DOMAIN DECOMPOSITION PRECONDITIONERS
FOR THIN RECTANGULAR

$p$-VERSION FINITE ELEMENTS

by

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B.A., University of Colorado at Denver, 1982

A thesis submitted to the
Faculty of the Graduate School of the
University of Colorado in partial fulfillment
of the requirements for the degree of

Doctor of Philosophy
Department of Mathematics

1990
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The use of the p-version finite element method to solve an elliptic partial differential equation results in a very large linear system which must be solved numerically. We consider iterative solution, using the conjugate gradients method, preconditioned by domain decomposition preconditioners, and analyze the condition number of two recently developed such preconditioners for problems in \( \mathbb{R}^2 \).

The p-version finite element method achieves accuracy by increasing the degree \( p \) of the elements, rather than by decreasing their diameter, as in the more familiar h-version. Thus, there are many degrees of freedom on each element. The preconditioners considered here treat each element as a subdomain, allowing much of the work of generating and solving the preconditioner to be performed on each element independently.

Most published analyses of preconditioners for the finite element method do not consider the case of elements with high aspect ratios, a case which happens often in practice, and consider only elements which are nearly, e.g. square. We show that, for the first preconditioner considered here, the condition number
grows with the square of the aspect ratio, and thus its efficacy degrades significantly. We also show that, for the second preconditioner, which adapts to elements of different aspect ratios, the condition number is bounded independently of the aspect ratio. Thus the second preconditioner can be more effective than the first for the case of very thin elements. We accomplish this analysis using energy seminorms which are defined especially for thin subdomains, and basis polynomials which are orthogonal with respect to these seminorms. This approach gives a greater insight into the behaviour of these methods than previously published methods.

The form and content of this abstract are approved. I recommend its publication.

Signed

Jan Mandel
For my wife,

Elizabeth
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CHAPTER 1

INTRODUCTION

Elliptic partial differential equations appear frequently in physical models in science and engineering. Frequently, we must make use of digital computers to approximate their solutions. The numerical solution of partial differential equations requires two main choices. Firstly, the method of discretizing the equation must be chosen, such as finite difference, finite element, or finite spectrum methods. Secondly, the method of solving the resulting discrete system of equations must be selected. This investigation will deal primarily with the second aspect of this problem. We assume that the discretization technique is the $p$-Version finite element method, and concentrate on analyzing some recently formulated preconditioners for iteratively solving the resulting linear systems.

Our model problem will be the linear boundary value problem

\[ \Delta u = f, \quad \text{in} \quad \Omega \]
\[ u = 0, \quad \text{on} \quad \partial \Omega \]

where $\Omega$ is a bounded, piecewise smooth domain in $\mathbb{R}^2$ with boundary $\partial \Omega$. 
The standard weak form of this problem is

\[ u \in V : \quad \int_\Omega v \Delta u \, dx \, dy = \int_\Omega f \, dx \, dy, \quad \forall v \in V \]

where \( V = H^1_0(\Omega) \). The solution to the weak form is known to be the same as that for the original boundary value problem, when \( u \) is sufficiently smooth. It is useful to be able to change this problem, using Green’s Theorem, to the symmetric form:

\[ u \in V : \quad \int_\Omega \nabla v^T \nabla u \, dx \, dy = \int_\Omega f \, dx \, dy, \quad \forall v \in V. \quad (1.1) \]

In the Galerkin finite element method [49] we choose an \( n \)-dimensional subspace, \( S \), of \( V \), and approximate the weak form (1.1) as

\[ u \in S : \quad a(u, v) = f(v), \quad \forall v \in S, \quad (1.2) \]

where

\[ a(u, v) = \int_\Omega (\nabla v)^T \nabla u \, dx \, dy, \]

\[ f(v) = \int_\Omega f \, dx \, dy. \]

Henceforth, the bilinear form \( a(\cdot, \cdot) \) will be understood to be defined on \( S \times S \), and the functional, \( f \), will be defined on \( S \). Once a basis, \( \{\phi_1, \phi_2, \ldots, \phi_n\} \), for \( S \) has been selected, we transform problem (1.2) into a linear system of equations, \( Ax = b \), where \( A = (a_{ij}), a_{ij} = a(\phi_i, \phi_j) \). We will study the solution of this system by the preconditioned conjugate gradients method (PCG). In particular,
we shall analyze some effective preconditioners for this method. A preconditioner can be defined as a second symmetric bilinear form, \( c(\cdot, \cdot) \) on \( S \times S \), such that the system of the form \( c(w, \phi_j) = f(\phi_j), j = 1, \ldots, n \), is solved at each iteration of PCG.

The first domain decomposition method has been credited to Schwarz [45], who used the convergence of his algorithm to prove the existence of solutions to elliptic partial differential equations on domains which were unions of simpler subdomains. The idea of domain decomposition algorithms today is to approximate the solution to a large, ill-conditioned system by solving smaller systems on subdomains of the original problem. The interest in these methods flowered in the 1980’s with the development of parallel computers, and many new algorithms have been developed and analyzed. Many of the algorithms borrow ideas from other methods, including substructuring [11, 12, 13, 15, 17, 18, 19] and multigrid methods [55, 54, 37, 38]. The methods appear to be divided into two basic categories. The first category, of the Schwarz-type, involves solutions to problems on overlapping subdomains. The second category involves the solution to problems on nonoverlapping subdomains, but requires the solution to a small global system. Both types of methods are used, and hybrid methods have been developed [13, 47]. The use of domain decomposition techniques to solve problems where refinement is used to improve the accuracy of the solution was an important development (see, e.g., [14, 37, 38]), which improved the usability of locally refined
finite difference and finite element meshes. The $p$-version finite element method, considered in our work here, is distinguished from the $h$-version finite element method and finite difference methods only in the way in which refinement is achieved. The $p$-version increases the degree of few finite elements, rather than increasing the number of elements. Thus, the idea of using one (or few) $p$-version elements as subdomains of domain decomposition algorithms seems natural.

It is known experimentally that the effectiveness of these preconditioners is adversely effected by the aspect ratio of the subdomains [47, 34]. Few theoretical results have been published, except in the case of the Schwarz-type methods, for which it is well established that the shape of the overlapping regions determines the rate of convergence of the method. This paper gives the results of previous work [36], and improves on the published theory, giving some results which explain the behavior of certain domain decomposition methods for the practical case of high aspect ratio subdomains.

In Chapter 2, we shall define the space $S$, the basis (or shape) functions, $\{\phi_1, \phi_2, \ldots, \phi_n\}$ used in the $p$-Version finite element method, describe the PCG algorithm, and state several useful properties of the algorithm. We will discuss the desirable properties of a good preconditioner. We shall also quote various results from the literature which will be needed later.

In Chapter 3, we shall describe a recently introduced domain decomposition preconditioner for the PCG algorithm, which treats each $p$-Version element as a
subdomain. We shall show that the performance of this preconditioner degrades under certain very realistic assumptions, namely when the subdomains have high aspect ratios. It will be shown, as was done in [36] that the condition number degrades as the square of the aspect ratio.

In Chapter 4, we shall modify the preconditioner from Chapter 3, and show that the new preconditioner has superior spectral properties to the original. The condition number for this preconditioner was introduced in [36], where we showed that the condition number degrades no faster than the aspect ratio. Here we will show that the condition number is bounded independently of the aspect ratio.
CHAPTER 2

PRELIMINARIES

In this chapter we will introduce some notation, define our problem, define the Preconditioned Conjugate Gradients Algorithm, and state some results of functional analysis on thin domains.

2.1 Function Spaces on Thin Domains

In this section, we define the norms and function spaces used in this paper, and state several propositions concerning functional spaces on a thin element.

2.1.1 Sobolev Spaces

We first define some norms and seminorms. We shall work with these explicitly given seminorms and norms for various domains (cf. [53, 39]).

On a domain $\mathcal{O} \subset R^2$, define the usual $L_2$-norm of a function $u$ on $\mathcal{O}$, $\|u\|_{L_2(\mathcal{O})}$, by

$$\|u\|_{L_2(\mathcal{O})}^2 = \iint_{\mathcal{O}} u^2 \, dx \, dy.$$
On a segment $s$ of $\partial \Omega$, define

$$\|u\|_{L^2(s)}^2 = \int_s u^2 \, ds.$$  

Now, consider the Sobolev seminorm [53]

$$|u|_{1, \Omega} = \left( \iint_\Omega |\nabla u|^2 \, dx \, dy \right)^{1/2},$$  \hspace{1cm} (2.1)

and the trace seminorm

$$|u|_{1/2, s} = \left( \int_s \left( \frac{u(t) - u(r)}{|t - r|} \right)^2 \, dr \, dt \right)^{1/2}. $$  \hspace{1cm} (2.2)

The corresponding norms are given by

$$\|u\|_{1, \Omega}^2 = \|u\|_{L^2(\Omega)}^2 + |u|_{1, \Omega}^2,$$

$$\|u\|_{1/2, s}^2 = \|u\|_{L^2(s)}^2 + |u|_{1/2, s}^2.$$  

We further denote

$$\|u\|_{k, \infty, \Omega} = \sum_{|\alpha| \leq k} \text{ess sup}_{\Omega} |D^\alpha u|$$

and

$$\|u\|_{k, \infty, s} = \sum_{|\alpha| \leq k} \text{ess sup}_{s} |D^\alpha u|.$$  

and, finally,

$$\|u\|_{0, s} = \|v\|_{1/2, \partial \Omega}$$

where $v$ is defined to be equal to $u$ on $s$ and extended by zero to all of $\partial \Omega$.

