MAXIMAL-CLIQUE PARTITIONS

by

Chariya Uiyyasathain

B.S., Chulalongkorn University, 1996

M.S., University of Illinois at Urbana-Champaign, 1998

A thesis submitted to the

University of Colorado at Denver

in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Applied Mathematics

2003

This thesis for the Doctor of Philosophy

degree by

Chariya Uiyyasathain

has been approved

by

Kathryn L. Fraughnaugh

William E. Cherowitzo

Harvey J. Greenberg

Michael S. Jacobson

Date

Uiyyasathain, Chariya (Ph.D., Applied Mathematics)

Maximal-clique Partitions

Thesis directed by Professor Kathryn L. Fraughnaugh

ABSTRACT

This dissertation discusses maximal-clique partition problems with emphasis on existence problems. We obtain a necessary and sufficient condition for a line graph to have a maximal-clique partition. There are three clique parameters of a graph G: the clique covering number $cc(G)$, the clique partition number $cp(G)$, and the maximal-clique partition number $mcp(G)$. We evaluate all three clique parameters for line graphs. In addition, we discuss the complexity of the problem of finding maximal-clique partitions for line graphs.

We then investigate graphs with maximal-clique partitions of different sizes. The graph $L(K_5)$ has two maximal-clique partitions of different sizes. We confirm that there are no graphs with maximal-clique partitions of different sizes of fewer vertices than $L(K_5)$. Also, for each natural number n, we give a clique-inseparable graph with n maximal-clique partitions of n different sizes.

We investigate the question of the existence of a graph G with $cc(G) = cp(G) < mcp(G)$. We achieve infinitely many graphs satisfying this property.

Finally, we solve another existence problem regarding graphs with $cc(G) < cp(G) < mcp(G)$. In 1982, Pullman, Shank and Wallis showed a graph of 11 vertices satisfying $cc(G) < cp(G) < mcp(G)$ and asked whether or not there exists a graph of fewer vertices with $cc(G) < cp(G) < mcp(G)$. We confirm that there is no such graph.

This abstract accurately represents the content of the candidate's thesis. I recommend its publication.

Signed ______________________________

Kathryn L. Fraughnaugh

ACKNOWLEDGMENT

Special thanks are due to my advisor, Professor Kathryn Fraughnaugh, for introducing me to this interesting topic and advice, for her guidance and encouragement. Without her continued support, this thesis would not be possible. I am also grateful to and very appreciative of my committee members, Associate Professor William Cherowitzo, Professor Harvey Greenberg, Professor Michael Jacobson, Professor Richard Lundgren, and Assistant Professor Ross McConnell, for their valuable comments and suggestions, and Dr. Sylvia Monson, for her kind comments about Chapter 4.

I would like to acknowledge and thank for their financial support: the DPST (Development and Promotion of Science and Technology Talent Project) scholarship from IPST (The Institute for the Promotion of Teaching Science and Technology), Thailand and the department of Mathematics at the University of Colorado at Denver for the doctoral fellowship.

Lastly, I would also like to express thanks to my parents and friends for their support during my stay in the USA.

CONTENTS

FIGURES

Figure

TABLES

Table

1. Introduction

1.1 Definitions

For our purpose, graphs are simple. *Cliques* are complete subgraphs of a graph that are not necessarily maximal. The number of vertices in a clique is its *order*. A clique of order n is also called an n-clique or K_n. A 3-clique is also referred to as a *triangle*. The triangle on the set of vertices $\{a, b, c\}$ is represented by $\triangle(a, b, c)$. The 4-clique on the set of vertices $\{a, b, c, d\}$ is represented by $\boxtimes(a, b, c, d)$. A *clique covering* of G is a family $\mathcal{C}$ of cliques of G such that every edge of G lies in at least one member of $\mathcal{C}$. The number of cliques in a clique covering is its *size*. The smallest size among all possible clique coverings of G is called the *clique covering number of* G and denoted by $cc(G)$. A *minimum clique covering* is a clique covering of size $cc(G)$. A *clique partition* is a clique covering in which each edge belongs to exactly one clique. The smallest size of all possible clique partitions of a graph G is the *clique partition number* denoted by $cp(G)$. A *minimum clique partition* is a clique partition of G having size $cp(G)$.

A *maximal clique* is a clique that is not contained in any larger clique. If every element in a clique partition $\mathcal{P}$ of G is maximal, then $\mathcal{P}$ is a *maximal-clique partition* of G (or briefly, an MCP of G). A maximal-clique partition with the smallest size among all possible maximal-clique partitions of G is a *minimum maximal-clique partition* of G and its size is denoted $mcp(G)$. We refer to $cc(G)$, $cp(G)$ and $mcp(G)$ as cc, cp and mcp, respectively, if there is no ambiguity.

Every graph has at least one clique covering and clique partition, namely the edge set of the graph. However, a graph does not necessarily have a maximal-clique partition. For example, when $n \geq 4$, $K_n \setminus K_2$, which is the graph obtained by deleting one edge from K_n, does not have a maximal-clique partition. (Because all maximal cliques in $K_n \setminus K_2$ are $(n-1)$-cliques and each pair of them share an edge.) In fact, many graphs do not have a maximal-clique partition.

1.2 History and Overview

The maximal-clique partition problem is one of many problems of covering and partitioning the edge set of a graph with a minimum number of cliques. The algebraic problem of factoring binary matrices, the geometric problem of minimum perfect pair coverings, and the set theoretic problem of finding a smallest set having a family of subsets with prescribed intersection properties are intimately connected to graph theoretic problems.

The subject of clique coverings of graphs has its origins in the problem of intersection sequences. Erdős, Goodman, and Pósa in 1966 [10] and Lovász in 1968 [16] considered the problem graph theoretically. In the early 1970's, H. Ryser [36] studied it as a problem in matrix theory. In 1977, J. Orlin emphasized the graph theoretic approach and obtained several results [26]. F. Roberts compiled a survey paper listing many applications of these and related ideas in 1985 [35]. The first survey of clique coverings of graphs for the early works was written by N.Pullman [27] in 1982. More recently, clique coverings and partitions have been studied by Wallis and co-authors [38, 37, 39, 41, 42, 40, 43, 44, 45], Cacetta *et al.* [3], Monson *et al.* [25], and others. An excellent survey by Monson, Pullman, and Rees [24] appeared in 1995 and updates and extends the 1982 survey [27]. It collects most related works and is considered to be one of the most valuable recent works.

The problem can be thought of as three subproblems: clique covering, clique partition and maximal-clique partition problems. They yield three clique numbers $cc(G)$, $cp(G)$ and $mcp(G)$ of a graph G. The difference in their values may be zero or extremely large depending on the graph. For example, all three clique numbers of K_n are the same which is 1. However, the graph G_n of $n+2$ vertices in Figure 1.1 has a minimum clique covering composed of n triangles: $\triangle(a, b, v_i)$, $i = 1, \ldots, n$, and a minimum clique partition composed of one triangle and the remaining $2(n-1)$ edges. Hence, $cc(G_n) = n$ and $cp(G_n) = 2n-1$. The difference between $cc(G_n)$ and $cp(G_n)$ is $n-1$.

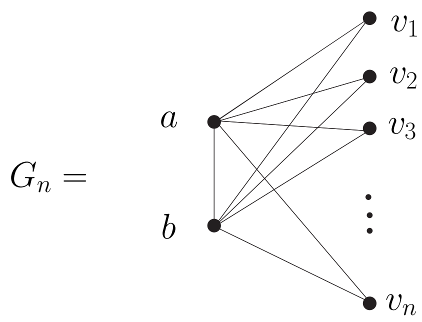


Figure 1.1: $cc(G_n) = n$ and $cp(G_n) = 2n-1$

Clique covering problems have been studied widely during the 1990's for many classes of graphs by a variety of authors, for instance, complements of cliques [31], regular graphs [5, 29], cubic graphs [4], threshold graphs [18], chordal graphs [17], generalized chordal graphs [41], complements of paths and cycles [8], line graphs [21], and self-complementary graphs [7].

In parallel, clique partition problems have been investigated. Pullman and de Caen [30] studied clique partitions of regular graphs, the cartesian product of graphs and graphs of maximal degree at most three in 1981. Later, Wallis and co-authors studied clique partitions of many classes of graphs including split graphs [40], chordal graphs [19, 11], threshold graphs [18], squares of trees [42], complements of triangles [43], complements of cycles [37], graphs of the form $G \vee \overline{K_n}$ [45], and complements of one-factors [12, 38]. In addition, McGuinness, Sean and Rees [21] studied clique partitions of line graphs, Lonc [15] clique partitions of graphs without odd chordless cycles and Rees [34] clique partitions of norm three.

Because clique covering and clique partition numbers are somewhat related and one can be much harder to determine than the other, some authors have tried to find relationships between them in order to help determining the value of the harder one. In 1985, Caccetta *et al.* [3] examined the clique covering and clique partition numbers in terms of their difference, which is called the *spread*, and in 1988, Erdős *et al.* [9] investigated them in terms of their ratio. The clique partition problem is also related to several other concepts. Two such articles concern clique partitions and pairwise balanced designs by Rees [33] and clique partitions and finite planes by Wallis [39]. The clique covering problem is also extended to the biclique covering problem: see biclique covering number of $K_n \setminus K_m$ and complete t-partite graphs [13]. Also, the effect of vertex and edge deletion on clique numbers [23, 2] and linear operators preserving partition numbers of graphs [1] are more recent related topics.

The maximal-clique partition problem is developed further from the clique partition problem. Unlike clique partition and clique covering problems, there have only been a few articles regarding maximal-clique partitions. A paper [32] written by Pullman, Shank and Wallis in 1982 covers some main concepts of the problem giving a few open problems. Since the existence of a maximal-clique partition requires maximality of all cliques in a partition, many graphs do not have a maximal-clique partition. Problems related to maximal-clique partitions are hard to solve. Results for only a few classes of graphs have been published so far: graphs with maximum degree at most four [32], triangulated planar graphs [32], irreducible graphs [44], interval graphs [20] and threshold graphs [18].

This dissertation emphasizes maximal-clique partitions. We study graphs with a maximal-clique partition and examine some relationships between the maximal-clique partition number and other clique numbers of a graph.

We will start by providing simple results about the existence of maximal-clique partitions in Section 1.3 to provide the reader with some intuition about this problem.

In Chapter 2, we obtain a necessary and sufficient condition for a line graph to have a maximal-clique partition, along with all three clique numbers and the complexity of the maximal-clique partition problems for line graphs. Although the fact that the line graph of the n-clique $L(K_n)$ has two maximal-clique partitions are well known, we provide details and investigate $L(K_5)$ further. These results will be used throughout the rest of the dissertation.

Chapter 3 involves graphs having at least two maximal-clique partitions of different sizes. When n is at least 5, $L(K_n)$ has two maximal-clique partitions of different sizes. Pullman, Shank and Wallis (PSW) [32] asked whether or not there is a graph with maximal-clique partitions of different sizes of fewer vertices than $L(K_5)$. We confirm that there is none. In [32], PSW also found a clique-inseparable graph (defined in Section 1.3) with n maximal-clique partitions. However, all of them have the same size. We will show that there exists a clique-inseparable graph with n maximal-clique partitions of n different sizes. The details of this will be found in Section 3.3. These graphs are obtained by gluing many copies of a variety of line graphs of complete graphs together.

