ordering of candidates or alternatives. A situation is cited in which no ordering is possible and a method of finding such situations is demonstrated.

Preliminaries of voting theory which are important to the developments in the paper are given in chapter 2. Chapter 3 describes the Condorcet phenomenon and related topics in detail. The tournament representation of Condorcet voting is shown in chapter 4 with several results from graph theory which have direct consequences to the Condorcet problem. In chapter 5 is a discussion leading to the problem of finding voting profiles which lead to disjoint hamiltonian cycles. This chapter also describes the results of a computer search to examine the cyclic structure of certain square voting profiles. Chapter 6 shifts gears to a discussion on the structure of latin squares which is preliminary to the construction of a voting profile with disjoint hamiltonian cycles. Chapter 7 introduces p-groupoids and shows that these structures decompose complete graphs of odd order into disjoint hamiltonian cycles. This chapter also shows the relationship between p-quasigroups and idempotent symmetric latin squares which is the main tool used in constructing the desired voting profiles. And finally, chapter 8
shows the constructions of the desired square voting profiles of order 5 and of order 7.
2. Preliminaries

Let \( A = \{a_1, a_2, \ldots, a_m\} \) be the set of \( m \) alternatives or \( m \) candidates presented to a society for decision (or priority ordering). The set \( A \) is also called the choice set. The society, committee, or voting body, is indexed by a subset of \( N \), the natural numbers, \( \{1, 2, \ldots, n\} \) where \( n \) is the size of the society.

Assume that an individual in the society can rationally order the alternatives in accordance with her own personal preferences and that such an ordering is independent of the orderings by the other voters. Each such preference order, \( p \), is a total order (linear order) on \( A \) which is a relation \( R \) on \( A \) which is

- irreflexive: \( (a_i, a_i) \notin R \) for all \( a_i \in A \);
- antisymmetric: \( (a_i, a_j) \in R \) implies \((a_j, a_i) \notin R \); and
- transitive: \( (a_i, a_j), (a_j, a_k) \in R \) implies \((a_i, a_k) \in R \).

Note that antisymmetry implies that "ties" are not allowed in the preference order. The usual notation is \( a_i > a_j \) if \((a_i, a_j) \in R \) and is read "\( a_i \) beats \( a_j \)" or "\( a_i \) is preferred to \( a_j \)." The preference order, \( p_i \), for the \( i^{th} \) voter, can be represented as the column vector
\[
    p_i = \begin{pmatrix}
    a_{i1} \\
    a_{i2} \\
    \vdots \\
    a_{in}
    \end{pmatrix},
\]

where the candidate on top is the most preferred by voter \( i \). A preference profile is an assignment of preference orders by the members of the society from the set of permutations of the choice set (candidates). A preference profile will be represented by an \( m \times n \) matrix where each column, \( i \), is the preference order for the \( i^{th} \) voter.

The major goal in studying social choice or voting theory is to produce a social ordering of the alternatives from the preference profile that reflects "the will of the people." This social ordering is a preference order with the same qualities as the individual preference order above, although antisymmetry is usually relaxed, that is, tie results are allowed (the order is actually a partial order with symbol \( \leq \)). The notion of "the will of the people" rests not only upon the confidence of the voters that the result is true but also on the voter's trust in the honesty of the results, that is, that the results were not altered by manipulation due to the choice of the voting
method or by strategic voting, which means that some voters vote other than how they feel to obtain a desired result.

A social choice function is a function from the set of preference profiles to the set of permutations of the choice set, A. The social choice function is the voting method that produces the desired social ordering. These functions are also called social welfare functions. Since a social choice function produces a social preference order on the choice set, then any subset of the choice set will also be ordered. Examples of social choice functions are plurality, the Borda count, the Hare system, and a dictator.

In plurality voting, given a preference profile, the alternative with the most first place votes has the first place in the social preference order. Another round of plurality voting on a reduced preference profile fixes the second place of the social order. This second preference profile is made up of the same individual profiles with the winner removed. A third round on another reduced profile fixes the third place of the social order and so on.

It is easy to see that ties cannot be avoided in plurality voting. In practice, another social choice function would be used to break ties when necessary: for example, a runoff election or vote, or even a coin toss.

The Borda count is a weighted voting scheme, or a scoring function [11]. The position of an alternative in an individual's preference list
determines the weight assigned to that alternative. The sum of all weights assigned to the alternatives by the entire profile determines the social ordering. In the case of the Borda count the scoring vector is 

\((m - 1, m - 2, m - 3, \ldots, 2, 1, 0)\) where \(m\) is the number of candidates. The alternative in first place in a preference list would receive \(m - 1\) points and the second \(m - 2\) points, while the last place candidate receives no points. The points for each candidate are totaled with the order established by the point totals. Simply, \(a_i > a_j\) if and only if candidate \(a_i\) has more points than candidate \(a_j\).

The Hare system is a sequential system which in modified form is in use at present to decide elections in several countries. If one candidate has more than half of the first place votes, then that candidate is assigned first place on the social preference order. If not, then all candidates which do not have a first place vote in the profile are eliminated from further consideration for first place and the reduced profile is examined. The candidates with the least number of first place votes are eliminated. The resulting reduced profile is examined for a candidate that now has more than half of the first place votes. If there is no such candidate, again, the candidates with the least number of first place votes are eliminated. This procedure is continued until a winner is produced. Then the original profile is reduced by the
candidate(s) that have already been selected. The entire Hare process is repeated for each position on the social choice preference list until the choice set has been prioritized.