We now define several spaces of functions. $L^2(\Omega) = H_0(\Omega)$ is the space of (equivalence classes of) functions $u$ for which $\|u\|_{L^2(\Omega)}^2 < \infty$. The space $H^1(\Omega)$
is the space of (equivalence classes of) functions $u$ for which $\|u\|_{l,\mathcal{O}}^2 < \infty$, for $l = 0, 1/2, 1$. The space $C_0^\infty(\mathcal{O})$ consists of infinitely differentiable functions with support contained in a subdomain $\mathcal{O}$ of $\mathcal{O}$. We denote by $V$ the completion of $C_0^\infty(\mathcal{O})$ with respect to the $H_1(\mathcal{O})$-norm.

2.1.2 Invariance and Equivalence of Seminorms

Consider the family of domains $\hat{K}_\varepsilon = (-1,1) \times (-\varepsilon, \varepsilon)$, $0 < \varepsilon \leq 1$ (see Fig. 2.1). We shall use the associated family of bijections

$$\hat{K}_1 \rightarrow \hat{K}_\varepsilon, \quad F_\varepsilon(x,y) = (x, \varepsilon y).$$

We will also occasionally use the domains $\hat{K}_{ee} = (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)$ and the corresponding mappings

$$\hat{K}_1 \rightarrow \hat{K}_{ee}, \quad F_{ee}(x,y) = (\varepsilon x, \varepsilon y).$$

Let $\hat{d}_{1,\varepsilon} = \{-1\} \times (-\varepsilon, \varepsilon)$ be one short side of $\hat{K}_\varepsilon$ as in Fig. 2.1. By a simple
computation, we have for any \( u \in H^1(\hat{K}_\varepsilon) \) the invariance of seminorms,

\[
|u \circ F^{-1}_{\varepsilon}|_{1, \hat{K}_\varepsilon} = |u|_{1, \hat{K}_\varepsilon}, \tag{2.3}
\]

\[
|u \circ F^{-1}_{\varepsilon}|_{1/2, \hat{s}_{1, \varepsilon}} = |u \circ F^{-1}_{\varepsilon}|_{1/2, \hat{s}_{1, \varepsilon}} = |u|_{1/2, \hat{s}_{1, \varepsilon}}. \tag{2.4}
\]

Because

\[
\iint_{K_\varepsilon} |\nabla (u \circ F^{-1}_{\varepsilon})|^2 = \varepsilon \iint_{K_\varepsilon} \left( \frac{\partial u}{\partial x} \right)^2 + \frac{1}{\varepsilon} \iint_{K_\varepsilon} \left( \frac{\partial u}{\partial y} \right)^2,
\]

we have the equivalence of seminorms,

\[
\varepsilon^{1/2} |u|_{1, \hat{K}_\varepsilon} \leq |u \circ F^{-1}_{\varepsilon}|_{1, \hat{K}_\varepsilon} \leq \varepsilon^{-1/2} |u|_{1, \hat{K}_\varepsilon}. \tag{2.5}
\]

It is easy to see that the constants in (2.5) are sharp by considering the functions \( u(x, y) = x \) and \( u(x, y) = y \). Finally note that

\[
\iint_{K_\varepsilon} u = \frac{1}{\varepsilon} \iint_{K_\varepsilon} u \circ F^{-1}_{\varepsilon} = \frac{1}{\varepsilon^2} \iint_{K_{\varepsilon_2}} u \circ F^{-1}_{\varepsilon}. \tag{2.6}
\]

### 2.1.3 Short Side Trace Theorem

With the definitions (2.1) and (2.2), we get a version of the trace theorem for the family of domains \( \hat{K}_\varepsilon \).

**Theorem 2.1** [36] There is a constant \( C \) independent of \( \varepsilon \) such that for all \( \varepsilon \in (0, 1) \),

\[
|u|_{1/2, \hat{s}_{1, \varepsilon}} \leq C |u|_{1, \hat{K}_\varepsilon}, \quad \forall u \in H^1(\hat{K}_\varepsilon).
\]
Proof: Let \( u \in H^1(\tilde{K}_\varepsilon) \), \( \int_{\tilde{K}_\varepsilon} u = 0 \). Put \( v = u \circ F_\varepsilon \). Then \( v \in \tilde{K}_1 \), and by (2.6), \( \int_{\tilde{K}_1} v = 0 \). From the trace theorem (see, e.g., [53]),
\[
|v|_{1/2, \tilde{K}_1, \varepsilon} \leq \|v\|_{1/2, \tilde{K}_1, \varepsilon} \leq C \|v\|_{1, \tilde{K}_1} \leq C |v|_{1, \tilde{K}_1},
\]
and using (2.3) and (2.4) we get
\[
|u|_{1/2, \tilde{K}_1, \varepsilon} \leq C |u|_{1, \tilde{K}_1}, \quad \forall u \in H^1(\tilde{K}_\varepsilon).
\]

It remains to note that \( |u|_{1, \tilde{K}_1, \varepsilon} \leq |u|_{1, \tilde{K}_1} \). \( \square \)

2.1.4 Polynomial Spaces on \( \tilde{K}_\varepsilon \)

Let \( P_p(\tilde{K}_\varepsilon) \) be the space of all real tensor product polynomials of degree \( p \) on \( \tilde{K}_\varepsilon \), that is, \( u \in P_p(\tilde{K}_\varepsilon) \) if and only if
\[
u(x, y) = \sum_{i,j=0}^{p} c_{ij} x^i y^j, \quad c_{ij} \in \mathbb{R}.
\]
The following proposition, from [36], is concerned with bounding the \( | \cdot |_{1, \tilde{K}_1} \) seminorm of a bilinear function from its vertex values.

Lemma 2.1 Let \( u_0 \in P_1(\tilde{K}_\varepsilon) \), and \( \hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{v}_4 \) be the vertices of \( \tilde{K}_\varepsilon \) as in Fig. 2.1. Then
\[
|u_0|_{1, \tilde{K}_1, \varepsilon} \leq C \varepsilon^{-1/2} \left( |u_0(\hat{v}_1) - u_0(\hat{v}_4)| + |u_0(\hat{v}_2) - u_0(\hat{v}_3)| + \varepsilon^{1/2} \sum_{i=1}^{4} |u_0(\hat{v}_i)| \right),
\]
where \( C \) does not depend on \( u_0 \) or \( \varepsilon \in (0,1] \).
In [7, Corollary 6.3], it was proved that

\[ \|v\|_{0,\infty, \partial \tilde{K}_1} \leq C(1 + \log^{1/2} p)\|v\|_{1,\tilde{K}_1}, \quad \forall v \in P_p(\tilde{K}_1). \]  

(2.7)

This result was extended to a family of thin domains, again in [36].

**Theorem 2.2** There is a constant \( C \) independent of \( \varepsilon \) such that

\[ \|u\|_{0,\infty, \partial \tilde{K}_\varepsilon} \leq C\varepsilon^{-1/2}(1 + \log^{1/2} p)|u|_{1,\tilde{K}_\varepsilon} \]

for all \( u \in P_p(\tilde{K}_\varepsilon) \) such that \( \int_{\tilde{K}_\varepsilon} u = 0 \), and all \( \varepsilon \in (0,1] \).

*Proof:* Write \( u = v \circ F_\varepsilon^{-1} \), and use (2.5), (2.6), and (2.7). \( \square \)

On the short sides \( \tilde{s}_{1,\varepsilon} \) and \( \tilde{s}_{3,\varepsilon} \) we have the following bound independent of \( \varepsilon \).

**Theorem 2.3** [36] There is a constant \( C \) independent of \( \varepsilon \) such that

\[ \max_{r,s \in \tilde{s}_{1,\varepsilon}} |u(r) - u(s)| \leq C(1 + \log^{1/2} p)|u|_{1,\tilde{K}_\varepsilon}, \quad \forall u \in P_p(\tilde{K}_\varepsilon). \]  

(2.8)

for all \( \varepsilon \in (0,1] \) and all \( p \).

*Proof:* Let \( \varepsilon = 1 \). Because the inequality (2.8) is invariant to adding a constant function to \( u \), suppose without loss of generality that \( \int_{\tilde{K}_1} u = 0 \); then (2.8) follows from (2.7). For a general \( \varepsilon \in (0,1] \), (2.3) and (2.4) give

\[ \max_{r,s \in \tilde{s}_{1,\varepsilon}} |u(r) - u(s)| \leq C(1 + \log^{1/2} p)|u|_{1,\tilde{K}_{\varepsilon}}, \]

and it remains to note that \( |u|_{\tilde{K}_{\varepsilon}} \leq |u|_{\tilde{K}_\varepsilon} \). \( \square \)

We will also need the following discrete extension theorem.
Theorem 2.4 [36] Let $u \in P_p(\hat{K}_\varepsilon)$ such that $u(\hat{v}_1) = u(\hat{v}_4) = 0$. Then there exists a function $v \in P_p(\hat{K}_\varepsilon)$ such that $v = u$ on $\hat{s}_{1,\varepsilon}$, $v = 0$ on $\partial \hat{K}_\varepsilon \setminus \hat{s}_{1,\varepsilon}$, and

$$|v|_{1,\hat{K}_\varepsilon} \leq C \varepsilon^{-1/2} \left( |u|_{1/2,\hat{s}_{1,\varepsilon}} + (1 + \log^{1/2} p) ||u||_{0, \infty, \hat{s}_{1,\varepsilon}} \right) \tag{2.9}$$

for all $\varepsilon \in (0, 1]$, and the constant $C$ does not depend on $\varepsilon$ or $u$.