Chapter 4 and Chapter 5 will emphasize the three clique parameters cc, cp and mcp. Since a clique partition is a clique covering, $cc \leq cp$. Since a maximal-clique partition is a clique partition, $cp \leq mcp$. Hence, it is always true that the three clique parameters of a graph satisfy $cc \leq cp \leq mcp$. These clique parameters of a graph are not always equal. However, all graphs with at most nine vertices have the same three clique parameter values. This fact was proved by Monson in her Ph.D. Dissertation [22]. Thus, these three parameters could be different when a graph has ten or more vertices. In Chapter 2, we show that if a line graph has a maximal-clique partition, then it has $cc = cp = mcp$. Interval graphs [20] and graphs with norm at most 4 [32] are also examples of graphs having $cc = cp = mcp$.

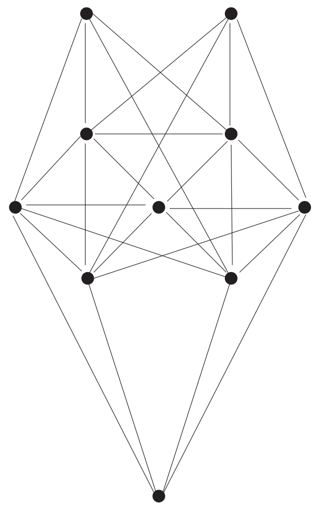


Figure 1.2: Monson's graph

As $cc \leq cp \leq mcp$, there are three more relationships among the three clique parameters, which are $cc < cp = mcp$, $cc = cp < mcp$ and $cc < cp < mcp$. Monson found the graph of ten vertices with $cc < cp = mcp$ shown in Figure 1.2. This graph G has $cc(G) = 8 < cp(G) = mcp(G) = 9$. Since ten vertices is the minimum number of vertices needed to get different values for the three clique parameters, Monson's graph has the minimum number of vertices for a graph satisfying $cc < cp = mcp$.

The question of the existence of graphs with $cc = cp < mcp$ has been investigated by Monson [22] who conjectured [personal communication] that no such graph exists. Chapter 4 answers this open problem. In fact, we construct infinitely many graphs with $cc = cp < mcp$. They will be constructed by using the glue operator (defined in Chapter 3) and their clique numbers will be verified.

The smallest graph among those constructed in this fashion has 68 vertices and 234 edges.

The next chapter involves the last relationship of clique parameters $cc < cp < mcp$. Pullman, Shank and Wallis [32] found a graph of 11 vertices satisfying this relationship. The graph is shown in Figure 1.3 with $cc = 5 < cp = 8 < mcp = 10$. They asked whether or not there exists a graph of fewer vertices satisfying $cc < cp < mcp$. In order to find such a graph, since graphs of at most nine vertices give $cc = cp = mcp$, our candidate graphs must have ten vertices. Chapter 5 studies related issues for graphs of ten vertices. The first part provides some characteristics of clique partitions and maximal-clique partitions of graphs with ten vertices. The second part of the chapter shows that in fact a graph of ten vertices with $cp < mcp$ does not exist. Hence, PSW's graph is a graph having the minimum number of vertices satisfying $cc < cp < mcp$. We conclude that graphs satisfying $cc < cp < mcp$ or $cc = cp < mcp$ have at least 11 vertices.

Finally, the conclusion and some open problems for future work are located in Chapter 6. Since many graphs do not have a maximal-clique partition, we suggest a way to investigate how much a graph deviates from having a maximal-clique partition.

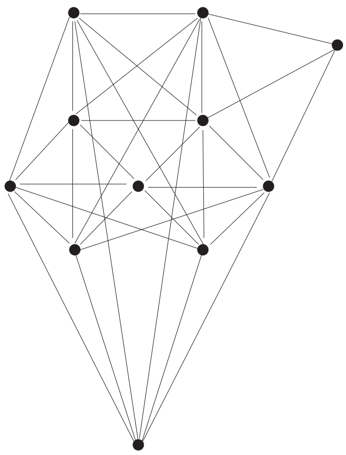


Figure 1.3: Pullman, Shank and Wallis (PSW)'s graph

1.3 The Existence of Maximal-clique Partitions

Definition 1.1 Let the graphs shown in Figure 1.4 be called the *kite* and the *ocular* graph.

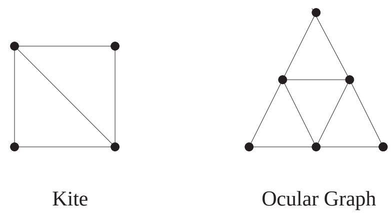


Figure 1.4: Kite and ocular graph

Lemma 1.1 *For any graph G, if G has a maximal-clique partition, then each kite in G must be contained in an ocular graph in G.*

Proof Assume G has a maximal-clique partition $\mathcal{M}$. Let K be a kite in G whose vertices are labelled a, b, c and d as in Figure 1.5.

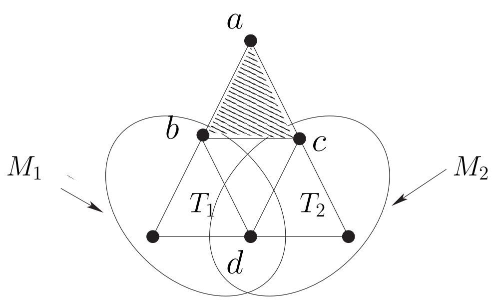


Figure 1.5: The kite K is contained in an ocular graph

Because $\triangle(a, b, c)$ and $\triangle(b, c, d)$ share an edge, both of them cannot be in $\mathcal{M}$ at the same time, say $\triangle(b, c, d)$ is not in $\mathcal{M}$. It follows that edges bd and cd must be contained in other maximal cliques in $\mathcal{M}$, say M_1 and M_2, respectively. M_1 is not edge bd because edge bd is not maximal. Thus, M_1 contains a triangle T_1 sharing bd with the kite K. Similarly there exists a triangle T_2 sharing cd with the kite K. Therefore, T_1 and T_2 together with $\triangle(a, b, c)$ and $\triangle(b, c, d)$ form an ocular graph in G containing the kite K. ■

A kite in a graph not satisfying the condition in Lemma 1.1 prevents the existence of a maximal-clique partition of the graph. There are a number of graphs containing kites that do not satisfy the condition. However, the condition in

Lemma 1.1 is not sufficient. Some graphs still do not have a maximal-clique partition though they satisfy the condition in the lemma. An example is shown in Figure 1.6.

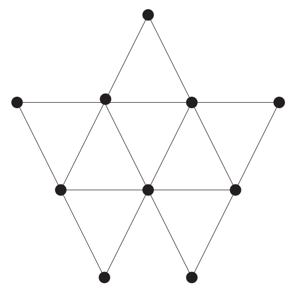


Figure 1.6: Each kite is contained in an ocular graph while the graph contains no MCP.

Determining whether a graph has a maximal-clique partition can be confined to determining whether certain of its subgraphs has a maximal-clique partition as follows:

Definition 1.2 A subgraph of G is a *proper subgraph of* G if it does not contain all edges of G and it is a *nonempty subgraph of* G if it contains at least one edge of G. If a subgraph H has the property that for every clique K in G, either every edge of K or no edge of K lies in H, then we say that H *separates the cliques of* G. If a proper nonempty subgraph separates the cliques of G, we say G is *clique-separable*. Otherwise G is *clique-inseparable*. If a subgraph B separates the cliques of G, but no proper nonempty subgraph of B does so, we call B a *clique-block* of G.

Example 1.1 Let G_1 and G_2 be graphs in Figure 1.7.

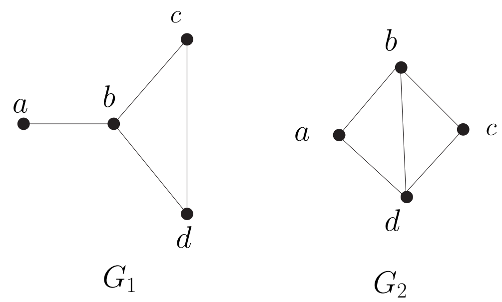


Figure 1.7: Example shows clique-separable and clique-inseparable graphs

Consider subgraphs H_1 and H_2 of G_1 where H_1 is composed of the edge ab and H_2 is composed of two edges ab and bc. H_2 does not separate cliques of G_1 because $\triangle(b, c, d)$ has edge bc in H_2 but edges bd and cd are not in H_2. It is easy to verify that H_1 separates the cliques of G_1. Hence, G_1 is a clique-separable graph. G_1 has two clique-blocks, the edge ab and the triangle $\triangle(b, c, d)$. In contrast, there is no proper nonempty subgraph separating the cliques of G_2. Thus, G_2 is a clique-inseparable graph.

Note that a clique-block is clique-inseparable in itself. Therefore, a subgraph B is a clique-block of G if and only if B is a clique-inseparable graph and B is not a subgraph of any other clique-inseparable subgraph of G. Note that if G has isolated vertices, then they form one clique-block of G. It was shown in [28] that the family $\mathcal{B}(G)$ of clique-blocks partitions the edge set of G.

Lemma 1.2 (Lemma 2.1 in [32])
The graph G has a maximal-clique partition if and only if each of its clique-blocks has a maximal-clique partition, in which case

$$mcp(G) = \sum_{B \in \mathcal{B}(G)} mcp(B)$$

Therefore, the study of maximal-clique partitions may be confined to clique-inseparable graphs.

2. Maximal-clique Partitions of Line Graphs

2.1 Line Graphs

Let G be a connected graph. The *line graph* $L(G)$ *of* G is the graph generated from G by $V(L(G)) = E(G)$ and for any two vertices $e, f \in V(L(G))$, vertex e and vertex f are adjacent in $L(G)$ if and only if edge e and edge f share a common vertex in G. If H is the line graph of G, we call G the *root graph* of H.

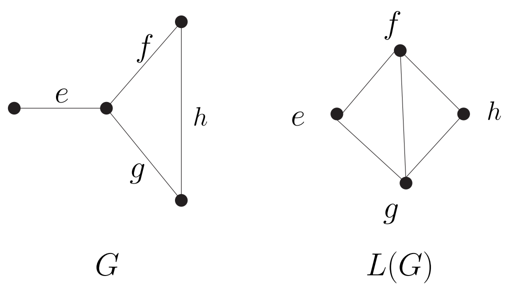


Figure 2.1: Example of a line graph

Because edges in $L(G)$ are induced by a pair of edges in G sharing a common endpoint, each vertex $v \in V(G)$ with $d(v) \geq 2$ generates a clique, denoted $Q(v)$. Vertices in $Q(v)$ correspond to edges incident to v in G; thus, $Q(v)$ has order $d_G(v)$. Let $V_2(G)$ be the set of vertices in G of degree at least 2. Then, $\{Q(v) : v \in V_2(G)\}$ partitions the edge set of $L(G)$.