If one of the voter's preference list always coincides with the social preference list no matter what social choice function is used then that person is a dictator.
3. Condon the Condor, the Condor the Winner Criterion, and the Condor the Condor Condo (Condor the Condo Cycles)

The Condorcet voting procedure was first discovered by Jean Charles Borda but was later popularized by the Marquis de Condorcet and hence bears his name. Given a preference profile, the Condorcet procedure selects alternative \( a_i \) to be preferred over \( a_j \) in the social choice preference list if and only if \( a_i > a_j \) in more than half of the preference lists. If no such preference exists then the social preference list determined by Condorcet voting does not exist. For example, alternative 2 is the Condorcet winner, first place in the social preference list, in the following profiles.

\[
\begin{bmatrix}
2 & 2 & 2 \\
3 & 1 & 3 \\
1 & 3 & 1
\end{bmatrix}
\quad
\begin{bmatrix}
2 & 2 & 3 & 1 \\
1 & 3 & 4 & 5 \\
3 & 1 & 5 & 4 \\
5 & 4 & 1 & 3 \\
4 & 5 & 3 & 2 & 2
\end{bmatrix}
\]

(3.1)

The alternative, 2, is the Condorcet winner since it beats every other alternative in pair wise comparisons 3 to 0 in the \( 3 \times 3 \) preference profile. In the \( 5 \times 5 \) profile alternative 2 beats every other alternative 3 to 2 since it is first in three preference lists and last in two lists.
A related concept is that of the Condorcet loser. The Condorcet loser doesn't beat any of the other alternatives in the pairwise comparisons over all the preference lists, that is, \( a_i \) is the Condorcet loser if and only if \( a_j > a_i \) in more than half of the preference lists.

The Condorcet winner criterion is a property applied to the comparison of social choice functions. Simply, a voting procedure satisfies the Condorcet winner criterion if whenever the Condorcet winner exists then the social choice function in question also picks that same alternative for a winner. This appears to be a completely reasonable expectation of prospective social choice functions. Unfortunately, the social choice functions mentioned in the previous section do not satisfy the Condorcet winner criterion [10]!

Theorem 3.1: The following voting procedures do not satisfy the Condorcet winner criterion:

a) the Hare system,

b) the dictator,

c) the Borda count, and

d) the plurality procedure.
Proof: By contradictions: Consider this preference profile made up of 5 alternatives with 17 voters:

\[
\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 & 5 \\
2 & 2 & 2 & 2 & 2 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 4 & 5 & 4 & 4 & 4 & 5 & 1 & 3 & 3 & 3 & 3 & 3 & 1 \\
4 & 4 & 5 & 3 & 4 & 5 & 5 & 1 & 4 & 5 & 5 & 5 & 4 & 4 & 4 & 3 \\
5 & 5 & 4 & 5 & 3 & 1 & 1 & 5 & 1 & 4 & 1 & 1 & 1 & 1 & 1 & 4 \\
\end{array}
\] (3.2)

Alternative 2 is the Condorcet winner since it beats all other alternatives in pairwise comparisons, by at least 12 to 5 (over alternative 1) and by at most 14 to 3 (over alternatives 3 and 4).

Consider this same profile under the Hare procedure. There is no alternative that has a clear majority. So we must iterate through the Hare process. Every alternative has at least one first place vote, so we proceed by eliminating the alternative with the least number of first place votes, which is alternative 2. Since alternative 2 is eliminated it cannot be the winner under the Hare procedure. Therefore the Condorcet winner is not the Hare winner.

Consider this preference profile made up of three voters deciding among three alternatives.
\[
\begin{bmatrix}
1 & 3 & 3 \\
2 & 2 & 2 \\
3 & 1 & 1
\end{bmatrix}
\] (3.3)

If the voter with preference list in the first column is a dictator then the dictator chooses alternative 1 as the winner; however, the Condorcet winner is alternative 3 since it beats 1 and 2 in two lists. So a dictator does not always pick the Condorcet winner. In fact, in this case, the dictator has picked the Condorcet loser.

Now, consider this preference profile of five voters on four candidates.

\[
\begin{bmatrix}
1 & 1 & 1 & 2 & 2 \\
2 & 2 & 2 & 3 & 3 \\
3 & 3 & 3 & 1 & 1 \\
4 & 4 & 4 & 4 & 4
\end{bmatrix}
\] (3.4)

Alternative 1 is the Condorcet winner since it beats alternatives 2 and 3, by 3 to 2 in the pairwise comparisons and alternative 4, by 5 to 0. Under the Borda count alternative 1 has 11 points while alternative 2 has 12 points. So alternative two is the Borda count winner. And the Borda count does not satisfy the Condorcet winner criterion. Notice the presence of the decisive Condorcet loser.
Finally, consider the preference profile of nine preference lists on three candidates.

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\
2 & 2 & 2 & 2 & 3 & 3 & 3 & 2 & 2 \\
3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\] (3.5)

Candidate 2 is the Condorcet winner since it beats both candidate 1, by 5 to 4, and candidate 3, 5 to 2. Candidate 1 is chosen by the plurality procedure since it has the most first place votes. So plurality does not satisfy the Condorcet winner criterion.

Is there any social choice function that does satisfy the Condorcet winner criterion?

Sequential pairwise voting is a social choice procedure by which an agenda is used to determine the order that the pairs are compared. The winner of each pairwise comparison moves onto the next comparison and the winner of the final comparison is the winner selected by the procedure. For example, consider the $3 \times 9$ preference profile given in (3.5) with an agenda of (1, 2, 3). First 1 is compared to 2 with the result that alternative 2 wins the comparison 5 to 4. Then the winner, alternative 2 is compared to 3
with alternative 2 winning by a margin of 7 to 2. So alternative 2 is the social choice. [9]

Theorem 3.2: Sequential pairwise voting with a fixed agenda satisfies the Condorcet winner criterion [10].

Proof: Suppose $a_i$ is the Condorcet winner in a given preference profile. Then $a_i$ already beats any $a_j$, $j \neq i$ in more than half of the preference lists. So $a_i$ will win all of its comparisons with any other alternative. Therefore the Condorcet winner will always be the social choice with sequential pairwise voting. ■

The Condorcet winner may not exist as seen in the following profile

\[
\begin{bmatrix}
1 & 1 & 2 & 2 \\
2 & 3 & 1 & 3 \\
3 & 2 & 3 & 1
\end{bmatrix}
\]

(3.6)

where candidates 1 and 2 are tied for first place in the social preference list by beating each other, 2 to 2 and by both beating alternative 3, by 3 to 1. This situation is only avoided by considering odd numbers of voters, since for every pairwise comparison there will be a clear winner. Thus, for even numbers of voters, Condorcet voting may not yield a social preference.
list. But even odd numbers do not guarantee that Condorcet voting yields a social preference profile as illustrated in the following simple $3 \times 3$ profile.