Proof: Let $\varepsilon = 1$. It was proved in [7, Theorem 6.6] that

$$||u||_{0, \hat{s}_{1,1}} \leq C \left( ||u||_{1/2, \hat{s}_{1,1}} + (1 + \log^{1/2} p) ||u||_{0, \infty, \hat{s}_{1,1}} \right)$$

Noting that

$$||u||^2_{1/2, \hat{s}_{1,1}} = ||u||^2_{1/2, \hat{s}_{1,1}} + ||u||^2_{1/2, \hat{s}_{1,1}} \leq ||u||^2_{1/2, \hat{s}_{1,1}} + ||u||^2_{0, \infty, \hat{s}_{1,1}},$$

we put $v = u$ on $\hat{s}_{1,1}$, $v = 0$ on $\partial \hat{K}_{1} \setminus \hat{s}_{1,1}$, and extend by [7, Theorem 7.5] to a function $v \in P_p(\hat{K}_{1})$ to get (2.9) for $\varepsilon = 1$. The general case then follows using (2.3), (2.4), and (2.5). □

2.2 The Discrete Form of the Model Problem

We consider solution of (1.1) by the usual Galerkin finite element method [27], except that we use a family of reference elements defined by $\hat{K}_\varepsilon = (-1, 1) \times (-\varepsilon, \varepsilon)$, $\varepsilon \in (0, 1]$, rather than a single reference element.

Let $\Lambda = \{K\}$ be a partition of $\Omega$ into nonoverlapping elements $K$, where each $K$ is the image of one reference element $\hat{K}_\varepsilon$,

$$\bar{\Omega} = \bigcup_{K \in \Lambda} \bar{K}, \quad K = G_K(\hat{K}_\varepsilon), \quad \varepsilon_K \in (0, 1],$$

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where $G_K \in P_i(\hat{K}_e)$ are bijections such that

$$|G_K|_{1,\infty,\hat{K}_e} \leq C h_K,$$

(2.10)

$$|G_K^{-1}|_{1,\infty,\hat{K}_e} \leq C h_K^{-1},$$

(2.11)

and

$$|J_{G_K}|_{0,\infty,\hat{K}_e} \leq C h_K^2,$$

(2.12)

$$|J_{G_K}^{-1}|_{0,\infty,\hat{K}_e} \leq C h_K^{-2},$$

(2.13)

where $J_{G_K}$ is the Jacobian of $G_K$, $h_K = \text{diam}(K)$. Roughly, this means that each element $K$ is nearly one of the reference elements, $\hat{K}_e$. We then define the finite element space

$$V_\Lambda = \{ u \in V : u|_K \circ G_K \in P_p(\hat{K}_e), \ \forall K \in \Lambda \},$$

and the discretization of (1.1) is

$$u \in V_\Lambda : \ a(u,v) = f(v), \ \forall v \in V_\Lambda.$$  

(2.14)

### 2.3 The Basis (Shape) Functions for the p-Version Finite Element Method

In order to solve (2.14) numerically, we must select a basis for $P_p(\hat{K}_e)$. A hierarchical basis, described in, e.g. [6], has three types of shape functions defined on a square reference element $\hat{K}_1$: 

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• the nodal shape functions;

• the side shape functions;

• the internal shape functions.

The nodal shape functions, \( N_i, i = 1, 2, 3, 4 \), are the usual bilinear functions determined by their values at the nodes (vertices) of the element. Now, define \( p_j, j = 1, \ldots \) by

\[
p_j(x) = \sqrt{\frac{2j + 1}{2}} \int_{-1}^{x} l_j(\xi) d\xi,
\]

where \( l_j \) is the \( j \)th Legendre polynomial. The side functions, \( S_{1,j}, j = 1, \ldots, p - 1 \), associated with the side \( \{(x, y) \in \{-1\} \times (-1, 1)\} \) are defined by

\[
S_{1,j}(x, y) = \frac{1 - x}{2} p_j(y), \quad \forall (x, y) \in \hat{K}_1.
\]

The other side functions are defined similarly. The interior functions, \( B_{j,k}, j, k = 1, \ldots, p - 1 \), are defined by

\[
B_{j,k}(x, y) = p_j(x) p_k(y), \quad \forall (x, y) \in \hat{K}_1.
\]

These basis functions will not be used further here, but give a motivation for the formulation of the preconditioners in later chapters. Note that there are \((p + 1)^2\) degrees of freedom on each element. The \( p \)-version finite element method achieves convergence by increasing the degree \( p \) of the elements (see, e.g., [6] and the references cited therein). The more familiar \((h\text{-version})\) finite element method
achieves convergence by decreasing the size $h$ of elements, with $p$ remaining fixed. Published theoretical results (e.g., [5, 25, 6]) indicate that solutions of high accuracy can be obtained with the $p$-version at lower computational cost than with the $h$-version. The $p$-version finite element method is quite similar to the spectral methods [24].

2.4 The Preconditioned Conjugate Gradient Algorithm

Once a basis, $\{\phi_1, \phi_2, \ldots, \phi_n\}$ for $V_A$ has been chosen, the solution to the discrete problem can be written

$$u = \sum_{i=0}^{n} c_i \phi_i.$$

The discrete problem (2.14) can be written as the linear system

$$c_1, c_2, \ldots, c_n : \quad a(\phi_i, \phi_j) = f(\phi_j), \quad i, j = 1, \ldots n. \tag{2.15}$$

This system can be written in matrix form,

$$Ax = b,$$

where $a_{ij} = a(\phi_i, \phi_j)$, $b_j = f(\phi_j)$ and $x_i = c_i$.

It is this problem which we will solve by the method of Preconditioned Conjugate Gradients (PCG) [22]. We must first define a preconditioning bilinear form, $c(\cdot, \cdot)$, also symmetric and positive-definite, with corresponding matrix,
\[ M = (m_{ij}), m_{ij} = c(\phi_i, \phi_j). \] We then define \( x_0 = 0, \) and \( r_0 = b, k = 1 \) and perform

\[ z_{k-1} : \quad Mz_{k-1} = r_{k-1} \quad \quad \quad \quad \quad \quad \quad (2.16) \]

\[ \beta_k = z_{k-1}^T r_{k-1} / z_{k-2}^T r_{k-2}, \quad (\beta_1 = 0) \quad \quad \quad \quad \quad \quad \quad (2.17) \]

\[ d_k = z_{k-1} + \beta_k d_{k-1}, \quad (d_1 = z_0) \quad \quad \quad \quad \quad \quad \quad (2.18) \]

\[ \alpha_k = z_{k-1}^T r_{k-1} / d_k^T Ad_k \quad \quad \quad \quad \quad \quad \quad (2.19) \]

\[ x_k = x_{k-1} + \alpha_k d_k \quad \quad \quad \quad \quad \quad \quad (2.20) \]

\[ r_k = r_{k-1} - \alpha_k Ad_k \quad \quad \quad \quad \quad \quad \quad (2.21) \]

until \( r_k = 0. \)

### 2.4.1 Spectral Bounds on the Error for PCG

Convergence \((r_k = 0, \) using exact arithmetic\) occurs in \( k \leq n \) steps. Usually, however, we treat this direct method as an iterative one, and stop with \( k \ll n. \)

We define the error \( e_k \) by \( e_k^2 = (x - x_k)^T A(x - x_k). \) We state a spectral bound on \( e_k. \) Suppose there exist positive \( m_1, m_2 \) such that

\[ m_1 c(v, v) \leq a(v, v) \leq m_2 c(v, v), \forall v \in V_h. \quad \quad \quad \quad \quad \quad \quad (2.22) \]

Then (see [22])

\[ e_k \leq 2e_0 \left( \frac{\sqrt{m_2} - \sqrt{m_1}}{\sqrt{m_2} + \sqrt{m_1}} \right)^k \quad \quad \quad \quad \quad \quad \quad (2.23) \]
When we have \( m_1, m_2 \) that satisfy the above and the ratio \( m_2/m_1 \) is the smallest possible, we call \( \kappa = m_2/m_1 \) the condition number of the preconditioned system, and write the condition number bound on the error

\[
e_j \leq 2e_0 \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^{2k}.
\]

We call \( c(\cdot, \cdot) \) the preconditioner.
CHAPTER 3

THE BILINEAR PRECONDITIONER

In this chapter, we give a general form for subspace decomposition preconditioners, as in Mandel [34]. We then define the first of two preconditioners, which uses the space of continuous piecewise bilinear functions as one of the subspaces. This preconditioner was first proposed in [7].