Table 2.1 contains a summary of properties of the vertices and edges in G and of their corresponding elements in $L(G)$:

Table 2.1: Relationships between G and $L(G)$

In G	In $L(G)$
• edge e	• vertex e
• vertex v of degree $d_G(v)$	• clique $Q(v)$ of order $d_G(v)$
• $V_2(G)$= the set of vertices of degree at least two in G	• $\{Q(v) : v \in V_2(G)\}$= a clique partition of $L(G)$

2.2 The Existence of Maximal-clique Partitions of Line Graphs

Consider the three motivating examples in Figure 2.2. Example 1 illustrates a line graph $L(G_1)$ that does not have a maximal-clique partition. Example 2 shows a line graph $L(G_2)$ with a maximal-clique partition. Note that the root graph G_1 has a vertex of degree two, but the root graph G_2 does not contain a vertex of degree two. However, the root graph G_3 in Example 3 contains a vertex of degree two, but $L(G_3)$ has a maximal-clique partition. Hence, a vertex of degree two is not the only factor that effects the existence of maximal-clique partitions of a line graph. Although both G_1 and G_3 contain a vertex of degree two, the vertex of degree two in G_1 is in a triangle whereas the one in G_3 is not. This idea leads to our necessary and sufficient condition on a root graph G that guarantees the existence of a maximal-clique partition of the line graph $L(G)$.

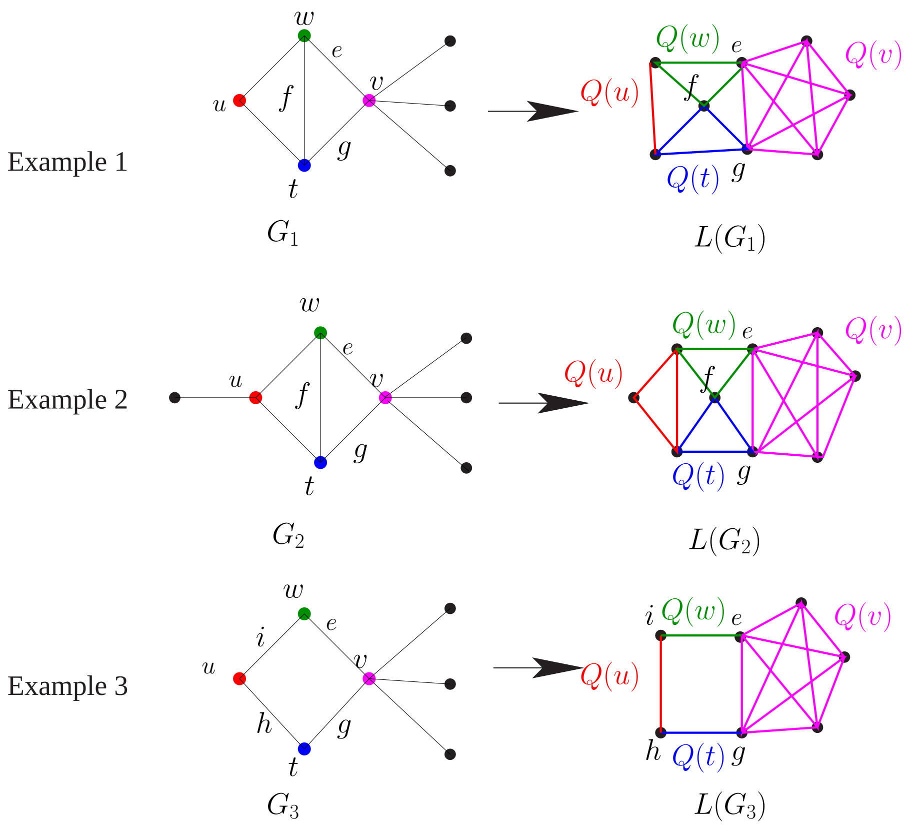


Figure 2.2: Motivating examples

Theorem 2.1 *Let $G \neq K_3$ be a connected graph. Then,*

$L(G)$ has a maximal-clique partition if and only if no vertex of degree two in G lies in a triangle.

Moreover, if $L(G)$ has a maximal-clique partition, then the set of cliques generated by vertices of degree at least two in G forms a maximal-clique partition of $L(G)$.

Proof Let $G \neq K_3$ be a connected graph.

($\Rightarrow$) Assume $L(G)$ has a maximal-clique partition $\mathcal{M}$. Suppose there exists a triangle T in G containing a vertex v of degree 2. As in Figure 2.3, let u, v, and w be the vertices of T and let e_1, e_2 and e_3 be the edges uv, vw, and uw, respectively. Since G is not a 3-clique and is connected, T contains a vertex of degree at least 3, say vertex u. Then u is incident to an edge outside T, say edge $e_4 = ut$. Figure 2.3 illustrates this situation.

Now, consider $L(G)$. As we observed in Table 2.1, T in G becomes a triangle $\triangle(e_1, e_2, e_3)$ in $L(G)$; also, u, v, and w in G produce cliques $Q(u), Q(v)$, and $Q(w)$ in $L(G)$, respectively. Since v has degree two, $Q(v)$ in $L(G)$ is the edge e_1e_2. Figure 2.3 illustrates the triangle T in G and its corresponding subgraph in $L(G)$.

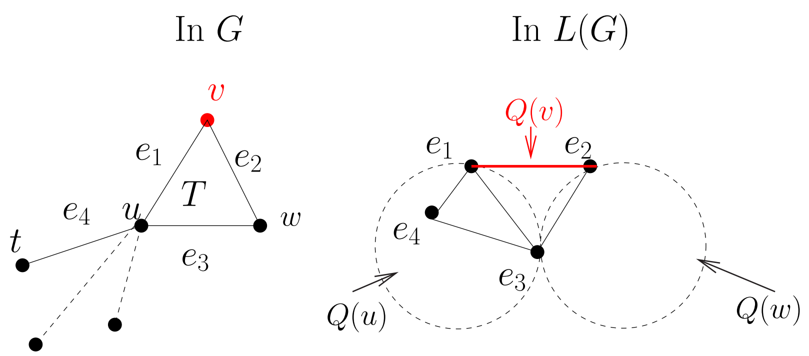


Figure 2.3: Triangle T and corresponding subgraph in $L(G)$

In G, since v has degree two and G is simple, e_3 is the only edge adjacent to both edges e_1 and e_2 at the same time. That is, the vertex e_3 in $L(G)$ is the only vertex adjacent to both vertices e_1 and e_2. Hence, $\triangle(e_1, e_2, e_3)$ is the only maximal clique in $L(G)$ containing edge e_1e_2. It follows that $\triangle(e_1, e_2, e_3)$ must be in $\mathcal{M}$. However, $Q(u)$ contains edge e_1e_3, while e_1e_3 is covered by $\triangle(e_1, e_2, e_3)$. So, $Q(u)$ cannot be in $\mathcal{M}$. Now, let K be the clique in $\mathcal{M}$ containing edge e_1e_4. (So, K is maximal.) If all vertices in K correspond to edges in G that share vertex u with e_1 and e_4, then K is a subgraph of $Q(u)$, contradicting the maximality of K. Hence, there exists a vertex in K corresponding to an edge in G that shares the other endpoint (which is not u) with e_1 and e_4. It forces the edge vt in G, which yields another edge in G incident to v besides edges e_1 and e_2. This contradicts $d(v) = 2$.

($\Leftarrow$) Conversely, assume that no vertex of degree two in G lies in a triangle. We will show that $L(G)$ has a maximal-clique partition.

Since $\{Q(v) : v \in V_2(G)\}$ is a clique partition of $L(G)$, it is sufficient to show that $Q(v)$ is maximal in $L(G)$ for all $v \in V_2(G)$.

Case 1 $d(v) > 2$. Then $Q(v)$ is a clique in $L(G)$ of order at least 3. Consider any vertex in $L(G)$ not contained in $Q(v)$. Let $e \in V(L(G) \setminus Q(v))$. Since e is not in $Q(v)$, v is not an endpoint of e. Then as in Figure 2.4 (G is simple), edge e in G is incident to at most two edges in $L(G)$ that contain v. Hence, in $L(G)$ the vertex e can be adjacent to at most two vertices in $Q(v)$. Since $Q(v)$ has order more than two, e is nonadjacent to a vertex of $Q(v)$. Hence, $Q(v)$ is a maximal clique. This completes the proof for this case.

Case 2 $d(v) = 2$. Let $e_1 = vx$ and $e_2 = vy$ be the two edges incident to v in G. If x is incident to y, the set $\{v, x, y\}$ forms a triangle in G containing the vertex v of degree 2. This contradicts the assumption. Thus, x is not adjacent

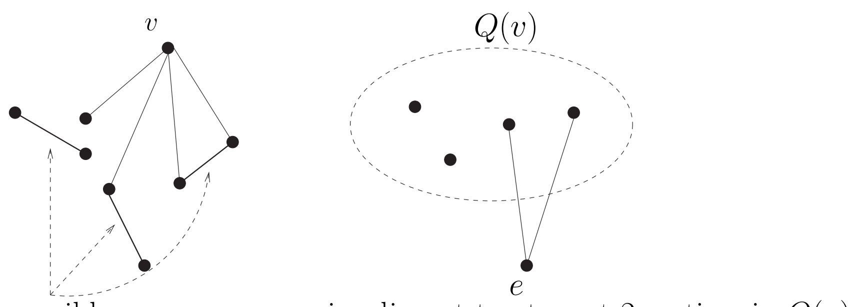


Figure 2.4: $e \in V(L(G))$ can be adjacent to at most two vertices in $Q(v)$

to y in G. Since $x \not\leftrightarrow y$ and $d(v) = 2$, there is no edge incident to both e_1 and e_2 in G. It follows that there is no vertex in $L(G)$ adjacent to both vertices e_1 and e_2. Hence, the edge e_1e_2 is not contained in other cliques in $L(G)$. Note that $Q(v)$ consists of just the edge e_1e_2. Thus, it is maximal. This completes the proof in this case.

In both cases we have shown that each $Q(v)$, where $v \in V_2(G)$, is a maximal clique in $L(G)$. Therefore, $\{Q(v) : v \in V_2(G)\}$ is a maximal-clique partition of $L(G)$. ■

Theorem 2.2 (Orlin, 1977 [26]) *Let G be a connected graph but not a 3-clique. Let m be the number of triangles with exactly two vertices of degree two in G. Then*

$$cp(L(G)) = |V_2(G)| \text{ and}$$

$$cc(L(G)) = |V_2(G)| - m.$$

We extend Theorem 2.2 to include the maximal-clique partition number for line graphs.

Theorem 2.3 *Let G be a connected graph. If $L(G)$ has a maximal-clique partition, then*

$$cc(L(G)) = cp(L(G)) = mcp(L(G)) = \begin{cases} 1 & \text{if } G \text{ is a 3-clique,} \\ |V_2(G)| & \text{if } G \text{ is not a 3-clique.} \end{cases}$$

Proof Let G be a connected graph. Assume $L(G)$ has a maximal-clique partition. By Theorem 2.1, G is a 3-clique or each triangle in G contains no vertices of degree two. If G is a 3-clique, $L(G)$ is also a 3-clique. Then $cc(L(G)) = cp(L(G)) = mcp(L(G)) = 1$. Otherwise, each triangle in G contains no vertices of degree two. That is, G does not contain any triangles with exactly two vertices of degree two. Hence, by Theorem 2.1 and Theorem 2.2, $cc(L(G)) = cp(L(G)) = mcp(L(G)) = |V_2(G)|$. ■

Theorem 2.1 gives a characterization of the root graph of a given line graph with a maximal-clique partition. Next we will provide a condition of the line graph that guarantees a maximal-clique partition. The following is straight forward, but we need it to prove our next theorem.