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{bmatrix}
\] (3.7)

Condorcet voting produces a result of 1 beats 2, by 2 to 1; 2 beats 3, by 2 to 1; and 3 beats 1, also by 2 to 1. This situation is a clear violation of transitivity and is known as the voting paradox or the Condorcet paradox and was known by both Condorcet and de Borda.

Of the voting procedures previously mentioned only the dictator always produces a clear winner and sequential pairwise voting produces a clear winner that is dependent on the agenda. To see this, note that for any pairwise comparison on (3.7) the winner of that comparison is defeated by the remaining alternative. To guarantee the winner it is only necessary that given the situation of the Condorcet paradox the desired winner be placed last in the agenda. Sequential pairwise voting is commonly used in the United States Congress.

The Condorcet paradox result, $a_1 > a_2$, $a_2 > a_3$, $a_3 > a_1$, is also called a Condorcet cycle.
4. **Tournament Representation of Condorcet Voting**

A tournament $T_n$ (on $n$ vertices) is an orientation of the complete graph $K_n$ which is the graph on $n$ nodes such that every pair of vertices is joined by an edge. An orientation of a graph is an assignment of direction to each edge in the graph.

A tournament gives a graph theoretical representation of any voting method which uses some system of paired comparisons in which the most preferred of each pair is indicated [5].

Condorcet voting by an odd number of voters on $m$ alternatives defines a tournament as follows: Let the vertices of the tournament $T_m$ be labeled by the $m$ alternatives. Let the edge joining a pair of vertices be oriented away from the vertex most preferred by Condorcet voting. If $a_i > a_j$, by the pairwise comparisons, then the edge between the two vertices is oriented from $a_i$ and toward $a_j$. For example, the results of the $3 \times 3$ profile given by (3.7), 1 beats 2, 2 beats 3, and 3 beats 1, defines the tournament in Figure 4.1. The cycle indicated by the Condorcet paradox is apparent.

Another example is the tournament defined by the $5 \times 5$ profile in (3.1) in Figure 4.2. Notice that the Condorcet winner, 2, has all the edges pointing
Figure 4.1 A Condorcet tournament on three alternatives.

Figure 4.2 A Condorcet tournament on five alternatives.
away, that is, it has maximum outdegree. The presence of the Condorcet loser is indicated by vertex 5 which has all edges coming in or maximum indegree.

Condorcet voting with an even number of voters may not have a tournament representation since there may be ties. And hence some edges may not be oriented.

We already know that the Condorcet paradox implies a failure of transitivity, and that the presence of a voting cycle is the mark of a Condorcet paradox. Before giving a related result for tournaments, notice that the following result on tournaments makes it not really surprising that Condorcet cycles exist.

A path is a sequence of vertices alternating with edges beginning at one vertex and ending at another such that both the vertices and the edges are distinct. A Hamiltonian path is a path such that all the vertices in the graph are contained in the sequence. In a tournament the direction of the path is determined by the directed edges. The edges in the sequence defining the path are usually suppressed.

Theorem 4.1: Every tournament has a Hamiltonian path [6].

Proof: By mathematical induction. This is clear for a tournament on 2
vertices. Assume that the theorem is true for a tournament with \( k \) vertices. Let \( T_{k+1} \) be an arbitrary tournament with \( k + 1 \) vertices. Let \( a_i \) be an arbitrary vertex in \( T_{k+1} \). Consider the subgraph \( T_{k+1} - \{a_i\} \). This subgraph is still a tournament since it is an orientation of the complete graph on \( k \) vertices. By assumption this subgraph has a hamiltonian path. Suppose the hamiltonian path is the sequence \( \{a_1, a_2, ..., a_k\} \). If there is an edge directed from \( a_i \) to \( a_i \) then \( \{a_i, a_1, a_2, ..., a_k\} \) is a hamiltonian path on \( T_{k+1} \). If not, then there is an edge directed from \( a_i \) to \( a_i \). Let \( j \) be the greatest integer such that there is an edge directed from \( a_j \) to \( a_i \). If \( j = k \), then \( \{a_1, a_2, ..., a_k, a_i\} \) is a hamiltonian path on \( T_{k+1} \). But if \( j < k \), then there is a directed edge from \( a_j \) to \( a_i \) and a directed edge from \( a_i \) to \( a_{j+1} \). And \( \{a_1, a_2, ..., a_j, a_i, a_{j+1}, ..., a_k\} \) is a hamiltonian path on \( T_{k+1} \).

Since a hamiltonian path exists in every tournament then the existence of a hamiltonian cycle depends on the direction of the edge between the vertex at the start of the path and the vertex at the end of the path. Further, the existence of a cycle depends on the direction of the edge between any two pairs on the path.

This result implies that a hamiltonian path exists but does not say that the path is unique. The existence of a unique path in a tournament defined
by Condorcet voting implies the existence of a social preference list, that is
that the Condorcet voting procedure has successfully produced the social
preference list. What are the conditions that such a path exists? Transitivity
in a tournament is defined similarly as transitivity in a preference list. A
tournament is transitive if for every set of three vertices \{a_i, a_j, a_k\} in the
tournament an edge directed from \(a_i\) to \(a_j\), and an edge directed from \(a_j\) to \(a_k\)
implies that a directed edge exists from \(a_i\) to \(a_k\).

Theorem 4.2: In a tournament, the following statements are equivalent [8]:

a. There is a unique hamiltonian path.

b. There are no cycles of length 3.

c. The tournament is acyclic.

d. The tournament is transitive.