3.1 Preconditioning by Subspace Decomposition

We now describe a general principle for constructing the preconditioners in this paper. For each element $K$, define the local space

$$V_K = \{ u|_K : u \in V \}.$$ 

By integration in (1.2) over the elements $K$, we can write

$$a(u,v) = \sum_K a_K(u,v), \quad a_K(u,v) = \int_K (\nabla u)^T \nabla v.$$ 

Denote the energy norm on $K$ by

$$|u|_K = (a_K(u,u))^{1/2}.$$ 

Now let $V_K$ be split in a direct upper sum of $n_K + 1$ subspaces

$$V_K = V_{u_K} \oplus \cdots \oplus V_{n_K},$$
so every \( v_K \in V_K \) can be written uniquely as the sum of \( n_K + 1 \) components,

\[ v_K = v_{K,0} + \ldots + v_{K,n_K}, \quad v_{K,i} \in V_i. \]  

(3.1)

We then define

\[ c(u, v) = \sum_K c_K(u, v), \quad c_K(u, v) = a_K(u_{K,0}, v_{K,0}) + \ldots + a_K(u_{K,n_K}, v_{K,n_K}). \]  

(3.2)

On each element, \( c_K(\cdot, \cdot) \) is a block diagonal approximation to \( a_K(\cdot, \cdot) \), with \( n_K \) blocks, each corresponding to one of the subspaces \( V_{K,i} \) of \( V_K \). \( c(\cdot, \cdot) \) is the global assembly of the local contributions from each element, and retains some of the block diagonal structure from the local parts. Each of the blocks is better conditioned than the original global system, and can be solved in parallel.

Obviously, if for all \( K \in \Lambda \),

\[ m_{K,1} c_K(u, u) \leq a_K(u, u) \leq m_{K,2} c_K(u, u), \quad \forall u \in V_K, \]  

(3.3)

then (2.22) holds with some

\[ m_1 \geq \min_K m_{K,1}, \quad m_2 \leq \max_K m_{K,2}. \]

Thus the condition number \( \kappa = m_2/m_1 \) can be bounded using local bounds from each element, independently of the number of elements. Let \( m_{K,1} \) and \( m_{K,2} \) be the largest and smallest, respectively, such that (3.3) holds, and denote the local condition number

\[ \kappa_K = \frac{m_{K,2}}{m_{K,1}}. \]
Figure 3.1: Thin element $K$.

The following theorem, given in [33, Theorem 4.1], shows that to bound the condition number, it is necessary and sufficient to bound the decomposition (3.1) in energy. For other versions and related results, see [7, 29, 34, 52, 47].

**Theorem 3.1** Let $b_K$ be the least number such that

$$
\sum_{i=1}^{n_K} |u_{K,i}|^2_K \leq b_K |u|^2_K, \quad \forall u \in V_K.
$$

Then

$$
b_K \leq \kappa_K \leq (n_K + 1)b_K.
$$

In this and the following chapters, we give specific examples of the abstract decomposition (3.2) and use Theorem 3.1 to estimate the condition number $\kappa_K$ for each preconditioner. The preconditioners differ only in the choice of the subspaces, $V_{K,i}$.

### 3.2 Formulation of the Bilinear Preconditioner

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Let $K = G_K(K_{\varepsilon K}) \in \Lambda$ be an element as in Fig. 3.1. Define the decomposition of the local space $V_K$ by specifying the associated projections of an arbitrary $u \in P_p(\hat{K})$. Let

$$u_K = \sum_{i=0}^{5} u_{K,i}, \quad u_K \in V_K,$$

where

- $u_{K,0}$ is the mapped bilinear interpolant of $u$; that is, $u_{K,0}(v_i) = u_K(v_i)$, $i = 1, 2, 3, 4$, and $u_{K,0} \circ G_K \in P_1(\hat{K}_K)$;

- for $i = 1, 2, 3, 4$, $u_{K,i} = u - u_{K,0}$ on the side $s_i$, $u_{K,i} = 0$ on $\partial K \setminus s_i$, $u_{K,i} \in V_K$, and $|u_{K,i}|_K$ is minimal for all such functions;

- $u_{K,5} = u - \sum_{i=0}^{4} u_{K,i}$.

By summation over all elements, we can see that the preconditioner $c(u, v)$ can be written as

$$c(u, v) = c(u_0, v_0) + \sum_{e} c(u_e, v_e) + \sum_{K} c(u_{K,5}, v_{K,5}),$$

where $u = u_0 + \sum_{e} u_e + \sum_{K} u_{K,5}$, and

- $u_0$ is the piecewise mapped bilinear interpolant of $u$;

- $u_e$ is a function determined by its values on one edge $e$ of the finite element mesh, nonzero only on the two elements adjacent to the mesh;
• $u_{K,3}$ is a function nonzero only on element $K$.

The system (2.16) thus decouples into one subproblem for each side $e$, one subproblem for the interior of each element $K$, and a subproblem, the matrix of which is the stiffness matrix of the discretization of (1.1) by bilinear elements. Thus, one can see the analogy between this preconditioner and the element shape functions described in Chapter 2. For variants of this method, see [7, 33]. The implementation of the method and comparisons with other methods from the literature will be given in Chapter 5.

3.3 The First Condition Number Bound

This preconditioner was proposed and studied in [7], where it was proved that $\kappa_K \leq C(1 + \log^2 p)$ in the case $\varepsilon = 1$. Here we show how $\kappa_K$ deteriorates for $\varepsilon \to 0$. This analysis first appeared in [36].

**Theorem 3.2** For the preconditioner defined by the decomposition (3.5), it holds that

$$C_1 \varepsilon^{-2} \leq \kappa_K \leq C_2 \varepsilon^{-2}(1 + \log^2 p),$$

where $C_1, C_2 > 0$ do not depend on $\varepsilon$ or $p$.

**Proof:** In view of the assumptions on the mapping $G_K$, we may assume without loss of generality that

$$K = \hat{K}_e.$$
In view of Theorem 3.1, we need only to show that for the minimal $b_K$ in (3.4),

\begin{align}
  b_K &\leq C\varepsilon^{-2}(1 + \log^2 p), & (3.7) \\
  b_K &\geq C_1\varepsilon^{-2}, \quad C_1 > 0. & (3.8)
\end{align}

In [7], it was proved that (3.7) holds for $\varepsilon = 1$. We give the proof here for the sake of completeness. Let $u \in V_K$. Because adding a constant to $u$ will change only the component $u_{K,0}$ by the same constant, we may assume without loss of generality that $\int_K u = 0$. It follows from Theorem 2.2 and Lemma 2.1 that

\begin{align}
  |u_{K,0}|_{0,\infty,K} &\leq C(1 + \log^{1/2} p)|u|_{1,K}, \\
  |u_{K,0}|_{1,K} &\leq C(1 + \log^{1/2} p)|u|_{1,K}.
\end{align}

Now $u(v_i) = u_{K,0}(v_i)$, $i = 1, 2, 3, 4$, and from the Trace Theorem and Theorem 2.4, there exist functions $w_i \in P_p(K)$, $i = 1, 2, 3, 4$, so that $w_i = u - u_{K,0}$ on $s_i$, $w_i = 0$ on $\partial K \setminus s_i$, and

\[ |w_i|_{1,K} \leq C(1 + \log p)|u|_{1,K}. \]

By the definition of $u_{K,i}$, we have $|u_{K,i}|_{1,K} \leq |w_i|_{1,K}$. The estimate of $u_{K,5}$ follows from the triangle inequality, and we may conclude that (3.4) holds with $b_K \leq C(1 + \log^2 p)$ for $\varepsilon = 1$. The general case of (3.7) follows using (2.5).

To prove (3.8), consider a polynomial $f(x)$ such that $f(-1) = f(1) = 0$ and let $u(x, y) = f(x)$ on $K$. Then the component $u_{K,0} = 0$, and

\begin{equation}
  |u|_{1,K}^2 = \varepsilon \int_{-1}^{1} |f'|^2 dx = \varepsilon |f|_{1,(-1,1)}^2. \tag{3.9}
\end{equation}
Let \( v \in H^1(K) \) be determined by the boundary conditions

\[
v = f \text{ on } s_1, \quad v = 0 \text{ on } \partial K \setminus s_1; \tag{3.10}
\]

and by the requirement that \( |v|_{1,K} \) is minimal. Then the component \( u_{K,1} \) satisfies

\[
|u_{K,1}|_{1,K} \geq |v|_{1,K}, \tag{3.11}
\]

and \( v \) is the solution of the boundary value problem \( \Delta v = 0 \) in \( K \) with the boundary conditions (3.10). Expanding \( f \) in a sine series,

\[
f(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x,
\]

we can write \( v \) as

\[
v(x, y) = \sum_{n=1}^{\infty} a_n \frac{\sinh(n\pi(\varepsilon - y))}{\sinh 2n\pi \varepsilon} \sin n\pi x.
\]

Now, using Green's theorem,

\[
|v|_{1,K}^2 = \iint_K (\nabla v)^T \nabla v = \int_K v \frac{\partial v}{\partial n} - \int_{\partial K} \left. v \frac{\partial v}{\partial y} \right|_{y=-1} \, dx.
\]

By orthogonality of sine functions, it follows that

\[
|v|_{1,K}^2 = \frac{1}{2} \sum_{n=1}^{\infty} n\pi a_n^2 \frac{\cosh 2n\pi \varepsilon}{\sinh 2n\pi \varepsilon} \geq \frac{1}{\varepsilon} \sum_{n=1}^{\infty} \frac{a_n^2}{2} = \frac{1}{\varepsilon} \|f\|_{0,(0,1)}^2
\]

using the inequality

\[
\frac{\cosh t}{\sinh t} > \frac{1}{t}, \quad t > 0,
\]

and Parseval's equality. Thus using (3.9), we get

\[
\frac{|v|_{1,K}^2}{|u|_{1,K}^2} \geq \frac{1}{\varepsilon^2} \frac{\|f\|_{0,(-1,1)}^2}{\|f\|_{1,(-1,1)}^2}.
\]
Figure 3.2: Condition numbers for linear preconditioning.