Lemma 2.1 *Let G be a connected graph. For $e = uv \in V(L(G))$, e is contained in $Q(x)$ if and only if $x = u$ or $x = v$. Moreover, $d_{L(G)}(e) = d_G(v) + d_G(u) - 2$.*

Proof Since edge e has two endpoints in G, vertex e can be contained in only two cliques in $L(G)$. As we observed in Table 2.1, $Q(v)$ and $Q(u)$ have order

$d_G(v)$ and $d_G(u)$, respectively. Hence, vertex e in $L(G)$ is adjacent to $d_G(v) - 1$ vertices in $Q(v)$ and $d_G(u) - 1$ vertices in $Q(u)$. ■

Theorem 2.4 *Let H be a line graph. If no maximal triangle $S = \triangle(v, u, w)$ in H has vertices satisfying $d_H(u) + d_H(w) - d_H(v) = 2$, then H has a maximal-clique partition.*

Proof Let H be a line graph and let G be the root graph of H, i.e., $H = L(G)$. If H is a triangle, then H itself is a maximal triangle. Hence, assume $H \neq K_3$ and that H does not have a maximal-clique partition. By Theorem 2.1, there exists a triangle $T = \triangle(a, b, c)$ in G containing a vertex a of degree 2. The triangle $S = \triangle(ab, bc, ac)$ in H comes from the edges of T in G. Note that there is no edge in G that can be incident to all three edges in T. So, there is no vertex in H adjacent to all three vertices in S. Hence, S is a maximal triangle. Now by Lemma 2.1, we have $d_H(ab) = d_G(a) + d_G(b) - 2$, $d_H(bc) = d_G(b) + d_G(c) - 2$, and $d_H(ac) = d_G(a) + d_G(c) - 2$. Thus,

$$\begin{aligned}
&d_H(ab) + d_H(ac) - d_H(bc) \\
&= [d_G(a) + d_G(b) - 2] + [d_G(a) + d_G(c) - 2] - [d_G(b) + d_G(c) - 2] \\
&= [2 + d_G(b) - 2] + [2 + d_G(c) - 2] - [d_G(b) + d_G(c) - 2] \\
&= 2.
\end{aligned}$$

Hence, H contains a maximal triangle $\triangle(ab, bc, ac)$ whose vertices satisfy $d_H(ab) + d_H(ac) - d_H(bc) = 2$. Therefore, the theorem is proved as desired. ■

Our condition in Theorem 2.4 is sufficient to guarantee a maximal-clique partition of a line graph without finding the root graph of the line graph. However, the existence of a maximal triangle $S = \triangle(u, v, w)$ in the line graph whose vertices satisfy $d(u) + d(w) - d(v) = 2$ cannot guarantee that the given line graph does not have a maximal-clique partition. As in Figure 2.5, H is a line graph of G, and $S = \triangle(u, v, w)$ is a maximal triangle in H whose vertices satisfy $d(u) + d(w) - d(v) = 2$, but H has a maximal-clique partition composed of two 3-cliques and two 2-cliques.

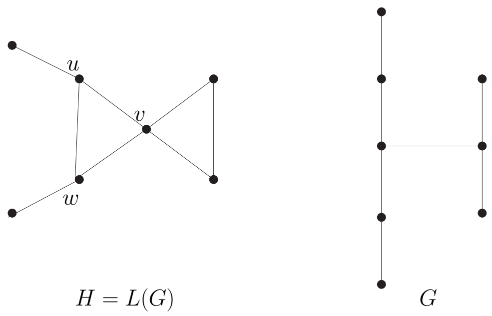


Figure 2.5: H contains a triangle whose vertices satisfy the condition, but H has a maximal-clique partition.

Generally, the maximal-clique partition problem is NP-complete [24]. However, the condition in Theorem 2.1 leads to an algorithm with linear time. Given a graph H, Lehot [14] presents a linear time algorithm for testing whether H is a line graph and retrieving the unique simple graph G such that $H = L(G)$. Then to test whether there exists a maximal-clique partition only requires finding vertices of degree two in G and verifying whether they are in a triangle (by checking whether the endpoints of the two edges incident to a vertex of degree two are adjacent). Hence, the condition in Theorem 2.1 can be checked in linear time. Therefore, the maximal-clique partition problem for line graphs has a linear time complexity in terms of the input size.

2.3 Maximal-clique Partitions of $L(K_n)$

For $n \geq 5$, the line graph of n-clique $L(K_n)$ has a special property. It contains exactly two maximal-clique partitions and their sizes are different. We first illustrate the two maximal-clique partitions of $L(K_5)$ to get the general idea for $L(K_n)$. Then we define two maximal-clique partitions $\mathcal{M}_n$ and $\mathcal{N}_n$ of $L(K_n)$. We shall then verify that $\mathcal{M}_n$ and $\mathcal{N}_n$ have different sizes when n is at least 5.

As K_5 has ten edges, $L(K_5)$ has ten vertices. In Figure 2.6, each vertex in $L(K_5)$ is labelled according to its corresponding edge in G. Then two distinct vertices in $L(K_5)$ are adjacent if their labels share a digit.

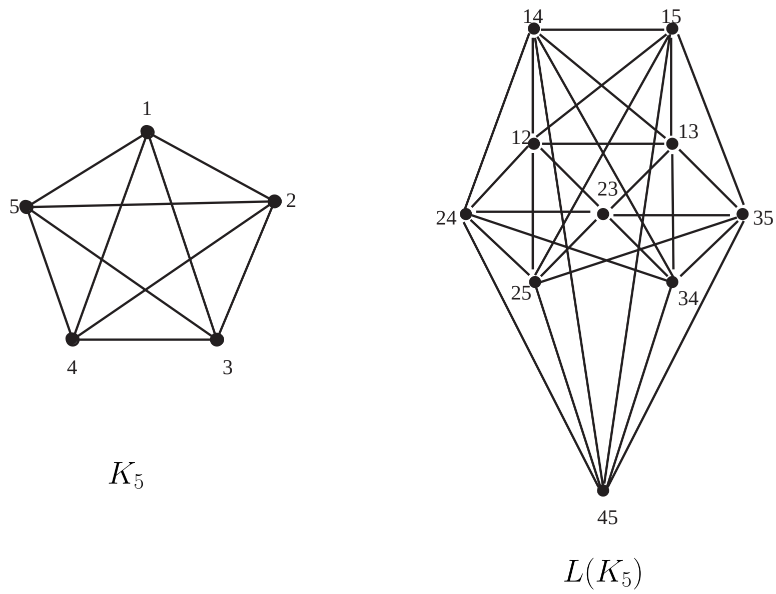


Figure 2.6: $L(K_5)$: the line graph of K_5

Let $u \in \{1, 2, 3, 4, 5\}$ be a vertex in K_5. Recall that $Q(u)$ is a clique in $L(K_5)$ derived from u. Since $d_{K_5}(u) = 4$, $Q(u)$ is a 4-clique. Then $V(Q(u)) = \{u1, \ldots, u(u-1), u(u+1), \ldots, un\}$. Let $\mathcal{M}_5 = \{Q(u) : u = 1, \ldots, 5\}$. By Theorem 2.1, $\mathcal{M}_5$ is a maximal-clique partition of $L(K_5)$. We refer to $\mathcal{M}_5$ as the *vertex-induced partition of* $L(K_5)$.

Next, consider the triangles of K_5. They may be identified with the 3-sets of $\{1, 2, 3, 4, 5\}$. There are $\binom{5}{3}$=10 such subsets. Let $\{x, y, z\}$ be any 3-set of $\{1, 2, 3, 4, 5\}$. Then vertices xy, yz and xz form a triangle in $L(K_5)$. Consider any vertex outside the triangle $\triangle(xy, yz, xz)$. Its label can contain at most one digit in $\{x, y, z\}$. Thus, it is adjacent to at most two vertices in $\triangle(xy, yz, xz)$. That is, $\triangle(xy, yz, xz)$ is maximal. Hence, a 3-set of $\{1, 2, 3, 4, 5\}$ induces a maximal triangle in $L(K_5)$. Furthermore, the ten maximal triangles induced from the ten 3-sets partition the edge set of $L(K_5)$, forming another maximal-clique partition of $L(K_5)$. Let $\mathcal{N}_5$ be the collection of ten maximal triangles induced from these ten 3-sets. It is referred to as the *triangle partition of* $L(K_5)$. Therefore, $L(K_5)$ has two maximal-clique partitions, $\mathcal{M}_5$ of size 5 and $\mathcal{N}_5$ of size 10.

In general, $L(K_n)$ is the intersection graph of the 2-sets of an n-set. The vertex set of $L(K_n)$ is the collection of all 2-sets of the n-set. Hence, $L(K_n)$ has $\binom{n}{2}$ vertices, and any two vertices of $L(K_n)$ are adjacent if and only if they share exactly one element. The graph K_1 has no edges, so $L(K_1)$ is an empty graph. When $n = 2$, then K_2 has one edge which implies that $L(K_2)$ is K_1. When $n = 3$, then K_3 is a triangle. It is easy to see that $L(K_3)$ is also a triangle. In the same fashion as in our explanation of $L(K_5)$, the triangle partition $\mathcal{N}_3$

consists of the triangle $L(K_3)$ itself, and the vertex-induced partition $\mathcal{M}_3$ is the set of three edges in $L(K_3)$. However, each edge in $\mathcal{M}_3$ is not maximal; hence, $\mathcal{M}_3$ is not a maximal-clique partition of $L(K_3)$. Hence, for our purposes, we consider $L(K_n)$ when n is at least 4.

Now let n be any integer at least 4. As before, let $u \in \{1, \ldots, n\}$ be a vertex in K_n. Then $Q(u)$ is an $(n-1)$-clique in $L(K_n)$ induced by u. The set $\{Q(u) : u = 1, \ldots, n\}$ is the *vertex-induced partition of* $L(K_n)$ which is denoted $\mathcal{M}_n$. By Theorem 2.1, $\mathcal{M}_n$ is a maximal-clique partition of $L(K_n)$. Clearly, $|\mathcal{M}_n| = n$. Next, we get the *triangle partition of* $L(K_n)$, or $\mathcal{N}_n$, by considering 3-sets of the n-set. Each of these 3-sets produces a triangle in $L(K_n)$. Since there are $\binom{n}{3}$ 3-sets of the n-set, there are $\binom{n}{3}$ of such triangles; so, they cover $3\cdot\binom{n}{3}=\frac{n\cdot(n-1)\cdot(n-2)}{2}$ edges in $L(K_n)$. Because $L(K_n)$ is composed of n copies of K_{n-1}, $L(K_n)$ has $n \cdot |E(K_{n-1})| = n\cdot \binom{n-1}{2}=\frac{n\cdot(n-1)\cdot(n-2)}{2}$ edges. Therefore, the collection of all of such triangles partitions the edge set of $L(K_n)$. And in the same fashion as $L(K_5)$, we can conclude that each of these triangles is maximal in $L(K_n)$. Hence, $\mathcal{N}_n$, which is the collection of these maximal triangles, is another maximal-clique partition of $L(K_n)$. Therefore, for $n \geq 4$, $L(K_n)$ always has exactly two maximal-clique partitions $\mathcal{M}_n$ and $\mathcal{N}_n$ where $|\mathcal{M}_n| = n$ and $|\mathcal{N}_n|= \binom{n}{3} = \frac{n(n-1)(n-2)}{6}$.