Proof: By proving that \(a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow a\). To prove \(a \Rightarrow b\), suppose there is
a unique hamiltonian path \(\{a_1, a_2, \ldots, a_m\}\). Show that for \(i < j\) there is an
edge from \(a_j\) to \(a_i\). Suppose this is false, then pick the smallest \(i\) such that
there is a \(j\) so that there is an edge from \(a_i\) to \(a_j\). Now pick the largest such \(j\).
Assume \(1 < i\) and that \(j < m\), for now. There is an edge from \(a_{i-1}\) to \(a_{i+1}\)
since \(i\) was chosen as small as possible. There is also an edge from \(a_i\) to \(a_{j+1}\)
since \(j\) was chosen as large as possible. Therefore, the sequence
\{a_1, a_2, \ldots, a_{i-1}, a_{i+1}, a_{i+2}, \ldots, a_i, a_{i+1}, a_{i+2}, \ldots, a_m\} is a second hamiltonian path, a contradiction. If i = 1 then \{a_{i-1}, a_{i+2}, \ldots, a_j, a_{j+1}, a_{j+2}, \ldots, a_m\} is the second hamiltonian path. And if j = m then \{a_i, a_2, \ldots, a_{i-1}, a_{i+1}, a_{i+2}, \ldots, a_j, a_i\} is the second hamiltonian path. Therefore for i < j there is an edge from a_j to a_i implying there are no cycles of length 3.

For b \Rightarrow c, let a tournament have no cycles of length 3. By induction, show that the presence of a cycle of length k > 3 implies the presence of a 3-cycle which provides the contradiction. Suppose that the sequence \{a_1, a_2, a_3, a_4\} is a cycle in the tournament. Consider the subtournament induced by the vertices in the 4-cycle. Consider one of the chords, say the edge between a_2 and a_4. Then since that edge is directed (the subgraph is a tournament), either \{a_1, a_2, a_4\} or \{a_2, a_3, a_4\} is a 3-cycle on both the subtournament and the original tournament.

Now assume that a cycle of length k in a tournament implies the presence of a 3-cycle. Show that a cycle of length k + 1 implies the presence of a 3-cycle. Let \{a_1, a_2, \ldots, a_k, a_{k+1}\} be a cycle of length k + 1 in the tournament. Consider the subtournament induced by the vertices in the k+1-cycle. Consider two vertices separated by a third vertex, say for example a_1 and a_3. The edge between is directed since we are in a tournament. So either \{a_1, a_2, a_3\} is a 3-cycle or \{a_1, a_3, a_4, \ldots, a_k, a_{k+1}\} is a
k-cycle. Either way implies the presence of a 3-cycle which provides the contradiction.

For $c \Rightarrow d$, is immediate. No three cycles imply transitivity.

For $d \Rightarrow a$, let the tournament be transitive and suppose that there are two different hamiltonian paths. Then there must be a pair of vertices $a_i$ and $a_j$ such that $a_i$ follows $a_j$ on one path but $a_j$ follows $a_i$ on the other (if such a pair does not exist then the hamiltonian paths are identical to begin with).

By extending transitivity along the paths, the fact that $a_i$ follows $a_j$ implies that there is a directed edge from $a_i$ to $a_j$, while transitivity along the other path implies that there is a directed edge from $a_j$ to $a_i$, which is a contradiction. Therefore, the two hamiltonian paths are identical. ■

This theorem implies that the Condorcet paradox means that there may be many Condorcet cycles not just one and that only one three cycle is sufficient to preclude the social ordering from Condorcet voting since transitivity has broken down.

A further issue is that, although a preference profile with an odd number of voters under Condorcet voting defines a specific tournament, a tournament does not necessarily define a preference profile since the number of voters cannot be determined. Also, even if the number of voters is
known, the direction of an edge only indicates which alternative received a majority in the comparisons between the two endpoint vertices, the score of the pairwise comparison is not indicated at all. Indeed, these two $5 \times 5$ profiles yield the same tournament on 5 vertices, given in Figure 4.3.

$$
\begin{align*}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5
\end{bmatrix}
&= \\
\begin{bmatrix}
1 & 2 & 2 & 3 & 3 \\
2 & 1 & 1 & 1 & 5 \\
5 & 4 & 4 & 2 & 1 \\
3 & 3 & 5 & 4 & 2 \\
4 & 5 & 3 & 5 & 4
\end{bmatrix}
\end{align*}
$$

(4.1)

Figure 4.3 Tournament on five alternatives.
5. **Square Voting Profiles, Hamiltonian Cycles, Latin Squares, and Rotational Tournaments**

The restriction to studying profiles under Condorcet voting with only odd numbers of voters is due to the unavoidable tie results in the pairwise comparisons. These tie results preclude the desired total ordering of the alternatives by the society.

So the voting profiles are restricted to odd numbers of voters. It is a natural restriction to consider voting profiles with the number of candidates or alternatives equal to the number of voters. In the sense of determining tournaments by the method of paired comparisons all profiles with an odd number of voters on an odd number of alternatives can be thought of as equivalent to a square voting profile on the same (odd) number of alternatives if they have the same tournament representation.

The third limitation is in studying voting profiles which cause hamiltonian cycles under Condorcet voting. In the attempt to order all of the alternatives under question into a social preference list that is a total order, it is clear that a single 3-cycle causes a failure since transitivity is violated. This may not preclude Condorcet voting from arriving at a clear winner (or loser) on this profile. The presence of hamiltonian cycles prevents
"anything" from being obtained about such voting profiles and may be the most chaotic or unacceptable result possible.

There are 216 different $3 \times 3$ preference profiles. This number is computed by first noting that there are $3! = 6$ different preference lists, the number of permutations on three letters. Since there are three preference lists we have $6^3 = 216$ possible preference profiles.

A latin square of order $n$ is an $n \times n$ square matrix in which each of $n$ elements appears once in each row and once in each column. A preference profile already meets this criteria in its columns only.

Theorem 5.1: A $3 \times 3$ square preference profile produces a Condorcet paradox if and only if the preference profile is a latin square.

Proof: If one of the candidates or alternatives appears two or three times in the first row that candidate will be the Condorcet winner, beating the other candidates by at least 2 to 1. Hence, for a Condorcet cycle to exist, each of the candidates must only appear once each in the first row. Since the first row of the preference profile contains a particular candidate once, then the second row can have that candidate twice but in different columns than the first row appearance. If so, then that candidate is the Condorcet winner. So, for a hamiltonian cycle to exist the alternatives can appear only once in the
first and second rows. If a candidate can appear only once in each of the first two rows then that candidate can appear only once in the last row. This implies that each candidate appears only once in any of the three rows for a hamiltonian cycle to exist.