Choosing \( f(x) = (x - 1)(1 - x) \) and using (3.11), we get (3.8). □

Figure 3.2 shows the numerically calculated condition numbers for \( p = 2 \) to 16 and \( \varepsilon = 1 \) to \( \varepsilon = 2^{-13} \). We can see that the condition numbers indeed grow as \( \varepsilon^{-2} \). The influence of \( p \) on the condition number is seen only for relatively small aspect ratios; for very small \( \varepsilon \), the influence of \( \varepsilon \) prevails. The numerical results are in complete correspondence with Theorem 3.2.
CHAPTER 4

THE ADAPTIVE-LINEAR PRECONDITIONER

Examination of the proof of the lower bound in Theorem 3.2 shows that the degradation of the condition number is caused by decomposing shape functions from opposite long sides into different subspaces. This motivates a preconditioner which decomposes $P_p(\hat{K}_e)$ into fewer subspaces. The following preconditioner, proposed and analyzed in [36], has a much lower condition number than the Bilinear Preconditioner of the previous chapter. In this chapter, we will define the preconditioner, state the results of [36], and give a new analysis which shows that the new preconditioner has a condition number which is bounded independently of $\varepsilon$.

4.1 Formulation

For $K = G_K(\hat{K}_e)$, we now define the decomposition of $V_K$ by

$$u_K = \sum_{i=0}^{3} u_{K,i},$$

(4.1)

where
• $u_{K,0}$ is the mapped bilinear interpolant of $u$: $u_{K,0}(v_i) = u_K(v_i)$, $i = 1, 2, 3, 4$, and $u_{K,0} \circ G_K \in P_1(\hat{K}_\varepsilon)$;

• $u_{K,1}$, and $u_{K,3}$ are defined by the boundary conditions $u_{K,i} = u - u_{K,0}$ on $s_i$, $u_{K,i} = 0$ on $\partial K \setminus s_i$, $i = 1, 3$, and by the requirement that $u_{K,i} \in V_K$ and $|u_{K,i}|_K$ is the minimal possible;

• $u_{K,2} = u_{K,0} - u_{K,1} - u_{K,3}$.

4.2 The First Condition Number Bound

In [36] we showed that the condition number $\kappa_K$ deteriorates only as $\varepsilon^{-1}$ rather than as $\varepsilon^{-2}$. We give that proof here, primarily to contrast the result and the method with that of the next section.

Theorem 4.1 For the preconditioner defined by the decomposition (4.1), it holds that

$$C_1 \min\{p, \varepsilon^{-1}\} \leq \kappa_K \leq C_2 \varepsilon^{-1}(1 + \log^2 p),$$

where $C_1 > 0$ and $C_2$ do not depend on $\varepsilon$ or $p$.

Proof: As in the proof of Theorem 3.2, we may assume without loss of generality that

$$K = \hat{K}_\varepsilon.$$
Because of Theorem 3.1, we only need to show that the minimal \( b_K \) in (3.4) satisfies

\[
    b_K \leq C \varepsilon^{-1} (1 + \log^2 p) \quad (4.2)
\]

\[
    b_K \geq C_1 \min \{ p, \varepsilon^{-1} \}, \quad C_1 > 0. \quad (4.3)
\]

Let \( u \in V_K = P_p(K) = P_p(K_\varepsilon) \). Because adding a constant to \( u \) will change only the component \( u_{K,0} \) by the same constant, we may assume without loss of generality that \( \int_K u = 0 \). Then we have from Theorem 2.2,

\[
    |u(v_i)| \leq C \varepsilon^{-1/2} (1 + \log^{1/2} p)|u|_{1,K}, \quad i = 1, 2, 3, 4, \quad (4.4)
\]

and from Theorem 2.3,

\[
    |u(v_1) - u(v_3)| \leq C (1 + \log^{1/2} p)|u|_{1,K} \quad (4.5)
\]

\[
    |u(v_2) - u(v_3)| \leq C (1 + \log^{1/2} p)|u|_{1,K}. \quad (4.6)
\]

It follows from (4.4)–(4.6) and Lemma 2.1 that

\[
    |u_{K,0}|_{1,K} \leq C \varepsilon^{-1/2} (1 + \log^{1/2} p)|u|_{1,K}. \quad (4.7)
\]

Further, (4.5) and (4.6) give

\[
    |u_{K,0}|_{1/2, s_i} \leq C (1 + \log^{1/2} p)|u|_{1,K}, \quad i = 1, 3, \quad (4.8)
\]

because \( u_{K,0} \) is linear on all sides, and because of the scaling invariance property (2.4). From (4.8) and Theorem 2.1, we have

\[
    |u - u_{K,0}|_{1/2, s_i} \leq C (1 + \log^{1/2} p)|u|_{1,K}, \quad i = 1, 3. \quad (4.9)
\]
Now Theorem 2.3 and the definition of $u_{K,0}$ imply that

$$||u - u_{K,0}||_{0,\infty,s_{i}} \leq C(1 + \log^{1/2} p)|u|_{1,K}, \quad i = 1, 3.$$  \hspace{1cm} (4.10)

Theorem 2.4 and (4.9), (4.10) now give the existence of functions $w_{i} \in P_{p}(K)$ such that $w_{i} = u - u_{K,0}$ on $s_{i}$, $w_{i} = 0$ on $\partial K \setminus s_{i}, i = 1, 3$, and

$$|w_{i}|_{1,K} \leq C\varepsilon^{-1/2}(1 + \log p)|u|_{1,K}.$$  

By the definition of $u_{K,i}$, we have $|u_{K,i}|_{1,K} \leq |w_{i}|_{1,K}$, which together with (4.7) and the Triangle Inequality concludes the proof of (4.2).

To prove (4.3), consider the function

$$u(x, y) = \frac{x^{p} y}{\varepsilon}$$
on $K = \hat{K}_{\varepsilon}$. Then the bilinear interpolant of $u$ is $u_{K,0}(x, y) = xy/\varepsilon$. By a simple computation,

$$|u|^{2}_{1,K} = \frac{1}{\varepsilon^{2p + 1}} + \frac{\varepsilon p^{2}}{3(2p - 1)}.$$ 

Thus

$$|u_{K,0}|^{2}_{1,K} > \frac{1}{3\varepsilon}$$

and if $p = [1/\varepsilon],$

$$|u|^{2}_{1,K} \leq C,$$

which proves (4.3). \hspace{1cm}$\Box$

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Figure 4.1: Condition numbers for improved preconditioning.

Fig. 4.1 summarizes numerically evaluated numbers $\kappa_K$ for $p = 2$ to 16 and $\varepsilon = 1$ to $2^{-13}$. The computed results agree with the previous theoretical result, but seem to show that the upper bound (4.2) is not sharp for $\varepsilon \to 0$. In the following section we will improve this upper bound.
4.3 An Improved Condition Number Bound

It appears in figure 4.1 that, for a given \( p \), the condition number remains bounded as \( \varepsilon \to 0 \). This is indeed the case, and we now develop an upper bound for the condition number which grows as a function of \( p \), but is independent of \( \varepsilon \). We accomplish this via energy bounds as in section 8, but this time in terms of the seminorms, \( \| \frac{\partial u}{\partial x} \|_{0, K_1}^2 \) and \( \| \frac{\partial u}{\partial y} \|_{0, K_1}^2 \). The idea is motivated by the following, an operator interpolation theorem, in the spirit of [28].

**Theorem 4.2** Let \( T : V \to V \) be a linear operator on a space \( V \), and let \( |u|_A \), \( |u|_B \) be seminorms on \( V \), and for \( \varepsilon \in (0, 1] \) let

\[
|u|_{\varepsilon}^2 = \varepsilon |u|_A^2 + \frac{1}{\varepsilon} |u|_B^2
\]

define a new seminorm on \( V \), and define

\[
|T|_{\varepsilon} = \sup_{u \in V, \ |u|_{\varepsilon} \neq 0} \frac{|Tu|_{\varepsilon}}{|u|_{\varepsilon}}.
\]

Suppose, that \( T \) satisfies, for some \( C_1, C_2, C_3, C_4 > 0 \),

\[
|Tu|_A = 0, \quad \text{if} \ |u|_A = 0
\]

\[
|Tu|_B = 0, \quad \text{if} \ |u|_B = 0
\]

\[
|T|_1 \leq C_1
\]

\[
\frac{|Tu|_A}{|u|_A} \leq C_2, \quad \forall u \in V.
\]

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Then, for any $\varepsilon \in (0, 1]$,

$$|T|_\varepsilon \leq \max\{C_1, C_2\}.$$  \hspace{1cm} (4.11)