When $n = 4$, $|\mathcal{M}_4| = 4$ and $|\mathcal{N}_4|= \binom{n}{4} = \frac{4(4-1)(4-2)}{6} = 4$. Hence, $\mathcal{M}_4$ and $\mathcal{N}_4$ have the same size. When $n \geq 5$, we have $n-1 \geq 4$ and $n-2 \geq 3$, so $\frac{n(n-1)(n-2)}{6} \geq \frac{n\cdot(4)\cdot(3)}{6} = 2n$. Hence, for $n \geq 5, |\mathcal{M}_n| = n < 2n \leq |\mathcal{N}_n|$. Furthermore, consider possible maximal cliques containing an edge (xy, yz) in

$L(K_n)$. A vertex that can be adjacent to both xy and yz must be either vertex xz or its label containing y. The first case yields a maximal triangle $\triangle(xy, yz, xz)$ in $\mathcal{N}_n$. The latter case yields a maximal $(n-1)$-clique $Q(y)$ in $\mathcal{M}_n$. This shows that each edge in $L(K_n)$ is contained in exactly two maximal cliques and all possible maximal cliques in $L(K_n)$ are in $\mathcal{M}_n$ or $\mathcal{N}_n$. Now, let $\mathcal{M}$ be a maximal-clique partition of $L(K_n)$. If an $(n-1)$-clique $Q(t)$ in $\mathcal{M}_n$ is in $\mathcal{M}$, since $V(Q(t))$ is $\{t1, t2, \ldots, t(t-1), t(t+1), \ldots, tn\}$, for any $i \neq j \in \{1, \ldots, t-1, t+1, \ldots, n\}$, edge $e_1 = (ti, tj)$ is covered by $Q(t)$. But e_1 is also in $\triangle(ti, tj, ij)$; so, $\triangle(ti, tj, ij)$ is not in $\mathcal{M}$. Since edge $e_2 = (tj, ij)$ is in $\triangle(ti, tj, ij)$, it follows that $Q(j)$ must be in $\mathcal{M}$ to cover e_2. Since j is arbitrary, the remaining $(n-1)$-cliques in $\mathcal{M}_n$ must be in $\mathcal{M}$. We have shown that if a clique in $\mathcal{M}_n$ is in a maximal-clique partition of $L(K_n)$, then all cliques in $\mathcal{M}_n$ must also be in it. Therefore, $L(K_n)$ has exactly two maximal-clique partitions. In particular, when n is 4, $L(K_4)$ has exactly two maximal-clique partitions of the same size, and when n is 5 or more, $L(K_n)$ always has exactly two maximal-clique partitions, and they have different sizes.

Notice that, for $n \geq 4$, the largest order of a clique in $L(K_n)$ is $n-1$ and $\mathcal{M}_n$ is composed of $(n-1)$-cliques; so, $\mathcal{M}_n$ is a minimum clique covering of $L(K_n)$. Hence, we can conclude that for $n \geq 4$, $cc(L(K_n)) = cp(L(K_n)) = mcp(L(K_n)) = |\mathcal{M}_n| = n$.

In conclusion, the properties of $L(K_n)$ mentioned above are summarized in the following lemma for reference later:

Lemma 2.2 (Folklore) *(i) For any integer* $n \geq 4$, $L(K_n)$ *has exactly two maximal-clique partitions, the vertex-induced partition* $\mathcal{M}_n$ *of size* n *and the triangle partition* $\mathcal{N}_n$ *of size* $\frac{n(n-1)(n-2)}{6}$. *In particular,* $|\mathcal{M}_4| = |\mathcal{N}_4|$; *otherwise,* $\mathcal{M}_n$ *and* $\mathcal{N}_n$ *have different sizes.*

(ii) For any integer $n \geq 4$, $cc(L(K_n)) = cp(L(K_n)) = mcp(L(K_n)) = |\mathcal{M}_n| = n$.

Note that we can also derive Lemma 2.2(ii) from Theorem 2.3. Next, Lemma 2.3 shows that for $n \geq 4$, $L(K_n)$ is a clique-inseparable graph. (Clique-inseparable is defined in Definition 1.2.)

Lemma 2.3 *For any integer* $n \geq 4$, $L(K_n)$ *is a clique-inseparable graph.*

Proof Let $n \geq 4$. Let H be a nonempty subgraph of $L(K_n)$ separating cliques in $L(K_n)$. Let $e = (uv, vw)$, where $u \neq v \neq w \in \{1, \ldots, n\}$, be an edge in H. (The two endpoints of an edge in $L(K_n)$ have labels sharing a digit.) Recall that $Q(v)$ is an $(n-1)$-clique in the vertex-induced partition where $V(Q(v)) = \{v1, v2, \ldots, v(v-1), v(v+1), \ldots, vn\}$. Then $Q(v)$ contains e. Since e is in H which separates cliques in $L(K_n)$, $Q(v)$ must be a subgraph of H.

Let a be any vertex in K_n other than u or v. Since edge (va, vu) is in $Q(v)$, edge (va, vu) is in H. But edge (va, vu) is also contained in $\triangle(va, vu, ua)$; because H separates cliques of $L(K_n)$, $\triangle(va, vu, ua)$ must be a subgraph of H, i.e., (va, ua) is contained in H. Moreover, edge (va, ua) is also in $Q(a)$, it follows that $Q(a)$ must be a subgraph of H. Since a is arbitrary, all $(n-1)$-cliques in the vertex-induced partition are subgraphs of H. Hence, $H = L(K_n)$. Therefore, there is no proper nonempty subgraph of $L(K_n)$ separating cliques in $L(K_n)$. Thus, $L(K_n)$ is a clique-inseparable graph. ■

2.4 $L(K_5)$

The ten vertices of $L(K_5)$ correspond to the ten edges in K_5. Each vertex can be labelled by two digits corresponding to the two labels on the endpoints incident to the edge it comes from in K_5. Two vertices in $L(K_5)$ are adjacent if and only if they share a digit. Hence, $L(K_5)$ is also an intersection graph. It follows from Section 2.3 that $L(K_5)$ has exactly two maximal-clique partitions, the vertex-induced partition $\mathcal{M}_5$ and the triangle partition $\mathcal{N}_5$. All maximal triangles of $L(K_5)$ are in $\mathcal{N}_5$. Another interesting fact about $L(K_5)$ is that the complement of $L(K_5)$ is the Petersen graph. In this section we discuss the effect of adding edges of the complement of $L(K_5)$ into $L(K_5)$.

Let $e = (xx', yy')$ be an edge in the complement of $L(K_5)$. (So, $x \neq x' \neq y \neq y'$.) And let $L(K_5) + e$ be the graph that results from adding the edge e to $L(K_5)$. Consider any triangle T containing the edge e in $L(K_5) + e$. Since the third vertex in T is adjacent to both xx' and yy', one digit in the label of the third vertex must be either x or x' and the other digit must be either y or y'. Without loss of generality, say the third vertex is xy. That is, $T = \triangle(xx', yy', xy)$. Since vertex xy' is adjacent to all three vertices in $\triangle(xx', yy', xy)$, triangle T is not maximal in $L(K_5) + e$.

Furthermore, adding edge $e = (xx', yy')$ to $L(K_5)$ creates four 4-cliques in

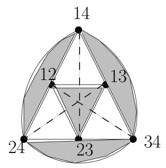
$L(K_5)+e$, namely $\boxtimes(xx',yy',xy,x'y)$, $\boxtimes(xx',yy',xy',x'y')$, $\boxtimes(xx',yy',xy,xy')$, and $\boxtimes(xx',yy',x'y,x'y')$. Hence, it destroys the maximality of four triangles in $L(K_5)$, namely $\triangle(xx',xy,x'y)$, $\triangle(xx',xy',x'y')$, $\triangle(yy',xy,xy')$, and $\triangle(yy',x'y,x'y')$.

For example, adding edge $(12,34)$ to $L(K_5)$ creates four 4-cliques $\boxtimes(12,34,13,23)$, $\boxtimes(12,34,14,24)$, $\boxtimes(12,34,13,14)$, and $\boxtimes(12,34,23,24)$, and destroys the maximality of triangles $\triangle(12,13,23)$, $\triangle(12,14,24)$, $\triangle(34,13,14)$, and $\triangle(34,23,24)$.

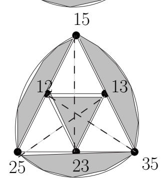
Let e be an edge in the set $\{(xx',yy'),\ (xy,x'y'),\ (xy',x'y)\}$. Then $\{\triangle(xx',xy,x'y),\ \triangle(xx',xy',x'y'),\ \triangle(yy',xy,xy'),\ \triangle(yy',x'y,x'y')\}$ be the set of all maximal triangles in $L(K_5)$ but not maximal in $L(K_5)+e$. This relation partitions the edge set of the complement of $L(K_5)$ into five classes $\mathcal{E}_1,\ldots,\mathcal{E}_5$ and the set of the maximal triangles in $L(K_5)$ into five classes $\mathcal{T}_1,\ldots,\mathcal{T}_5$ as in Figure 2.7. Then for $i=1,\ldots,5$, $e\in\mathcal{E}_i$ if and only if all triangles in $\mathcal{T}_i$ are not maximal in $L(K_5)+e$.

Next, we consider adding three edges of the complement of $L(K_5)$ into $L(K_5)$. Let the resulting graph be called $L(K_5)+3e$. Note that adding all three edges in the same class to $L(K_5)$ does not create any new maximal triangle in the resulting graph while it destroys the maximality of all triangles in $\mathcal{T}_i$. Hence, the graph obtained from adding at most three edges in the same class to $L(K_5)$ contains at most six maximal triangles. However, if not all of them are in the same class, then they may create at most two new maximal triangles in $L(K_5)+3e$. Since $|\mathcal{T}_i\cap\mathcal{T}_j|=1$, adding three edges not in the same class to

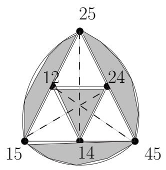
$L(K_5)$ destroys the maximality of seven triangles of $L(K_5)$. Hence, the graph obtained from adding three edges not all in the same class to $L(K_5)$ contains at most $10 + 2 - 7 = 5$ maximal triangles. We will put this fact in Remark2.1 and we will refer back to it in Chapter 5:



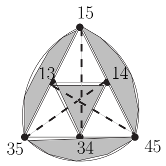
$\mathcal{E}_1 = \{(12, 34), (13, 24), (14, 23)\}$
$\mathcal{T}_1 = \{\triangle(12, 13, 23), \triangle(12, 14, 24), \triangle(13, 14, 34), \triangle(23, 24, 34)\}$



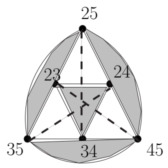
$\mathcal{E}_2 = \{(12, 35), (13, 25), (15, 23)\}$
$\mathcal{T}_2 = \{\triangle(12, 13, 23), \triangle(12, 15, 25), \triangle(13, 15, 35), \triangle(23, 25, 35)\}$



$\mathcal{E}_3 = \{(12, 45), (14, 25), (15, 24)\}$
$\mathcal{T}_3 = \{\triangle(12, 14, 24), \triangle(12, 15, 25), \triangle(14, 15, 45), \triangle(34, 35, 45)\}$



$\mathcal{E}_4 = \{(13, 45), (14, 35), (15, 34)\}$
$\mathcal{T}_4 = \{\triangle(13, 14, 34), \triangle(13, 15, 35), \triangle(14, 15, 45), \triangle(34, 35, 45)\}$



$\mathcal{E}_5 = \{(23, 45), (24, 35), (25, 14)\}$
$\mathcal{T}_5 = \{\triangle(23, 24, 34), \triangle(23, 25, 35), \triangle(24, 25, 45), \triangle(34, 35, 45)\}$

Figure 2.7: Each edge in $\mathcal{E}_i$ destroys the maximality of triangles in $\mathcal{T}_i$.

