Theoretical Fluid Mechanics

Laminar Flow Velocity Profile

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Pipe flows are divided into laminar flow, transitional flows, and turbulent flows, depending on the flow Reynolds number.

A. Laminar Flows and Reynolds Number

For a circular pipe, the laminar flow is defined to have the flow Reynolds number < 2000.

Reynolds Number $R_e = \frac{\rho VD}{\mu} = \frac{VD}{\nu}$ <2000 for laminar flow

For a non-circular conduit, the diameter has to be replaced by hydraulic radius defined as:

$R_h = \frac{A}{P}$ = Flow area to Wetted Perimeter ratio

For a flow full condition in a circular pipe, the hydraulic radius is four times the diameter.

$R_h = \frac{\pi D^2 / 4}{\pi D} = \frac{D}{4}$ or $D=4R_h$

$R_e = \frac{\rho V(4R_h)}{\mu} = \frac{VR_h}{\nu}$ <500 for laminar flow in non-circular conduit.

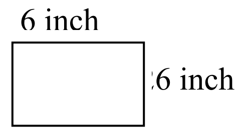
QA.1 At 35°F, Crude Oil (S=0.925) flows in a circular pipe of 12-inch in diameter. Determine the maximum flow rate to be a laminar flow.

At 35°F, the kinematic viscosity for the oil is: $\nu=3*10^{-3}$ ft^2/sec.

Reynolds Number $R_e = \frac{VD}{\nu} = \frac{V \times 1.0}{3 \times 10^{-3}} \le 2000$. So, V< 6.0 fps and Q <1.18 cfs.

QA.2 At 35°F, Crude Oil (S=0.925) flows in a 6-inch by 6-inch box conduit. Determine the maximum laminar flow rate.

(1) Flowing flow condion



*A = 6/12 * 6/12 = 0.25 sq ft*
P = 2 (6/12 + 6/12) = 2.0 ft

R_h = 0.25/2 = 0.125

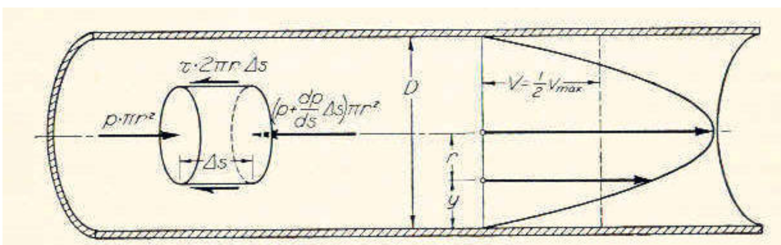
Reynolds Number $R_e = \frac{VR_h}{\nu} = \frac{V \times 0.125}{3 \times 10^{-3}} \leq 500$*, So V<12 fps and Q <3.0 cfs.*

(2) Flowing **<u>almost</u>** full (The water depth is 95 to 98% of the conduit height.)

A = 0.25 sq ft and P = 6/12+ 6/12 + 6/12 = 1.5 ft (only three sides are wetted.)
Can you calculate the flow velocity and flow rate?

B. Steady Flow in Circular Pipe

A steady flow in a circular pipe has a constant flow rate and unchanged velocity profile.



The coordinate system (x, r) is set up to have x = horizontal distance in the flow direction, r = vertical distance from the center line of the pipe. Let D be the diameter of the pipe and R be the radius of the pipe. The variable "r" varies from 0 (centerline) to R (on the wall).

$$0 \leq r \leq R(= D/2)$$

The forces applied to a steady laminar flow are in an equilibrium condition. In a horizontal pipe, the pressure force in the flow is balanced by the shear force on the wall.

$$-\frac{dP}{dx}\Delta x \pi r^2 = \tau 2\pi r \Delta x \quad \text{(Z = constant)}$$

The above is reduced to

$$\tau = -\frac{r}{2}\frac{dP}{dx} \qquad (1)$$

Define the pressure gradient as:

$$\frac{P_1}{\gamma} + Z + \frac{V^2}{2g} = \frac{P_2}{\gamma} + Z + \frac{V^2}{2g} + H_f \quad \text{Or } H_f = \frac{P_1 - P_2}{\gamma} = -\frac{\Delta P}{\gamma}.$$

Aided by H_f, Eq A becomes:

$$\tau = -\frac{r}{2}\frac{dP}{dx} = \frac{r}{2}\frac{\Delta P}{L} = \frac{r}{2}\gamma\frac{H_f}{L} \qquad (2)$$

C. Laminar Pipe Flows

The shear stress for laminar flow is linearly related to the fluid viscosity as:

$$\tau = \mu\frac{du}{dr}$$

Aided by the above relationship, Eq A becomes:

$$\mu\frac{du}{dr} = -\frac{r}{2}\frac{dP}{dx}$$

To integrate the above yields

$$u = \frac{1}{\mu}\frac{dP}{dx}\frac{r^2}{4} + C \qquad (3)$$

The integration constant, C, can be determined by u=0 at r =D/2 (on the solid boundary).

$$0 = \frac{1}{\mu}\frac{dP}{dx}\frac{R^2}{4} + C \qquad (4)$$

Substituting (4) into (3) yields

$$u = -\frac{1}{4\mu}\frac{dP}{dx}(R^2 - r^2) \text{ --- Parabolic relationship} \qquad (5)$$