**Proof:** Let $u \in V$, $|u|_\varepsilon \neq 0$, be given. Suppose first that $|u|_A = 0$. Then

$$\frac{|Tu|_\varepsilon}{|u|_\varepsilon} = \frac{|Tu|_1}{|u|_1} \leq C_1.$$  

So (4.11) holds in this case. Now suppose that $|u|_A \neq 0$. Then

$$\frac{|Tu|_\varepsilon^2}{|u|_\varepsilon^2} = \frac{\varepsilon |Tu|_A^2 + \frac{1}{\varepsilon} |Tu|_B^2}{\varepsilon |u|_A^2 + \frac{1}{\varepsilon} |u|_B^2} \equiv h(\varepsilon),$$

and $h$ is a monotone continuous function of $\varepsilon$. Thus, since $h(1) \leq C_1$, and

$$\lim_{\varepsilon \to 0} h(\varepsilon) = \frac{|Tu|_A^2}{|u|_A^2}$$

(4.11) follows. $\Box$

This result can be applied with $V = P_p(\tilde{K}_1)$, and

$$|u|_A^2 = \left\| \frac{\partial u}{\partial x} \right\|_{0, \tilde{K}_1}^2,$$

$$|u|_B^2 = \left\| \frac{\partial u}{\partial y} \right\|_{0, \tilde{K}_1}^2,$$

and with linear operators defined by, e.g., bilinear interpolation. Theorem 4.2 indicates that we should be able to bound

$$\left\| u \right\|_{0, \tilde{K}_\varepsilon}^2 = \varepsilon \left\| \frac{\partial u}{\partial x} \right\|_{0, \tilde{K}_1}^2 + \frac{1}{\varepsilon} \left\| \frac{\partial u}{\partial y} \right\|_{0, \tilde{K}_1}^2.$$
independently of $\varepsilon$, if we can bound $\|\partial u / \partial y\|^2_{0, \tilde{K}_1}$. We will now develop bounds for interpolation, trace, and extension operators in terms of these seminorms.

The following lemma makes use of the properties of Legendre polynomials (see, e.g. [1, 24]) to obtain a bound on the energy of bilinear interpolation in terms of the new seminorms.

**Lemma 4.1** Let $u \in P_p^2(\tilde{K}_1)$, and suppose $u_0$ is the bilinear interpolation of $u$ on $\tilde{K}_1$. Then

\[
\|\partial u_0 / \partial x\|^2_{0, \tilde{K}_1} \leq C p^2 \|\partial u / \partial x\|^2_{0, \tilde{K}_1}
\]

(4.12)

\[
\|\partial u_0 / \partial y\|^2_{0, \tilde{K}_1} \leq C p^2 \|\partial u / \partial y\|^2_{0, \tilde{K}_1}
\]

(4.13)

for some $C > 0$, independent of $p$. These inequalities are sharp, up to a constant factor.

**Proof:** To prove (4.12) note that $u$ can be written uniquely in the form

\[
\sum_{i=0}^{p} \sum_{j=0}^{p} a_{ij} l_i(x) L_j(y)
\]

where $l_i$ is the $i^{th}$ degree Legendre polynomial and

\[
L_j(y) = \begin{cases} 1, & \text{if } j = 0 \\ y, & \text{if } j = 1 \\ \int_{-1}^{1} l_{j-1}(\eta)d\eta, & j = 2, \ldots, p \end{cases}
\]
We use the following properties of Legendre polynomials

\[ \int_{-1}^{1} l_i(x) l_j(x) dx = \frac{2 \delta_{ij}}{2i + 1} \]  \hspace{1cm} (4.14)

\[ l_i(-1) = (-1)^i \]  \hspace{1cm} (4.15)

\[ l_i(1) = 1 \]  \hspace{1cm} (4.16)

\[ l'_i = \sum_{n=0}^{i-1} (2n + 1) l_n \]  \hspace{1cm} (4.17)

where \( \delta_{ij} \) is the Kronecker delta. From these it follows that

\[ \left\| \frac{\partial u}{\partial y} \right\|_{0,K_1}^2 = \int_{-1}^{1} \int_{-1}^{1} \left[ \sum_{i=0}^{p} \sum_{j=1}^{p} a_{ij} l_i(x) l_{j-1}(y) \right]^2 dx dy \]  \hspace{1cm} (4.18)

\[ = \sum_{i=0}^{p} \sum_{j=1}^{p} \frac{4a_{ij}^2}{(2i + 1)(2j - 1)} \]  \hspace{1cm} (4.19)

\[ \geq \sum_{i=0}^{p} \frac{4a_{i1}^2}{2i + 1} \]  \hspace{1cm} (4.20)

It also follows that if \( \omega_{ij} \) is the bilinear interpolation of \( l_i(x) L_j(y) \), then \( u_0 = \sum_{i=0}^{p} \sum_{j=0}^{p} a_{ij} \omega_{ij} \). By direct application of the properties of Legendre polynomials and their primitives (see above),

\[ \frac{\partial \omega_{ij}}{\partial y} = \begin{cases} 0, & \text{if } j \neq 1 \\ 1, & \text{if } j = 1, i \text{ even} \\ x, & \text{if } j = 1, i \text{ odd} \end{cases} \]

Thus,

\[ \frac{\partial u_0}{\partial y} = A + Bx, \]

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where,

$$A = \sum_{\substack{i=0 \atop i \text{ even}}}^{p} a_{i1},$$

$$B = \sum_{\substack{i=0 \atop i \text{ odd}}}^{p} a_{i1}.$$  

Then, using the Cauchy inequality for sums,

$$\left\| \frac{\partial u_0}{\partial y} \right\|_{0, K_1}^2 \leq 4(A^2 + B^2) \leq 4\left(\sum_{\substack{i=0 \atop i \text{ even}}}^{p} a_{i1}\right)^2 + 4\left(\sum_{\substack{i=0 \atop i \text{ odd}}}^{p} a_{i1}\right)^2 \leq 4\left(\frac{p + 2}{2}\right)\sum_{i=0}^{p} a_{i1}^2 \leq \frac{(p + 2)(2p + 1)}{2} \sum_{i=0}^{p} \frac{4a_{i1}^2}{2i + 1} \leq \frac{(p + 2)(2p + 1)}{2} \left\| \frac{\partial u}{\partial y} \right\|_{0, K_1},$$

where the last inequality follows by (4.20). Thus (4.13) holds, and by interchanging the roles of $x$ and $y$, we get (4.12).

To see that the bound is sharp, let

$$u(x, y) = \sum_{i=0}^{p} (2i + 1)l_i(x)y,$$

and $u_0(x, y) = (A + Bx)y$, where

$$A = \sum_{\substack{i=0 \atop i \text{ even}}}^{p} (2i + 1),$$

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\[ B = \sum_{i \geq 0, i \text{ odd}}^p (2i + 1). \]

Observe that \( A \geq c_1 p^2 \) and \( B \geq c_1 p^2 \) for some positive constant \( c_1 \). Then we have

\[
\left\| \frac{\partial u}{\partial y} \right\|_{0, \hat{K}_1}^2 = \sum_{i=0}^p (2i + 1)^2 \left( \frac{2}{2i + 1} \right) = 2 \sum_{i=0}^p (2i + 1) \leq c_0 p^2
\]

for some constant \( c_0 \), while

\[
\left\| \frac{\partial u_0}{\partial y} \right\|_{0, \hat{K}_1}^2 \geq A^2 + B^2 \geq 2c_1^2 p^4.
\]

Thus,

\[
\left\| \frac{\partial u}{\partial y} \right\|_{0, \hat{K}_1}^2 \leq \frac{c_0}{2c_1^2} \left\| \frac{\partial u_0}{\partial y} \right\|_{0, \hat{K}_1}^2.
\]

\[ \Box \]

The following is a discrete extension inequality, in the spirit of (2.4), but in terms of the new seminorms.

Lemma 4.2 (Extension from the side) Let \( f \) be a polynomial of degree \( p \). Then there is an \( \omega \in P_p(\hat{K}_1) \) such that

\[
\omega|_{s_1} = f
\]

\[
\omega|_{s_2} = 0
\]

\[
\left\| \frac{\partial \omega}{\partial y} \right\|_{0, \hat{K}_1}^2 \leq \frac{c_1}{p} |f|_{1, s_1}^2
\]

\[
\left\| \frac{\partial \omega}{\partial x} \right\|_{0, \hat{K}_1}^2 \leq c_2 p^2 |f|_{0, s_1}^2.
\]

If, in addition, \( f(-1) = f(1) = 0 \), then \( \omega|_{s_2} = \omega|_{s_1} = 0 \).
Proof: Let \( g \) be defined on \((-1, 1)\) by

\[
g(x) = \frac{(-1)^p}{2} [l_p(x) - l_{p-1}(x)],
\]

where \( l_i \) is the \( i^{th} \) Legendre polynomial. Then the following properties hold:

(4.15) and (4.16), imply

\[
g(-1) = \frac{(-1)^p}{2} [(-1)^p - (-1)^{p-1}] = 1,
\]

\[
g(1) = \frac{(-1)^p}{2} [1 - 1] = 0.
\]

From (4.14), we obtain

\[
\|g\|_{0,s_t}^2 = \frac{1}{4} \left[ \frac{2}{2p + 1} + \frac{2}{2p - 1} \right] \leq \frac{4}{3p}.
\]

Applying (4.17), we get

\[
|g|_{1,s_t}^2 = \sum_{n=0}^{p-1} (4n + 2) + \sum_{n+p-1 \text{ odd}}^{p-2} (4n + 2) = 2p^2.
\]