Remark 2.1 *The graph that results from adding three edges to* $L(K_5)$ *contains at most six maximal triangles.*

$L(K_5)$ is the smallest line graph of an n-clique having maximal-clique partitions of different sizes. In the next chapter we will show that $L(K_5)$ is in fact the smallest one among graphs with a maximal-clique partitions of different sizes.

3. Maximal-clique Partitions of Different Sizes

Chapter 3 is concerned with maximal-clique partitions of different sizes. By Lemma 2.2, when n is at least 5, $L(K_n)$ has two maximal-clique partitions of different sizes. In 1982, Pullman, Shank and Wallis [32] asked whether or not there is a graph with maximal-clique partitions of different sizes and fewer vertices than $L(K_5)$. This chapter confirms that there is no such graph. In the same paper [32], Pullman *et al.* exhibited a clique-inseparable graph with n maximal-clique partitions. However, all of them have the same size. We are interested in whether or not there exists a clique-inseparable graph with n maximal-clique partitions of n different sizes. Section 3.3 illustrates a graph satisfying this property. The graph is constructed by gluing many copies of various sizes of line graphs of complete graphs. A graph operation called the *glue operator* will be introduced in Section 3.2 and some of its properties will be investigated.

3.1 Smallest Graph with Maximal-clique Partitions of Different Sizes

We start by preparing a list of properties of graphs with at least two maximal-clique partitions. We will apply the classic theorem of de Bruijn and Erdős [6] as formulated in graph theoretic terms by Orlin in [26].

Lemma 3.1 *(i) The intersection of two different maximal cliques of a graph is empty or a clique of order less than the order of either.*

(ii) Any two cliques in the same clique partition of a graph share at most one vertex.

Proof The proofs are simple, but here are the details:

(i) Clearly the nonempty intersection of two cliques is a clique. If the intersection is a clique of the same order as one of the original cliques, then such clique is contained in the other one, contradicting its maximality.

(ii) If two cliques in a maximal clique partition $\mathcal{P}$ share two vertices, they will share the edge joining these vertices, contradicting the fact that $\mathcal{P}$ is a partition. ■

Theorem 3.1 (N.G. de Bruijn and P. Erdős [6], 1948)

If $\mathcal{C}$ is a clique partition of an n-clique and $1 < |\mathcal{C}| \leq n$, then $\mathcal{C}$ is composed of either

(i) one $(n-1)$-clique and $n-1$ copies of 2-cliques or

(ii) n copies of $(m+1)$-cliques if there exists an integer m such that $n = m^2 + m + 1$.

Corollary 3.1 *Let $\mathcal{M}$ be a maximal clique partition of a graph. Then each n-clique K in the graph that does not belong to $\mathcal{M}$ needs at least n different cliques in $\mathcal{M}$ to cover its edges.*

Moreover, if K is covered by n cliques in $\mathcal{M}$, say $M_1, M_2, \ldots, M_n$, then $\{K \cap M_i : i = 1, 2, \ldots, n\}$ is composed of either

(i) one $(n-1)$-clique and $n-1$ copies of 2-cliques or

(ii) n copies of $(m+1)$-cliques if there exists an integer m s.t. $n = m^2 + m + 1$.

Proof Let K be an n-clique that does not belong to $\mathcal{M}$. Consider cliques in $\mathcal{M}$ covering edges of K. Let $\mathcal{P} \subseteq \mathcal{M}$ be composed of maximal cliques covering edges of K. Let $\mathcal{P}' = \{K \cap C : C \in \mathcal{P}\}$. Notice that members in $\mathcal{P}'$ are cliques, and by Lemma 3.1(ii), they share at most one vertex. Thus, $\mathcal{P}'$ is a clique partition of K. Moreover, both types of clique partitions of K in Theorem 3.1 have size n, which means that K has no clique partitions of size i, where $1 < i < n$. Hence, $|\mathcal{P}| = |\mathcal{P}'| \geq n$.

If K is covered by exactly n maximal cliques in $\mathcal{M}$, then the second part of the corollary follows immediately from Theorem 3.1. ■

Theorem 3.2 *Let $\mathcal{M}$ be any maximal-clique partition of a graph. Let K be an n-clique in the graph that does not belong to $\mathcal{M}$.*

(i) If every two maximal cliques in $\mathcal{M}$ covering edges of K share a vertex in K, then there are at least n vertices not in K.

(ii) If there exist two maximal cliques in $\mathcal{M}$ covering some edges of K but not sharing a vertex in K, then there are at least six vertices not in K.

Proof (i) Let K be an n-clique that does not belong to $\mathcal{M}$ such that every two maximal cliques in $\mathcal{M}$ covering edges of K share a vertex of K. Let M_1 and M_2 be two maximal cliques in $\mathcal{M}$ covering edges of K. Then M_1 and M_2 share a vertex in K. By Lemma 3.1(i), each of M_1 and M_2 must contain a vertex that is not contained in K. However, since they share a vertex in K, by Lemma 3.1(ii), they cannot share vertices outside K. Hence, each maximal clique in $\mathcal{M}$ that covers edges of K has a distinct vertex outside K. By Corollary 3.1, K is covered by at least n cliques in $\mathcal{M}$; so there are at least n vertices outside K.

(ii) Let K be an n-clique in the graph that does not belong to $\mathcal{M}$. Assume there exist two maximal cliques, say M_1 and M_2, in $\mathcal{M}$ covering edges of K and not sharing a vertex in K. Then each of them contains at least one edge of K and, therefore, at least two vertices of K. Let ab and $cd \in E(K)$ be contained in M_1 and M_2, respectively. Because M_1 and M_2 do not share a vertex of K, edges ac, ad, bc and bd cannot be in M_1 or M_2. Moreover, if any two edges in $\{ac, ad, bc, bd\}$ are in the same maximal clique, then such a clique contains edge ab or cd, contradiction. Hence, edges in $\{ac, ad, bc, bd\}$ must be in different maximal cliques in $\mathcal{M}$. This yields a 4-clique $\boxtimes(a, b, c, d)$ which is a subset of K covered by six maximal cliques in $\mathcal{M}$. For convenience, for any $i, j \in \{a, b, c, d\}$, let M_{ij} be the maximal clique in $\mathcal{M}$ covering edge ij of K. Hence, M_1 and M_2 are renamed M_{ab} and M_{cd}, respectively. By Lemma 3.1(i), each M_{ij} contains an extra vertex not in K. If all such extra vertices are different, we have six extra vertices as desired. Otherwise, there are two maximal cliques sharing the same extra vertex not in K. By Lemma 3.1(ii), they cannot share another vertex, without loss of generality say M_{ac} and M_{bd} share the same extra vertex v_1. For all $i, j \in \{a, b, c, d\}$, let $P_{ij} = M_{ij} \cap K$. It follows that vertices in $\{v_1\} \cup V(P_{ac}) \cup V(P_{bd})$ form a larger clique containing the clique formed by $\{v_1\} \cup V(P_{ac})$. Since M_{ac} is maximal, it cannot be the clique composed of vertices in $\{v_1\} \cup V(P_{ac})$. Thus, M_{ac} must contain another vertex, say v_2. Similarly, vertices in $\{v_1\} \cup V(P_{ac}) \cup V(P_{bd})$ form a larger clique containing the clique formed by $\{v_1\} \cup V(P_{bd})$; hence, M_{bd} contains another vertex, say v_3. If v_3 is the same as v_2, by the same argument as above, M_{bd} must contain another extra vertex, which must eventually be a different vertex from v_2. Hence, we can assume that v_3 is different from v_2. Now, let v_4 and v_5 be extra vertices of M_{ab} and M_{ad}, respectively. Since edges ab and ad share a, by Lemma 3.1(ii), $v_4 \neq v_5$. Furthermore, ab and ad share a vertex with both ac and bd; hence, $v_4, v_5 \notin \{v_1, v_2, v_3\}$.

Next let u be an extra vertex of M_{bc} outside K. Then u cannot be v_1, v_2, v_3 or v_4 because bc shares a vertex with ac, bd and ab. If u is not v_5, we have six different extra vertices as desired. If u is v_5, vertices in $\{v_5\} \cup V(P_{bc}) \cup V(P_{ad})$ form a larger clique containing the clique formed by $\{v_5\} \cup V(P_{bc})$; hence, M_{bc}

must contain at least one more vertex outside $\{v_1, v_2, v_3, v_4, v_5\}$; thus, we get the sixth extra vertex. Hence, there are at least six extra vertices not in K. ■

Theorem 3.3 *The minimum number of vertices of graphs with at least two maximal-clique partitions of different sizes is 10.*

Proof Suppose the theorem is not true. Let G be a graph of at most nine vertices with at least two maximal-clique partitions of different sizes. Let $\mathcal{M}$ and $\mathcal{N}$ be two maximal-clique partitions of G of different sizes. If $\mathcal{M}$ and $\mathcal{N}$ have nonempty intersection, remove from G all cliques in the intersection to get a graph G' of at most nine vertices with two maximal-clique partitions $\mathcal{M} \setminus \mathcal{N}$ and $\mathcal{N} \setminus \mathcal{M}$. Because $\mathcal{M}$ and $\mathcal{N}$ are different and they partition the edges of G, neither $\mathcal{M} \setminus \mathcal{N}$ or $\mathcal{N} \setminus \mathcal{M}$ is empty. Hence, without loss of generality, assume that $\mathcal{M}$ and $\mathcal{N}$ have empty intersection. Note that neither $\mathcal{M}$ nor $\mathcal{N}$ can contain a 2-clique, because if a 2-clique is a maximal clique, it must be in every maximal-clique partition. Hence, maximal cliques of G in $\mathcal{M} \cup \mathcal{N}$ are cliques of order at least three. If $\mathcal{M}$ (or $\mathcal{N}$) contains a clique C of order at least five, then C is not contained in $\mathcal{N}$ (or $\mathcal{M}$). Since by Theorem 3.2, G would have at least ten vertices, this contradicts $|V(G)| \leq 9$. Hence, $\mathcal{M}$ and $\mathcal{N}$ cannot contain cliques of order greater than or equal to five. Thus, $\mathcal{M}$ and $\mathcal{N}$ are composed of 3-cliques and 4-cliques.