There are only twelve profiles that produce a Condorcet paradox or the associated hamiltonian cycle in the tournament produced by the paired comparisons. But if we re-order the preference lists so that the top row is \{1, 2, 3\} then we see that there are only two such preference lists, one given in (3.7) with the associated tournament given in Figure 4.1, and the other preference list given by the profile

\[
\begin{bmatrix}
1 & 2 & 3 \\
3 & 1 & 2 \\
2 & 3 & 1 \\
\end{bmatrix}
\]

and the related tournament is just the same as Figure 4.1 with the directions reversed.

Since the pairwise comparisons of Condorcet voting do not depend on the order of the individual preference lists, the reordered preference lists which are identical will have the same results and produce the same tournaments. This characteristic is called anonymity of the voters which is
more carefully defined as invariance of results under permutation of the voters [12].

When we look at the $5 \times 5$ square profiles the situation is much more complicated. There are $5! = 120$ different individual preference lists, therefore there are $120^5 = 24,883,200,000 = 2.5 \times 10^{10}$ preference profiles. That combinatorial explosion is evident is driven home by the approximate number of $7 \times 7$ square profiles: $8.2 \times 10^{35}$.

A computer program was utilized to study the $5 \times 5$ squares. To make the search more tractable, the program searches for cycles in all of the 5-squares with first lines of one appearance of each element, \{1, 2, 3, 4, 5\}, two appearances of one element and one appearance of 3 of the 4 remaining, \{1, 1, 2, 3, 4\}, and two appearances of two distinct elements and one out the other three, \{1, 1, 2, 2, 3\}. The 5-squares with 3 or more appearances of one element on the first line have Condorcet winners and therefore have no hamiltonian cycles. Anonymity of results allows for considering only one permutation of each first line category (the permutations presented here) since the results are the same as for the other permutations. And since all other profiles with the first line in one of these patterns are just re-labellings of the elements of the profile the results apply to all squares that do not automatically produce a Condorcet winner in the first line. This results in
only 23,887,872 5-squares to check. Additional testing within the program precluded matrices that would yield a Condorcet winner by combinations of elements of the first and second line and effectively eliminated about 20% of the squares to be tested.

The 5-squares had up to three hamiltonian cycles. Latin squares had either 2 or 3 cycles while non-latin squares had 0 - 3 cycles.

The surprise result is that there are over 60,000 of the profiles, with \{1, 2, 3, 4, 5\} as the first line, with three hamiltonian cycles in the tournament representation. For example, the profile

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
5 & 1 & 4 & 2 & 3 \\
3 & 4 & 1 & 5 & 2 \\
4 & 5 & 2 & 3 & 1 \\
2 & 3 & 5 & 1 & 4 \\
\end{bmatrix}
\] (5.2)

produces the three hamiltonian cycles \{1, 5, 3, 4, 2\}, \{1, 4, 2, 5, 3\}, and \{1, 4, 5, 3, 2\}, see Figure 5.1. It is interesting that these three cycles coincide on the edge from vertex 5 to vertex 3 and that pairwise the hamiltonian cycles coincide on two edges. Note that (5.2) is a latin square.
Figure 5.1 Tournament with three hamiltonian cycles.

The following profile is obviously not a latin square.

\[
\begin{bmatrix}
1 & 1 & 2 & 3 & 4 \\
2 & 2 & 5 & 2 & 5 \\
3 & 4 & 3 & 4 & 3 \\
4 & 5 & 4 & 5 & 1 \\
5 & 3 & 1 & 1 & 2
\end{bmatrix}
\]  

(5.3)

Nevertheless, (5.3) under Condorcet voting still produces the three hamiltonian cycles: \{1, 2, 3, 4, 5\}, \{1, 2, 4, 5, 3\}, and \{1, 2, 5, 3, 4\}, see Figure 5.2.
Are there any odd square profiles for which there is more than one hamiltonian cycle such that the cycles are disjoint in the tournament representation? Two cycles are disjoint if they do not share any of the same edges. A tournament on five vertices with two disjoint hamiltonian cycles must have a structure as in Figure 5.3. This is just a rotational tournament. A regular tournament is a tournament such that the outdegrees of all vertices are equal. A rotational tournament is a regular tournament such that the vertices of the tournament can be labelled $0, 1, 2, \ldots, m - 1$ in such a way that for some subset $S$ of $\{1, 2, \ldots, m\}$, vertex $i$ beats vertex $i + j \pmod{m}$ if and only if $j \in S$. $S$ is called the symbol of the tournament [6]. For the tournament in Figure 5.2, $S = \{1, 2\}$. The rotational tournaments have an
odd number of vertices. It is clear that the rotational tournaments have a cyclic structure. For those cycles to be hamiltonian the tournament must have a prime number of vertices.

![Diagram of a rotational tournament]

**Figure 5.3** A rotational tournament.

For example, consider the following voting profile which is a latin square of order 5.

\[
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
2 & 1 & 5 & 3 & 4 \\
3 & 4 & 1 & 5 & 2 \\
4 & 5 & 2 & 1 & 3 \\
5 & 3 & 4 & 2 & 1 \\
\end{array}
\] \hspace{1cm}(5.4)
The pairwise comparisons of Condorcet voting result in the tournament representation in Figure 5.4. But this is just a rotational tournament as can be seen by switching the labels on vertices 3 and 4 and redrawing to arrive at Figure 5.3.
6. Algebraic Structure of Latin Squares

Latin squares were discovered by Leonard Euler in 1782. The word "latin" refers to the Latin letters that he used in his examples.

A set $S$ forms a groupoid $(S, *)$ with respect to a binary operation $(*)$ if, each ordered pair of elements $a, b \in S$ is associated with a uniquely determined element $a \ast b \in S$ called their product. The definition of groupoid essentially establishes what an algebraic structure on a set is: a set with a binary operation defined on it is a groupoid. For example, this is the multiplication table of a groupoid.