Let \( \omega(x, y) = g(x)f(y) \) on \( \hat{K}_1 \). It follows immediately from these properties

\[
\omega(-1, y) = g(-1)f(y) = f(y)
\]

\[
\omega(1, y) = g(1)f(y) = 0
\]

of \( g \) that if \( f(-1) = f(1) = 0 \) then \( \omega(x, -1) = \omega(x, 1) = 0 \)

\[
\|\frac{\partial \omega}{\partial y}\|_{0,\hat{K}_1}^2 = \|g\|_{0,s_t}^2 |f|_{1,s_t}^2 \leq \frac{4}{3p} |f|_{1,s_t}^2
\]

\[
\|\frac{\partial \omega}{\partial x}\|_{0,\hat{K}_1}^2 = |g|_{1,s_t}^2 \|f\|_{0,s_t}^2 = p^2 \|f\|_{0,s_t}^2.
\]

\( \square \)

The following lemma is a trace inequality with respect to the new seminorms.
Lemma 4.3 (Trace Inequality) Let $v \in P_p(K_1)$, and let $f = v|_{s_1}$. Then

$$\left| f \right|_{1,s_1}^2 \leq c_2 \left\| \frac{\partial v}{\partial y} \right\|_{0,K_1}^2. \tag{4.21}$$

Proof: We can expand

$$v(x,y) = \sum_{i=0}^{p} \sum_{j=0}^{p} a_{ij} l_i(x)L_j(y)$$

where, $l_i, L_j$ are as in Lemma 4.1. Then

$$\left\| \frac{\partial v}{\partial y} \right\|_{0,K_1}^2 = \sum_{i=0}^{p} \sum_{j=0}^{p} a_{ij}^2 \left( \frac{2}{2i + 1} \right) \left( \frac{2}{2j - 1} \right).$$

Now,

$$f(y) = \sum_{i=0}^{p} \sum_{j=0}^{p} a_{ij} (-1)^j L_j(y).$$

Thus, we have

$$f'(y) = \sum_{j=1}^{p} \left[ \sum_{i=0}^{p} a_{ij} (-1)^j \right] l_{j-1}(y).$$

Therefore, by Cauchy's inequality for sums,

$$\left\| f' \right\|_{0,s_1}^2 = \sum_{j=1}^{p} \left[ \sum_{i=0}^{p} a_{ij} (-1)^j \right]^2 \left( \frac{2}{2j - 1} \right) \leq \sum_{j=1}^{p} (p + 1) \sum_{i=0}^{p} a_{ij}^2 \left( \frac{2}{2j - 1} \right) \leq (p + 1) \left( \frac{2p + 1}{2} \right) \sum_{j=1}^{p} \sum_{i=0}^{p} a_{ij}^2 \left( \frac{2}{2i + 1} \right) \left( \frac{2}{2j - 1} \right) \leq (p + 1) \left( \frac{2p + 1}{2} \right) \left\| \frac{\partial v}{\partial y} \right\|_{0,K_1}^2 \leq 3p^2 \left\| \frac{\partial v}{\partial y} \right\|_{0,K_1}^2.$$
Lemma 4.4 Let $f$ be a polynomial of degree $p$ on $[-1, 1]$, let $f_0$ be the linear interpolant of $f$, and let $g = f - f_0$. Then

$$|g|_{1,(-1,1)}^2 \leq |f|_{1,(-1,1)}^2.$$  

Proof: Let $L_i$, $i = 1, \ldots, p$ be defined as in Lemma 4.1. Then $f$ has the expansion

$$f = \sum_{i=0}^{p} b_i L_i$$

and

$$|f|_{1,(-1,1)}^2 = \sum_{i=1}^{p} b_i^2 \frac{2}{2i-1}.$$  

Because $L_i(-1) = L_i(1) = 0$, for $i > 1$, $L_0(y) = 1$, and $L_1(y) = y$, we have $f_0 = b_0 + b_1 y = b_0 L_0(y) + b_1 L_1(y)$. Therefore, $g = f - f_0$ has the expansion

$$g = \sum_{i=2}^{p} b_i L_i$$

and

$$|g|_{1,(-1,1)}^2 = \sum_{i=2}^{p} b_i^2 \frac{2}{2i-1} \leq |f|_{1,(-1,1)}^2.$$  

□

The following theorem gives a bound on the projections defined by the Adaptive-Linear decomposition.

Theorem 4.3 Let $u \in P_p(\hat{K}_1)$, $v = u \circ F_{\varepsilon}^{-1}$, and $v_i$, $i = 0, 1, 2, 3$ be defined on $\hat{K}_\varepsilon$ by

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• $v_0$ is the bilinear interpolant of $v$ on $\hat{K}_e$;

• $v_1$ minimizes $a_{\hat{K}_e}(\cdot, \cdot)$, subject to:
  \[
  \begin{cases}
  v_1|_{s_i} = (v - v_0)|_{s_i} & ; \\
  v_1|_{s_i} = 0, & i = 2, 3, 4
  \end{cases}
  \]

• $v_2$ minimizes $a_{\hat{K}_e}(\cdot, \cdot)$, subject to:
  \[
  \begin{cases}
  v_2|_{s_3} = (v - v_0)|_{s_3} & ; \\
  v_2|_{s_i} = 0, & i = 1, 2, 4
  \end{cases}
  \]

• $v_3 = v - (\sum_{i=0}^{2} v_i)$.

Then we have

\[
|v_0|^2_{1, \hat{K}_e} \leq c_0 p^2 |v|^2_{1, \hat{K}_e},
\]

\[
|v_i|^2_{1, \hat{K}_e} \leq c_1 (\epsilon p^4 + p) |v|^2_{1, \hat{K}_e}, \quad i = 1, 2
\]

\[
|v_3|^2_{1, \hat{K}_e} \leq c_2 (\epsilon p^4 + p^2) |v|^2_{1, \hat{K}_e}.
\]

**Proof:** For $v_0$, the bilinear interpolant of $v$ on $\hat{K}_e$, let $u_0$ be the bilinear interpolant of $u$ on $\hat{K}_1$. Then $v_0 = u_0 \circ F_\epsilon^{-1}$. Thus, by Lemma 4.1, we have

\[
|v_0|^2_{1, \hat{K}_e} = \epsilon \left\| \frac{\partial u_0}{\partial x} \right\|^2_{0, \hat{K}_1} + \frac{1}{\epsilon} \left\| \frac{\partial u_0}{\partial y} \right\|^2_{0, \hat{K}_1}
\]

\[
\leq c p^2 \epsilon \left\| \frac{\partial u}{\partial x} \right\|^2_{0, \hat{K}_1} + \frac{c p^2}{\epsilon} \left\| \frac{\partial u}{\partial y} \right\|^2_{0, \hat{K}_1}
\]

\[
= c p^2 |u|^2_{1, \hat{K}_e}.
\]

For $v_1$, let $u, u_0$ be as above, $g = u|_{s_1}$, and $f = (u - u_0)|_{s_1}$. By Lemmas 4.2, 4.3 and 4.4, there is a function $\omega \in P_p(\hat{K}_1)$ such that $v_1 = \omega \circ F_\epsilon^{-1}$ on $\partial \hat{K}_e$, and

\[
\left\| \frac{\partial \omega}{\partial x} \right\|^2_{0, \hat{K}_1} \leq c p^2 \|f\|^2_{0, s_1}
\]

\[40\]
\[ \left| \frac{\partial \omega}{\partial y} \right|_{0, \bar{K}_1}^2 \leq \frac{c}{p} |f|_{1, s_1}^2 . \]  \hspace{1cm} (4.26)

So, by the minimality of \( v_1 \),

\[ |v_1|_{1, \bar{K}_1}^2 \leq |\omega \circ F_\epsilon^{-1}|_{1, \bar{K}_1}^2 . \]

By (2.6),

\[ |\omega \circ F_\epsilon^{-1}|_{1, \bar{K}_1}^2 = \epsilon \left\| \frac{\partial \omega}{\partial x} \right\|_{0, \bar{K}_1}^2 + \frac{1}{\epsilon} \left\| \frac{\partial \omega}{\partial y} \right\|_{0, \bar{K}_1}^2 . \]

Lemma 4.3 gives

\[ \epsilon \left\| \frac{\partial \omega}{\partial x} \right\|_{0, \bar{K}_1}^2 + \frac{1}{\epsilon} \left\| \frac{\partial \omega}{\partial y} \right\|_{0, \bar{K}_1}^2 \leq cp^2 \epsilon |f|_{1, s_1}^2 + \frac{c}{p \epsilon} |f|_{0, s_1}^2 . \]

Since \( f(-1) = f(1) = 0 \),

\[ cp^2 \epsilon |f|_{1, s_1}^2 + \frac{c}{p \epsilon} |f|_{0, s_1}^2 \leq cp^2 \epsilon |f|_{1, s_1}^2 + \frac{c'}{p \epsilon} |f|_{1, s_1}^2 \]

\[ = (cp^2 \epsilon + \frac{c'}{p \epsilon}) |f|_{1, s_1}^2 . \]

From Lemma 4.4, we have

\[ (cp^2 \epsilon + \frac{c'}{p \epsilon}) |f|_{1, s_1}^2 \leq (cp^2 \epsilon + \frac{c'}{p \epsilon}) |g|_{1, s_1}^2 . \]