Re-arranging Eq 5 yields:

$$u = -\frac{1}{4\mu}\frac{dP}{dx}R^2[1-(\frac{r}{R})^2] \qquad (6)$$

At r = 0, the centerline velocity is calculated as:

$$u = U_m = \frac{R^2}{4\mu}\frac{dP}{dx} \qquad (7)$$

Substituting U_m into (7) yields the non-dimensional laminar flow velocity profile as:

$$\frac{u}{U_m} = 1 - (\frac{r}{R})^2 \text{ --- Parabolic curve} \quad (8)$$

The above demonstrates the fact that a laminar flow is parabolic in nature. For a parabolic velocity profile, the average velocity, V, is 1/2 the centerline velocity, U_m.

$$V = \frac{1}{2}U_m = \frac{R^2}{8\mu}\frac{dP}{dx} = \frac{\gamma D^2}{32\mu}\frac{P_1 - P_2}{L\gamma} = \frac{\gamma D}{32\mu}\frac{D}{L}H_f \quad (9)$$

Re-arrange the above to yield:

$$H_f = \frac{64}{R_e}\frac{L}{D}\frac{V^2}{2g} \qquad \text{for laminar flows only} \quad (10)$$

Discussions:

Where is the surface roughness in Eq 10? Does that mean a rough pipe will dissipate the same amount of energy as the smooth pipe? What is your explanation?

Eq 10 was derived for laminar flows. Let us generalize the loss equation to include both fluid property, R_e, and surface roughness height, e, as:

$$H_f = f\frac{L}{D}\frac{V^2}{2g} \qquad \text{for both laminar and turbulent flows} \quad (11)$$

$$f = fct(R_e, \frac{e}{D})$$

The above functional relationship has two extreme conditions. It implies that the laminar flows are dictated by *f = fct* (R_e) and the fully turbulent flows are dictated by *f =fct (e/D)*, and the transitional flows are dictated by *f = fct* $(R_e$*, and e/D)*. Substituting Eq 11 into Eq 2 yields:

$$\frac{\tau}{\rho} = \frac{fV^2}{8} \quad \text{or} \quad \tau = \frac{\rho f V^2}{8} \quad (12)$$

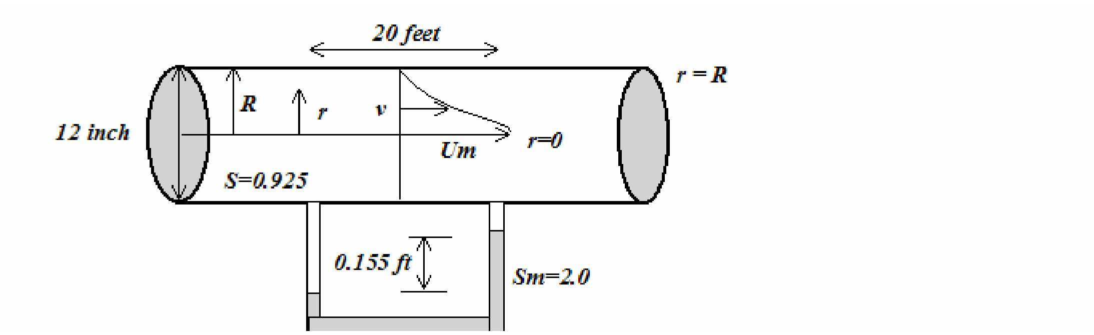
For convenience, the ratio of τ to ρ is termed shear velocity, u*, because it has a velocity unit as:

$$u^* = \sqrt{\frac{\tau}{\rho}} \text{ in fps or mps} \quad (13)$$

or

$$\frac{u^*}{V} = \sqrt{\frac{f}{8}} \quad (14)$$

QB-1 Crude oil (S=0.925) flows through a 12-inch circular pipe. Over a horizontal distance of 20 feet, the deflection on the manometer is 0.155 ft when using the working fluid with S=2.0. The flow rate is 2.36 cfs. Determine the velocity distribution, friction factor, and shear force on the wall.



$$V = \frac{Q}{A} = \frac{2.36}{\frac{3.1416 \times 1^2}{4}} = 3.0 \text{ fps}$$

$$H_f = (\frac{S_m}{S} - 1)\Delta h = (\frac{2.0}{0.925} - 1) \times 0.155 = 0.18 \text{ ft}$$

$$R_e = \frac{VD}{\nu} = \frac{3.0 \times 1}{3.0 \times 10^{-3}} = 1000. \text{ It is a laminar flow.}$$

$$f = \frac{64}{R_e} = 0.064$$

$$\text{Check: } H_f = 0.064 \times \frac{20}{1} \times \frac{3^2}{2 \times 32.2} = 0.18 \text{ ft}$$

$$U_m = 2V = 6.0 \text{ fps}$$

$$\frac{u}{U_m} = 1 - (\frac{r}{0.5})^2 \quad \text{For instance, at r = 0.25, u = 4.5 pfs}$$

$$\frac{\tau}{\rho} = \frac{fV^2}{8} = \frac{0.064 \times 3^2}{8} = 0.072 \text{ (f/s)}^2 \text{ or } \tau = 0.129 \text{ lb/ft}^2 \text{ on the wall}$$

As a result, for a pipe length (L) of 10 feet, the friction force on the pipe wall is

$$F_\tau = \tau\pi DL = 0.129 \times 3.1416 \times 1.0 \times 10 = 4.05 \text{ pounds}$$