Table 6.1 Multiplication Table of a Groupoid

<table>
<thead>
<tr>
<th>*</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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</thead>
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<td>3</td>
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<td>1</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>
A groupoid $S$ is a quasigroup if for all $a, b \in S$ the equations $a \ast x = b$ and $y \ast a = b$ each have exactly one solution. The two conditions are referred to as left divisibility and right divisibility, respectively. Many examples of a multiplication table of a quasigroup are given by the next result.

Theorem 6.1: The multiplication table of a groupoid is a latin square if and only if the groupoid is a quasigroup [2].

Proof: Suppose that $a_1, a_2, \ldots, a_n$ are the elements of a quasigroup $S$. Let the $a_{r,s}$ entry, the entry in row $r$ and column $s$, of the multiplication table be the product of $a_r$ and $a_s$, $a_r \ast a_s$. If the same element occurred twice in the same row then there would be two solutions to the equation $a_r \ast x = b$ for some $b \in S$, since we now have $a_{r,s} = a_{r,t} = b$ (where $s$ and $t$ indicate the column). This contradicts the definition of quasigroup. Similarly if the same entry occurred twice in a column there would be two solutions to the equation $y \ast a_s = c$ for some $c \in S$. Therefore, each element of $G$ occurs exactly once in each row and once in each column of the multiplication table. The unbordered multiplication table is therefore a latin square.

Conversely, suppose the multiplication table of a groupoid $S$ is a latin square. Then clearly the definition of quasigroup holds. ■
A matrix with \( n \) rows is referred to as column latin if each of \( n \) elements is in each column. There are no restrictions on the rows. Similarly, a matrix with \( n \) columns is referred to as row latin if each row is a permutation of the same \( n \) elements. A voting profile is column latin since the individual preference lists have each of the alternatives listed once and only once. Obviously, a latin square is both row latin and column latin.

A latin square is in standard form (or reduced form) if the elements of the first row and column are in their natural order top to bottom and left to right. These are obtained by permuting rows and columns of a given latin square. For example, in the following, the latin square on the right is the standard form of the latin square on the left after column permutations and then row permutations.

\[
\begin{bmatrix}
1 & 3 & 5 & 2 & 4 \\
5 & 2 & 4 & 1 & 3 \\
4 & 1 & 3 & 5 & 2 \\
3 & 5 & 2 & 4 & 1 \\
2 & 4 & 1 & 3 & 5 \\
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
5 & 1 & 2 & 3 & 4 \\
4 & 5 & 1 & 2 & 3 \\
3 & 4 & 5 & 1 & 2 \\
2 & 3 & 4 & 5 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 5 & 1 \\
5 & 1 & 2 & 3 & 4 \\
\end{bmatrix}
\]

For a square preference profile which is a latin square, Condorcet voting allows column permutations by anonymity. However, row permutations will change the individual preference lists.
An idempotent latin square of order n is a latin square in which each of the n elements appears on the main diagonal. These squares are also known as transversal squares. The idempotent law states that \( a * a = a \) for all a.

A quasigroup is commutative or symmetric if \( a * b = b * a \). The multiplication table of a symmetric quasigroup is a symmetric latin square.
7. Decomposition of Prime Order Latin Square Profiles into Disjoint Hamiltonian Cycles

A p-groupoid is a groupoid \((V, \ast)\) which has the following properties:

a. \(a \ast a = a\) for all \(a \in V\),

b. \(a \neq b\) implies \(a \neq a \ast b \neq b\) for all \(a, b \in V\), and

c. \(a \ast b = c\) implies and is implied by \(c \ast b = a\) for all \(a, b, c \in V\).

The leading "p" in p-groupoid stands for partition. Condition a means that a p-groupoid is idempotent. The multiplication table given in section 6 is an example of a p-groupoid. We first establish a correspondence between p-groupoids with \(n\) elements and decompositions of the complete undirected graph on \(n\) vertices, \(K_n\), into disjoint cycles.

Let \(G\) be a graph. A partition of \(G\) into disjoint cycles is a set, \(Z\), of cycles such that any edge of \(G\) belongs to exactly one cycle in \(Z\). If \(Z\) contains only one element then that graph is eulerian.

Let \(Z\) be a partition of \(G\) into disjoint cycles. Let \(H(i)\) be the set of all edges incident to vertex \(a_i\) in \(G\) and let \(d_i\) be the degree of \(a_i\), that is, the number of edges in \(H(i)\). When do such partitions exist?
Theorem 7.1: A partition of a connected graph $G$ into disjoint cycles exists if and only if all the $d_i$'s are even.

Proof: Suppose $Z$ is a partition of $G$ into disjoint cycles. Then each cycle contributes twice to the degree of each vertex it passes through. Therefore the $d_i$'s are all even.

Suppose all the $d_i$'s in a connected graph $G$ are even. Let $m$ be the number of edges in $G$. Suppose $m = 3$. This graph is $K_3$ and contains one 3-cycle. Suppose all connected graphs with number of edges less than $m$ and with all $d_i$'s even can be partitioned into disjoint cycles. For a graph $G$ with $m$ edges and all $d_i$'s even, there must be a cycle, $C$, since the total degree of $G$ is even. Let $G'$ be the graph obtained by deleting the edges of $C$ from $G$. Since $G'$ may be disconnected, consider the subgraphs of $G'$: $G_1, G_2, \ldots, G_k$. No two of the $G_i$'s have a common vertex. So, every vertex in each $G_i$ has even degree and each $G_i$ is connected. By the induction hypothesis each $G_i$ has a partition of disjoint cycles. The partition of $G$ into disjoint cycles is made up of all the disjoint cycles of each $G_i$ and $C$. ■

A transition of a $k$-cycle $W = \{a_1, e_1, a_2, e_2, \ldots, a_{k-1}, e_{k-1}, a_k, e_k\}$ through a vertex $a_i$ is the triple $\{e_{i-1}, a_i, e_i\}$. The transition through $a_i$, is $\{e_k, a_i, e_i\}$.

Let $d_i = 2c_i$ where $c_i$ is an integer. Within the cycles of $Z$ there are
The set of transitions through the vertex $a_i$ are the triples $\{e_i(k), a_i, f_i(k)\}$. The union of all the edges in the $c_i$ transitions for each vertex is just $H(i)$, the set of edges incident upon the vertex $a_i$. The edges of different transitions of a vertex are disjoint.