Now, by the trace lemma, (4.3),

\[ (cp^2 \epsilon + \frac{c'}{p \epsilon}) |g|_{1, s_1}^2 \leq (c_1 p^4 \epsilon + \frac{c_1 p^4 \epsilon}{\epsilon}) \left\| \frac{\partial u}{\partial y} \right\|_{0, \bar{K}_1}^2 \]

\[ = (c_1 p^4 \epsilon + \frac{c_1 p^4 \epsilon}{\epsilon}) \left[ \epsilon \left( \frac{1}{\epsilon} \left\| \frac{\partial u}{\partial y} \right\|_{0, \bar{K}_1}^2 \right) \right] . \]
\[ \leq (c_1 \epsilon + \frac{c_1 p}{\epsilon}) \left[ \epsilon \left( \frac{1}{\epsilon} \left\| \frac{\partial u}{\partial y} \right\|_{0,K_1}^2 + \epsilon \left\| \frac{\partial u}{\partial x} \right\|_{0,K_1}^2 \right) \right]. \]

Finally, by (2.6),

\[ (c_1 \epsilon + \frac{c_1 p}{\epsilon}) \left[ \epsilon \left( \frac{1}{\epsilon} \left\| \frac{\partial u}{\partial y} \right\|_{0,K_1}^2 + \epsilon \left\| \frac{\partial u}{\partial x} \right\|_{0,K_1}^2 \right) \right] = c_1 (\epsilon p + p) |u|_{1,K_1}, \]

with \( c, c', \) and \( c_1 \) constants, each independent of \( \epsilon, p, \) and the bound for \( v_1 \) is proved. Similarly, the bound for \( v_3 \) follows.

For \( v_2 \), we have, by the triangle inequality, Lemma 4.1 and Lemma 4.2,

\[ |v_2|^2_{1,K_1} = |v - (v_0 + v_1 + v_3)|^2_{1,K_1} \]

\[ \leq |v|^2_{1,K_1} + |v_0|^2_{1,K_1} + |v_1 + v_3|^2_{1,K_1} \]

\[ |v|^2_{1,K_1} + |v_0|^2_{1,K_1} + |v_1 + v_3|^2_{1,K_1} \leq [1 + c_0 p^2 + 2c_1 (\epsilon p^2 + p)] |v|^2_{1,K_1} \]

\[ \leq c_2 (\epsilon p^4 + p^2) |v|^2_{1,K_1}. \]

\[ \square \]

We cannot apply Theorem 4.2 directly to results given above, because the minimum energy extension operator is a different operator for each \( \epsilon \). We can, however, obtain a bound on the condition number which is independent of \( \epsilon \).

**Theorem 4.4** The preconditioner defined by the decomposition (4.1) has a condition number which is bounded by

\[ \kappa_{K_1} \leq C p^2 (1 + \log p) \quad (4.27) \]

where \( C \) is a positive constant, independent of \( \epsilon, p \).
Proof: As stated before, we lose no generality in assuming that $\dot{K} = \dot{K}_\varepsilon$. It follows immediately from Theorem 4.3 and Theorem 3.1 that there is a $C_0 > 0$, independent of $p, \varepsilon$ such that

$$\kappa_{\dot{K}} \leq C_0(p^2 + \varepsilon p^4),$$

from which we can immediately get the bound $\kappa_{\dot{K}} \leq C_0 p^4$, but we can do much better. From Theorem 4.1, we know that there is a $C_1 > 0$ such that

$$\kappa_{\dot{K}} \leq C_1 \frac{1 + \log^2 p}{\varepsilon}.$$

Thus, there is a positive constant, $C_3$, such that

$$\kappa_{\dot{K}} \leq C_3 \min\{p^2 + \varepsilon p^4, \frac{1 + \log^2 p}{\varepsilon}\}.$$

Since $p^2 + \varepsilon p^4$ is a monotonically increasing function of $\varepsilon > 0$, and $(1 + \log^2 p)/\varepsilon$ is monotonically decreasing, their minimum achieves its maximum value at a point $\varepsilon^* > 0$ where the two functions are equal:

$$p^2 + \varepsilon^* p^4 = \frac{1 + \log^2 p}{\varepsilon^*}.$$

This occurs at

$$\varepsilon^* = -1 \pm \frac{(5 + 4 \log^2 p)^{1/2}}{2p^2}.$$

Then we have

$$p^2 + \varepsilon^* p^4 = p^2 \frac{1 + (1 + \log^2 p)}{2(5 + 4 \log^2 p)^{1/2}} \leq C_4 p^2 (1 + \log p).$$
for some positive constant $C_4$. Therefore,

$$\kappa_K \leq C_3 C_4 p^2 (1 + \log p).$$

\[\square\]

The results of this chapter combine to give the condition number bound

$$\kappa_K \leq C \min\{ (1 + \log^2 p)/\epsilon, p^3 (1 + \log p)\}$$

which more fully predicts the behavior of the condition numbers given in figure 4.1.
CHAPTER 5

IMPLEMENTATION, VARIATIONS AND CONCLUSIONS

We propose a hybrid algorithm which combines the use of the preconditioners from Chapters 3 and 4. The choice of preconditioner can be made on each element independently; precondition nearly square elements with the simpler Bilinear Preconditioner, and precondition the more distorted elements with the more effective Adaptive-Bilinear Preconditioner.

With a certain ordering of the hierarchical basis functions using the shape functions from Chapter 2, the matrix $A$ corresponding to this problem has the block form:

$$
\begin{pmatrix}
  A_{11} & A_{12} & A_{13} \\
  A_{21} & A_{22} & A_{23} \\
  A_{31} & A_{32} & A_{33}
\end{pmatrix}
$$

The $A_{11}$ block corresponds to the stiffness matrix of the original problem, discretized with bilinear elements. The $A_{22}$ block is a block 7-diagonal system, with blocks of order $p$, each corresponding to the edge basis functions of a single edge, shared between two elements. The $A_{33}$ block is a block diagonal system, with blocks of order $(p - 1)^2$, each corresponding to the interior basis functions of a
single element.

Instead of the original shape functions, we change the side shape functions to satisfy the minimum energy condition of Chapter 3. This change-of-basis can be performed on each element independently. The basis functions are no longer hierarchical, so one of the advantages of $p$-refinement (e.g. [56]) is lost. The global matrix is then assembled in the usual way, giving the following matrix in block form:

$$
\begin{pmatrix}
A_{11} & B_{12} & A_{13} \\
B_{21} & B_{22} & \\
A_{31} & A_{33}
\end{pmatrix}.
$$

The matrix $M$ corresponding to preconditioner from Chapter 3 has the block form:

$$
\begin{pmatrix}
M_{11} \\

M_{22} \\

M_{33}
\end{pmatrix}.
$$

The blocks correspond to the original problem: $M_{11} = A_{11}$, $M_{22}$ decouples into small 'edge' problems, $M_{33} = A_{33}$ is block diagonal, with diagonal blocks corresponding to the interior functions of each element. This preconditioner is different from that proposed in [16], in that the interior functions are not eliminated, and the vertex basis functions are not changed at all. This allows the global (coarse grid) system to be solved a priori, or in parallel. This is simpler than the previous algorithm, but does not generalize well to
three dimensions, because of reasons cited in [7], namely that the energy of the
vertex functions grows very quickly as a function of $p$. The preconditioners in
[34, 7, 33, 36] advocate elimination of the unknowns, rather than orthogonaliza-
tion. The method in [31] uses a symbolic elimination, with iteration on a reduced
system. This is similar to the methods developed in [46, 47], which use variations
of the Schwarz algorithm on the reduced system. The methods presented here
bear some similarity to the element-by-element techniques of Hughes et al (see,
e.g., [26]), but the EBE methods appear to require many more iterations.

The interior unknowns of the Adaptive-Bilinear Preconditioner are coupled
to the edge functions corresponding to long sides of the corresponding element,
so the procedure is not quite as simple. One can, by eliminating the interior
variables from the edge functions, one can iterate on the reduced system. The
interior functions can be eliminated, by substructuring, on each element inde-
pendently. One obtains in this case a subproblem corresponding to a chain of
thin elements connected on long sides. Adjacent chains (and nearly square ele-
ments) are decoupled, however, and thus can be solved in parallel. Thus, we have
an algorithm which adapts to distorted elements, and which still decouples the
original problem into much smaller subproblems.

As was noted in [7], the method of Chapter 3 does not generalize well to three
dimensions, because the condition number grows too quickly as a function of $p$.
Other methods have been proposed for the 3-dimensional case [7] [35] [32] for
which the idea of combining subspaces for distorted elements appears promising; the analysis techniques of Chapters 3 and 4 should be applicable to those cases as well. The method of Chapter 3 was based upon a similar one for the $h$-version finite element method and finite difference methods [16]. The techniques developed here should also have their analogues for that case, when subdomains are distorted. The $p$-version finite element method is closely related to the spectral methods [24] [41] and the techniques given here may be applicable to analyzing and improving performance of domain decomposition methods (see, e.g., [42]). The technique of analyzing the behavior of numerical methods by considering the energy of subspaces in the one-sided energy seminorm of Chapter 4 appears to have wide applicability.
BIBLIOGRAPHY


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