Let m and n be the numbers of 4-cliques in $\mathcal{M}$ and $\mathcal{N}$, respectively, and let s and t be the numbers of 3-cliques in $\mathcal{M}$ and $\mathcal{N}$, respectively. Since $\mathcal{M}$ and $\mathcal{N}$ are clique partitions of G, counting the number of edges we have $6m + 3s = 6n + 3t$.

If $m = n$, then $s = t$. Then $|\mathcal{M}| = m + s = n + t = |\mathcal{N}|$, which contradicts $|\mathcal{M}| \neq |\mathcal{N}|$. Hence, $m \neq n$. We can assume without loss of generality that $m > n$. Then $m \geq 1$.

To prove the theorem, it suffices to show that there exists a 4-clique in $\mathcal{M}$ no two edges of which are covered by the same maximal clique in $\mathcal{N}$. If this occurs, we can apply Theorem 3.2(ii) to conclude that G has at least $4 + 6 = 10$ vertices, contradicting $|V(G)| \leq 9$. Then it follows that the theorem is true.

Suppose that each 4-clique in $\mathcal{M}$ contains two edges covered by the same maximal clique in $\mathcal{N}$. Since an intersection between two cliques is a clique, each 4-clique in $\mathcal{M}$ must share at least a triangle with some maximal clique in $\mathcal{N}$. Moreover, we can conclude by Lemma 3.1(i) that they share exactly one triangle and that such a maximal clique in $\mathcal{N}$ must be a 4-clique. Therefore, we have shown that each 4-clique in $\mathcal{M}$ shares a triangle with a 4-clique in $\mathcal{N}$. However, if each two 4-cliques in $\mathcal{M}$ share a triangle with the same 4-clique in $\mathcal{N}$, they must share at least two vertices, contradicting Lemma 3.1 (ii). Thus, each 4-clique in $\mathcal{M}$ shares a triangle with different 4-cliques in $\mathcal{N}$. Hence, the number of 4-cliques in $\mathcal{M}$ is at most the number of 4-cliques in $\mathcal{N}$ or $m \leq n$. This contradicts $m > n$. Hence, we have the desired result and the theorem is proved. ■

As $L(K_5)$ has ten vertices and two maximal-clique partitions of different sizes, $L(K_5)$ is a graph having the minimum number of vertices satisfying this property. Determining whether or not it is unique still remains an open problem.

The next section defines a natural graph operation called the *glue operator*. It will be used to construct our graphs in Section 3.3 and Chapter 4.

3.2 The Glue Operator

Let G_1 and G_2 be any two graphs. If there exist $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ such that $H_1 \cong H_2$ with an isomorphism f, then we will denote by $G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2$ the graph that results from combining G_1 with G_2 by identifying H_1 and H_2 with respect to the isomorphism f between H_1 and H_2. We call this the *glued graph of* G_1 *and* G_2 *at* H_1 *and* H_2 *with respect to* f.

Example 3.1 Let H_1 and H_2 be subgraphs of G_1 and G_2, respectively, where $H_1 \cong H_2$ as in Figure 3.1:

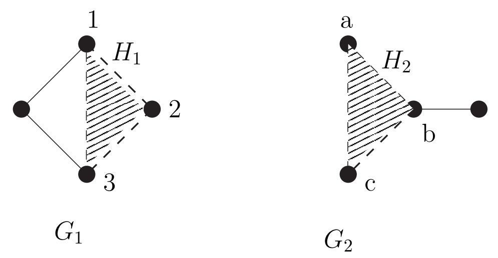


Figure 3.1: $H_1 = \triangle(1, 2, 3) \subseteq G_1$ and $H_2 = \triangle(a, b, c) \subseteq G_2$

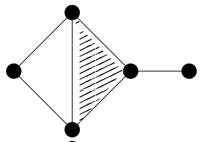
Consider three isomorphisms between H_1 and H_2, f, g, and h, as follows:

$f(1) = a, \quad f(2) = b, \quad f(3) = c,$

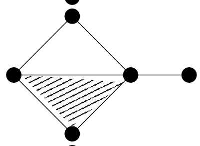
$g(1) = b, \quad g(2) = c, \quad g(3) = a$, and

$h(1) = c, \quad h(2) = b, \quad h(3) = a$. Then we have:

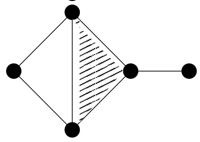
$$G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2 \quad =$$



$$G_1 \underset{H_1 \cong_g H_2}{\lhd\!\rhd} G_2 \quad =$$



$$G_1 \underset{H_1 \cong_h H_2}{\lhd\!\rhd} G_2 \quad =$$



Example 1 shows that different isomorphisms could give different or the same result. However, in some cases it is possible that all isomorphisms give the same resulting graph. The next example will illustrate this fact.

Example 3.2 Consider the following $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ as Figure 3.2.

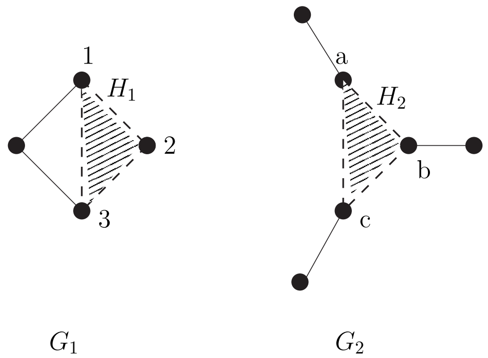


Figure 3.2: $H_1 = \triangle(1, 2, 3) \subseteq G_1$ and $H_2 = \triangle(a, b, c) \subseteq G_2$

There are six isomorphisms between H_1 and H_2, but all of them give the same graph as in Figure 3.3:

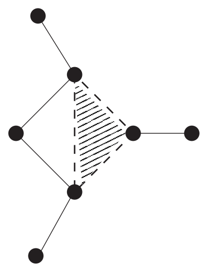


Figure 3.3: The unique resulting graph

In this particular case, we do not need to specify an isomorphism in the notation representing the glued graph, i.e., f can be omitted from the notation without ambiguity. Hence, the glued graph notation can be written as $G_1 \underset{H_1 \cong H_2}{\triangleleft\!\triangleright} G_2$.

Generally, a number of graphs with the isomorphic subgraphs can be glued together at the same time.

Definition 3.1 For $i = 1, 2, ..., n$, let H_i be a subgraph of G_i such that $H_i \cong H_j$, for all $i, j \in \{1, 2, ..., n\}$. If gluing all G_i together at H_i gives the isomorphic resulting graph, then write $\underset{H_1 \cong H_2 \cong ... \cong H_n}{\triangleleft\!\triangleright} (G_1, G_2, ..., G_n)$ as a notation for $(((G_1 \underset{H_1 \cong H_2}{\triangleleft\!\triangleright} G_2) \underset{H_2 \cong H_3}{\triangleleft\!\triangleright} ...) \underset{H_{n-1} \cong H_n}{\triangleleft\!\triangleright} G_n)$.

Now we will investigate some properties of graph gluing.

Theorem 3.4 *Let H_1 and H_2 be subgraphs of G_1 and G_2, respectively, and $H_1 \cong H_2$ for an arbitrary isomorphism f. The number of vertices in $G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2$ is $|V(G_1)| + |V(G_2)| - |V(H_i)|$ where $i = 1$ or 2.*

Proof Because H_1 and H_2 are identified, they are counted twice in the glued graph. ■

Definition 3.2 Let $\mathcal{C}(G)$ be the set of all cliques and $\mathcal{M}(G)$ be the set of all maximal cliques in a graph G.

Theorem 3.5 *Let $G_1 \lhd\!\rhd G_2$ be an arbitrary graph resulting from gluing G_1 and G_2 at any isomorphic subgraphs $H_1 \cong H_2$ with respect to any of their isomorphisms. Then*

$$\mathcal{C}(G_1 \lhd\!\rhd G_2) = \mathcal{C}(G_1) \cup \mathcal{C}(G_2)$$

Proof Because G_1 and G_2 are subgraphs of $G_1 \lhd\!\rhd G_2$, $\mathcal{C}(G_1) \subseteq \mathcal{C}(G_1 \lhd\!\rhd G_2)$ and $\mathcal{C}(G_2) \subseteq \mathcal{C}(G_1 \lhd\!\rhd G_2)$. Hence, $\mathcal{C}(G_1) \cup \mathcal{C}(G_2) \subseteq \mathcal{C}(G_1 \lhd\!\rhd G_2)$. Conversely, let $C \in \mathcal{C}(G_1 \lhd\!\rhd G_2)$. Note that if $v \in V(G_1 \setminus H_1)$ and $u \in V(G_2 \setminus H_2)$, then v is not adjacent to u. Hence, $C \in \mathcal{C}(G_1)$ or $C \in \mathcal{C}(G_2)$. Therefore, $\mathcal{C}(G_1 \lhd\!\rhd G_2) = \mathcal{C}(G_1) \cup \mathcal{C}(G_2)$. ■

That is, graph gluing does not create any new cliques or destroy any cliques in the original graphs, but it identifies each pair of corresponding cliques in H_1 and H_2. The next theorem determines the number of cliques in a glued graph.

Theorem 3.6 *Let $G_1 \lhd\!\rhd G_2$ be an arbitrary graph resulting from gluing G_1 and G_2 at any isomorphic subgraphs $H_1 \cong H_2$ with respect to any of their isomorphisms. Then*

$$|\mathcal{C}(G_1 \lhd\!\rhd G_2)| = |\mathcal{C}(G_1)| + |\mathcal{C}(G_2)| - |\mathcal{C}(H_i)| \text{ where } i = 1 \text{ or } 2.$$

Proof

$$\begin{aligned} |\mathcal{C}(G_1 \lhd\!\rhd G_2)| &= |\mathcal{C}(G_1) \cup \mathcal{C}(G_2)| \\ &= |\mathcal{C}(G_1)| + |\mathcal{C}(G_2)| - |\mathcal{C}(G_1) \cap \mathcal{C}(G_2)| \\ &= |\mathcal{C}(G_1)| + |\mathcal{C}(G_2)| - |\mathcal{C}(H_i)| \text{ where } i = 1 \text{ or } 2. \end{aligned}$$

■

However, graph gluing may destroy the maximality of cliques. Consider the next example.

Example 3.3 Let H_1 and H_2 be subgraphs of G_1 and G_2, respectively, where $H_1 \cong H_2$ as in Figure 3.4:

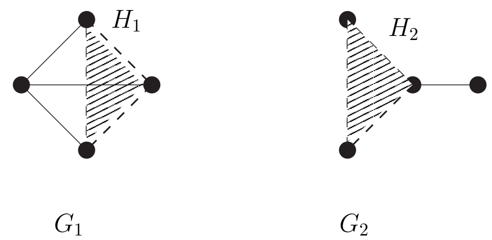
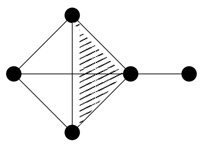


Figure 3.4: H_1 is not maximal in G_1 whereas H_2 is maximal in G_2

A glued graph of G_1 and G_2 at H_1 and H_2 with respect to an isomorphism is as follows:

$$G_1 \underset{H_1 \cong H_2}{\lhd\!\rhd} G_2 \quad = $$



H_2 is a maximal triangle in G_2. However, its copy in $G_1 \underset{H_1 \cong H_2}{\lhd\!\rhd} G_2$ is not maximal. Hence, graph gluing may destroy the maximality of a clique in the original graph.