The partition

$$D_1 = \left\{ \{e_i(1), f_i(1)\}, \{e_i(2), f_i(2)\}, ..., \{e_i(c_i), f_i(c_i)\} \right\} \tag{7.1}$$

of $H(i)$ into pairs of edges is called a $\delta$-partition in $a_i$ formed by $Z$. And $D = \{D_1, D_2, ..., D_n\}$ is called a $\delta$-system formed in $G$ by $Z$ and is denoted $D = \delta(Z)$.

Theorem 7.2: For every partition $Z$ there is a unique $\delta$-system in $G$, and for every $\delta$-system $D$ there is exactly one partition $Z$ of $G$ into disjoint cycles so that $D = \delta(Z)$.

Proof: Let $Z$ be a partition of $G$ into disjoint cycles. Suppose there are two $\delta$-systems, $D$ and $E$, formed in $G$. So $D = \{D_1, D_2, ..., D_n\}$ and $E = \{E_1, E_2, ..., E_n\}$ where $D_i$ and $E_i$ are partitions of $H(i)$ into pairs of edges formed by $c_i$ transitions for each vertex by the cycles of $Z$. Since $c_i$ counts the cycles through $a_i$, and therefore the transitions through $a_i$, the classes of $D_i$ and $E_i$ are identical.
Suppose that \( D \) is a \( \delta \)-system formed in \( G \) by \( Z \) and by \( X \) where \( Z \) and \( X \) are partitions of \( G \) into disjoint cycles. Since \( c_i \) counts the number of cycles through vertex \( a_i \), the number of cycles in \( Z \) and \( X \) through vertex \( a_i \) must be equal. Consider the cycles of \( Z \) through \( a_i \) and the cycles of \( X \) through \( a_i \). The transitions caused by \( X \) and the transitions caused by \( Z \) through \( a_i \) form a \( \delta \)-partition in \( a_i \). But since there is only one \( \delta \)-partition in \( a_i \) the transitions caused by \( X \) and caused by \( Z \) through \( a_i \) must be identical which implies that the cycles of \( X \) and the cycles of \( Z \) must be identical.

And therefore \( X = Z \). □

Theorem 7.3: Every \( p \)-groupoid of order \( n \) forms a \( \delta \)-system on \( K_n \), the complete undirected graph on \( n \) vertices.

Proof: Let \((V,*)\) be a \( p \)-groupoid. Let \( V = \{a_1, a_2, ..., a_n\} \) label the vertices of \( K_n \). Construct \( D_i \) of the set \( H(i) \) of all edges in \( K_n \) as follows: Let the edge \( e_{ij} \) joining vertices \( a_i \) and \( a_j \) be in the same class of \( D_i \) as the edge \( e_{ik} \) if and only if \( a_j * a_i = a_k \). Since \((V,*)\) is a \( p \)-groupoid we also have \( a_k * a_i = a_j \).

Every class of \( D_i \) contains two edges and \( D_i \) is a partition of \( H(i) \) into pairs of edges. Therefore \( D = \{D_1, D_2, ..., D_n\} \) is a \( \delta \)-system in \( K_n \). □
The last two results say that not only does every p-groupoid of order 
n form a δ-system on \( K_n \), but every p-groupoid also corresponds to one and 
only one partition of \( K_n \) into disjoint cycles.

Theorem 7.4: For a p-groupoid \((V,*)\) the number of elements \( n \) is odd, and 
\[ x * b = c \] is uniquely soluble for \( x \) [3].

Proof: Let \((V,*)\) be a p-groupoid with \( n \) elements. Label the vertices of the 
complete undirected graph on \( n \) elements, \( K_n \), with the elements of \( V \). Let 
the edges \((a,b)\) and \((b,c)\) belong to the same closed path in the graph if and 
only if \( a * b = c, \ a \neq b \) for all \( a, b, c \in V \). For example, the p-groupoid given 
in the multiplication table in section 6 results in the disjoint cycles of \( K_n \) in 
Figure 7.1.

Now since an arbitrary vertex, \( v \in K_n \), is part of at least one closed 
path determined by \( V \) the number of edges incident upon each vertex is an 
even number. This implies that \( v \) is connected to an even number of vertices 
hence the number of vertices of \( K_n \) determined by \( V \) is odd and the number 
of elements in \( V \) is odd.

For a p-groupoid \( x * b = c \) implies \( c * b = x \) and for a groupoid \( c \) and 
\( b \) have a unique product \( c * b \), and so \( x * b = c \) has a unique solution. \( \blacksquare \)
Corollary 7.5: The multiplication table of a p-groupoid is a column latin square.

Proof: Suppose that c appears twice in column b. Then \( x \ast b = c \) does not have a unique solution. A contradiction. And since the choice of c and b are arbitrary, c can only appear once in any column. ■

This result implies that the multiplication tables of p-groupoids can be considered as voting profiles. A p-groupoid which is also a quasigroup is called a p-quasigroup. The tools are almost available to construct square voting systems of prime order which result in disjoint hamiltonian cycles in a tournament representation of Condorcet voting.
Theorem 7.6: Let \((V, \cdot)\) be a \(p\)-quasigroup and let a groupoid \((V, *)\) be defined by the statement that \(a \cdot (a * b) = b\) holds for all \(a, b \in V\).

Then \((V, *)\) is an idempotent and commutative quasigroup. Also, with any given idempotent commutative quasigroup \((V, *)\) there is associated a \(p\)-quasigroup \((V, \cdot)\) related to \((V, *)\) by the correspondence \(a \cdot b = c\) if and only if \(a \ast c = b\) [3].

Proof: The operation \(\ast\) is well defined since \(a \cdot x = b\) is uniquely soluble for \(x\) since \((V, \cdot)\) is a quasigroup. \((V, \ast)\) is commutative since \(a \cdot x = b\) implies \(b \cdot x = a\), so \(a \cdot (a * b) = b\) implies that \(b \cdot (b * a) = a\), therefore \(a * b = b * a\). We also have \(a \cdot a = a\) because \((V, \cdot)\) is a \(p\)-quasigroup. This implies that \(a \cdot (a * a) = a\) and therefore \(a * a = a\) so \((V, *)\) is idempotent.

The association is proved in reverse by noting that \(a \ast (a \cdot b) = b\) for all \(a, b \in V\). ■

Note that the multiplication table of the new quasigroup \((V, \ast)\) in the above theorem is a symmetric idempotent latin square.

Theorem 7.7: Idempotent symmetric latin squares exist only for odd orders.

Proof: Idempotency implies that each element appears once on the diagonal. Symmetry implies that each element appears off the diagonal in pairs. Each element must then appear an odd number of times in the square. ■
8. The Construction

To construct a square voting profile with a tournament representation under Condorcet voting, that has disjoint hamiltonian cycles, begin with an idempotent symmetric latin square (the unbordered idempotent symmetric quasigroup multiplication table), for example

\[
\begin{array}{cccc}
1 & 4 & 2 & 5 \\
4 & 2 & 5 & 3 \\
2 & 5 & 3 & 1 \\
5 & 3 & 1 & 4 \\
3 & 1 & 4 & 2 \\
\end{array}
\]  \hspace{1cm} (8.1)

Now applying Theorem 7.6 to (8.1) defines the p-quasigroup

\[
\begin{array}{cccc}
1 & 3 & 5 & 2 \\
5 & 2 & 4 & 1 \\
4 & 1 & 3 & 5 \\
3 & 5 & 2 & 4 \\
2 & 4 & 1 & 3 \\
\end{array}
\]  \hspace{1cm} (8.2)

which partitions \( K_5 \) into disjoint hamiltonian cycles. Consider this square as a voting profile and column permute this square into
from which it is easy to see that Condorcet voting applied to this square results in the rotational tournament in Figure 8.1. It is also apparent that the disjoint hamiltonian tournament cycles follow the undirected cycles of the decomposition of $K_5$ from (8.2).

![Figure 8.1 A rotational tournament.](image)

The voting profile in (8.3) above also indicates that these tournaments with disjoint hamiltonian cycles can be easily produced since the voting
preference lists, the matrix columns, are rotations of the column to the left, while the first is a rotation of the last. This is the case for order 5, but for order 7 the situation is more complicated, since there is more than one way to decompose $K_7$ into disjoint cycles than just following the paths of the rotational tournament of order 7. Of the seven idempotent symmetric latin squares of order 7, only one has a rotational structure among the columns that is similar to the structure given in (8.1). As an example, at order 7, without such a structure, consider this idempotent symmetric quasigroup of order 7.

\[
\begin{bmatrix}
1 & 3 & 5 & 2 & 6 & 7 & 4 \\
3 & 2 & 6 & 5 & 7 & 4 & 1 \\
5 & 6 & 3 & 7 & 4 & 1 & 2 \\
2 & 5 & 7 & 4 & 1 & 3 & 6 \\
6 & 7 & 4 & 1 & 5 & 2 & 3 \\
7 & 4 & 1 & 3 & 2 & 6 & 5 \\
4 & 1 & 2 & 6 & 3 & 5 & 7 \\
\end{bmatrix}
\] (8.4)

By applying Theorem 7.6 we arrive at the following p-quasigroup of order 7.
which decomposes $K_7$ into the following disjoint cycles in Figure 8.2:

\{1, 2, 4, 6, 7\},

\{1, 3, 2, 7, 5, 6\},

\{1, 4, 7, 3, 5\}, and

\{3, 4, 5, 2, 6\}.

Figure 8.2 A Decomposition of $K_7$ into Disjoint Cycles.

Considering the latin square above in (8.5) as a voting profile and applying

Condorcet voting results in the rotational tournament of Figure 8.3.
Figure 8.3 Rotational tournament of order 7.

Notice that the disjoint cycles from the decomposition of K₇ listed above do not provide a path for the tournament cycles produced by the paired comparisons of Condorcet voting.

Kotzig has shown that there are only 7 distinct idempotent symmetric quasigroups, up to isomorphism. Each of these under Theorem 7.6 results in a p-quasigroup that, as a voting profile under Condorcet voting, defines the rotational tournament of order 7. It is a conjecture that this works for all prime orders. Kotzig has also conjectured that a decomposition of this nature works for all odd orders [4].

To see why the construction works in the case of the rotational column structure consider the idempotent symmetric quasigroup given by
(8.1) as a voting profile. Under the pairwise comparisons of Condorcet voting this profile results in the tournament given in Figure 8.4 below. This tournament has the peculiar characteristic that the two disjoint hamiltonian cycles rotate or circulate in opposite directions. Consider the outside cycle \{1, 2, 3, 4, 5\}. And notice that from the multiplication table (8.1) that we have: \(1 \cdot 2 = 4, 2 \cdot 3 = 5, 3 \cdot 4 = 1, 4 \cdot 5 = 2,\) and \(5 \cdot 1 = 3\) where we have taken the order of multiplication from the direction of the cycle and the products are the pairs in the order they occur in the cycle. If we apply the defining equation of Theorem 7.6, \(a \cdot (a \ast b) = b,\) to the five equations above we get the set of equations: \(1 \cdot 4 = 2, 2 \cdot 5 = 3, 3 \cdot 1 = 4, 4 \cdot 2 = 5,\) and \(5 \cdot 3 = 1,\) respectively where (\(\ast\)) is multiplication in the
p-quasigroup given in (8.2). If we rearrange these differently we have:

\[ 1 \cdot 4 = 2, \quad 4 \cdot 2 = 5, \quad 2 \cdot 5 = 3, \quad 5 \cdot 3 = 1, \quad \text{and} \quad 3 \cdot 1 = 4. \]

These are the equations for one of the undirected hamiltonian cycles \( \{1, 4, 2, 5, 3\} \) on \( K_5 \) determined by the p-quasigroup. The order of operation is determined by the direction of the cycle caused by the pairwise comparisons due to Condorcet voting. In a sense, the cycles in the tournament representation of the idempotent symmetric quasigroup under Condorcet voting are preserved under the equation that defines the p-quasigroup.
References