Theorem 3.7 *Let $G_1 \lhd\!\rhd G_2$ be an arbitrary graph resulting from gluing G_1 and G_2 at arbitrary isomorphic subgraphs $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ with respect to any of their isomorphisms. Then*

$$\mathcal{M}(G_1 \lhd\!\rhd G_2) \subseteq \mathcal{M}(G_1) \cup \mathcal{M}(G_2)$$

Moreover, $\mathcal{M}(G_1 \lhd\!\rhd G_2) = \mathcal{M}(G_1) \cup \mathcal{M}(G_2)$ if and only if for any pair of corresponding cliques C_1 in H_1 and C_2 in H_2, C_1 is maximal in G_1 and C_2 is maximal in G_2, or C_1 is not maximal in G_1 and C_2 is not maximal in G_2.

Proof Let M be a maximal clique in $\mathcal{M}(G_1 \lhd\!\rhd G_2)$. Since $\mathcal{M}(G_1 \lhd\!\rhd G_2) \subseteq \mathcal{C}(G_1 \lhd\!\rhd G_2)$, by Theorem 3.5, either $M \in \mathcal{C}(G_1)$ or $M \in \mathcal{C}(G_2)$. Assume $M \in \mathcal{C}(G_i)$, for $i = 1$ or 2. Because $G_i \subseteq G_1 \lhd\!\rhd G_2$ and M is maximal in $G_1 \lhd\!\rhd G_2$, then M is maximal in G_i. Therefore, $M \in \mathcal{M}(G_i)$ for $i = 1$ or 2, i.e., $M \in \mathcal{M}(G_1) \cup \mathcal{M}(G_2)$.

Now, to show a necessary and sufficient condition for equality, it suffices to show a necessary and sufficient condition for $\mathcal{M}(G_1) \cup \mathcal{M}(G_2) \subseteq \mathcal{M}(G_1 \lhd\!\rhd G_2)$.
($\Rightarrow$) Assume there exists a clique C_1 in H_1 that is maximal in G_1, but its

corresponding clique C_2 in H_2 is not maximal in G_2. Then there exists a larger clique X containing C_2. It follows that X contains C_1 in $G_1 \lhd\!\rhd G_2$. Hence, C_1 is not maximal in $G_1 \lhd\!\rhd G_2$. Thus, $C_1 \in \mathcal{M}(G_1)$, but $C_1 \notin \mathcal{M}(G_1 \lhd\!\rhd G_2)$, i.e., $\mathcal{M}(G_1) \cup \mathcal{M}(G_2) \nsubseteq \mathcal{M}(G_1 \lhd\!\rhd G_2)$.

($\Leftarrow$) Assume for any pair of corresponding cliques C_1 in H_1 and C_2 in H_2, C_1 and C_2 are maximal in G_1 and G_2, respectively, or C_1 and C_2 are not maximal in G_1 and G_2, respectively. Let $C \in \mathcal{M}(G_1) \cup \mathcal{M}(G_2)$. Without loss of generality, assume $C \in \mathcal{M}(G_1)$, i.e., C is maximal in G_1. If C contains a vertex outside H_1, then C is also maximal in $G_1 \lhd\!\rhd G_2$ because gluing does not effect any vertices outside H_1. Thus, assume C is in H_1. Let C' be the corresponding clique of C in H_2. Since C is maximal in G_1, no vertex in $G_1 \setminus C$ is adjacent to all vertices of C. By the assumption, C' is also maximal in G_2. Then no vertex in $G_2 \setminus C'$ is adjacent to all vertices of C'. Hence, identifying C and C' yields that C is maximal in $G_1 \lhd\!\rhd G_2$. Therefore, $C \in \mathcal{M}(G_1 \lhd\!\rhd G_2)$. Hence, $\mathcal{M}(G_1) \cup \mathcal{M}(G_2) \subseteq \mathcal{M}(G_1 \lhd\!\rhd G_2)$. ■

Theorem 3.8 *Let $G_1 \lhd\!\rhd G_2$ be an arbitrary graph resulting from gluing G_1 and G_2 at any isomorphic subgraphs $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ with respect to any of their isomorphisms. Let A be the set of cliques in H_1 that are maximal in G_1 and whose corresponding cliques are also maximal in G_2. Let B_1 be the set of cliques in H_1 that are maximal in G_1 but whose corresponding cliques in H_2 are not maximal in G_2. Let B_2 be the set of cliques in H_2 that are maximal in G_2 but whose corresponding cliques in H_1 are not maximal in G_1. Then*

$$|\mathcal{M}(G_1 \lhd\!\rhd G_2)| = |\mathcal{M}(G_1)| + |\mathcal{M}(G_2)| - |A| - |B_1| - |B_2|.$$

Proof We can derive the number of cliques in the glued graph $\mathcal{M}(G_1 \lhd\rhd G_2)$ by discarding the number of cliques in $|\mathcal{M}(G_1)| + |\mathcal{M}(G_2)|$ that are not counted in $|\mathcal{M}(G_1 \lhd\rhd G_2)|$. The result follows directly from Theorem 3.7. If both cliques of a pair of corresponding cliques are maximal, they are counted twice in the glued graph. Hence, one of them must be discarded by being in A. If a clique C is maximal in one original graph while its corresponding clique is not maximal in the other original graph, then C is not maximal in the glued graph. Hence, it will be discarded by being in either B_1 or B_2. ■

Theorem 3.9 *The glued graph of two clique-inseparable graphs is clique-inseparable.*

Proof Let G_1 and G_2 be graphs. Let $G_1 \lhd\rhd G_2$ be the glued graph between G_1 and G_2 at $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$. Suppose $G_1 \lhd\rhd G_2$ is clique-separable. That is, there exists a proper nonempty subgraph H of $G_1 \lhd\rhd G_2$ separating cliques in $G_1 \lhd\rhd G_2$. By Theorem 3.5, $\mathcal{C}(G_1 \lhd\rhd G_2) = \mathcal{C}(G_1) \cup \mathcal{C}(G_2)$. Then it follows that $H \cap G_1$ separates cliques in G_1 and $H \cap G_2$ separates cliques in G_2. If $H \cap G_1 = G_1$, then $H \cap G_2 = H_2$ and it follows that H_2 separates G_2. Since H_2 is a proper nonempty subgraph of G_2, G_2 is clique-separable. If $H \cap G_1 = \emptyset$, then $H \cap G_2$ is nonempty and $H \cap G_2 \neq G_2$. Thus, G_2 is clique-separable. Otherwise, H is a proper nonempty subgraph of G_1. It follows that G_1 is clique-separable. ■

3.3 Maximal-clique Partitions of k Different Sizes

When $n \geq 5, L(K_n)$ is a clique-inseparable graph having maximal-clique partitions of two different sizes. A disjoint union of a number of $L(K_n)$ has maximal-clique partitions of various sizes. The graph composed of a number of $L(K_n)$ all of which share a vertex is a connected graph with maximal-clique partitions of various sizes. However, neither of these examples are clique-inseparable. Pullman, Shank and Wallis (PSW) [32] showed a graph H_k with k maximal-clique partitions of the same size by using $k-1$ copies of $L(K_4)$. By Theorem 2.3 and Theorem 3.9, we can conclude that H_k is a clique-inseparable graph. The objective of this section is to construct a clique-inseparable graph with maximal-clique partitions of k different sizes. Our construction is similar in concept to PSW's construction. We will reintroduce H_k in terms of a construction using the glue operator for insight into the construction of a graph with k maximal-clique partitions of k different sizes.

3.3.1 Graphs with k Maximal-clique Partitions of the Same Size

From Lemma 2.2, $L(K_4)$ has two maximal-clique partitions $\mathcal{M}_4$ and $\mathcal{N}_4$ where $\mathcal{M}_4$ is the vertex-induced partition and $\mathcal{N}_4$ is the triangle partition shown in Figure 3.5. These two maximal-clique partitions have the same size, which is 4.

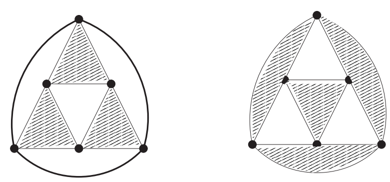


Figure 3.5: $L(K_4)$ with two maximal-clique partitions, $\mathcal{M}_4$ and $\mathcal{N}_4$

Consider $k-1$ copies of $L(K_4)$, called $L(K_4)^1, L(K_4)^2, ..., L(K_4)^{k-1}$. For $i = 1, \ldots, k-1$, let $\mathcal{M}_4^i$ and $\mathcal{N}_4^i$ be the vertex-induced partition and the triangle partition of $L(K_4)^i$, respectively, and for each i, let S_i be any triangle in the triangle partition $\mathcal{N}_4^i$.

Let S_i be the middle triangle in Figure 3.5, and for $i \neq j \in \{1, \ldots, k-1\}$, any isomorphism between S_i and S_j gives the same resulting graph. Then it is well-defined to define

$$H_k = \underset{S_1 \cong S_2 \cong \ldots \cong S_k}{\lhd\!\rhd} (L(K_4)^1, L(K_4)^2, ..., L(K_4)^{k-1})$$

That is, H_k is the graph obtained by gluing $k-1$ copies of $L(K_4)$ together at the middle triangle of each copy as shown in Figure 3.6. Rename the middle triangle in the resulting graph H_k as S. Hence, $S = S_1 = \ldots = S_{k-1}$ in H_k.

It is easy to see that for each i, covering $L(K_4)^i \setminus S$ in H_k by $\mathcal{N}_4^i \setminus \{S_i\}$ yields a maximal-clique partition of $H_k \setminus S$. Let $\mathcal{Y} = \{S\} \cup \bigcup_{i=1}^{k-1} (\mathcal{N}_4^i \setminus \{S_i\})$; then $\mathcal{Y}$ is a maximal-clique partition of H_k, and $|\mathcal{Y}| = |\{S\}| + \sum_{i=1}^{k-1}(|\mathcal{N}_4^i| - 1) = 1 + \sum_{i=1}^{k-1}(4-1) = 1 + 3k - 3 = 3k-2$. Now, we obtain other maximal clique partitions of H_k. Fix $t \in \{1, \ldots, k-1\}$ and cover $L(K_4)^t$ by its vertex-induced partition $\mathcal{M}_4^t$. Note that $\mathcal{M}_4^t$ has already covered S. Since other copies of $L(K_4)$ in H_k share S with $L(K_5)^t$, it follows that the remaining edges in $L(K_4)^i \setminus \{S_i\}$ with $i \neq t$ are in the unique maximal-clique partition which is the collection of nine triangles in $\mathcal{N}_4^i \setminus \{S_i\}$. That is, for $t = 1, \ldots, k-1$, let $\mathcal{X}_t = \mathcal{M}_4^t \cup \bigcup_{i \neq t, i=1}^{k-1} (\mathcal{N}_4^i \setminus \{S_i\})$. Thus, $\mathcal{X}_t$ is a maximal-clique partition of H_k, and $|\mathcal{X}_t| = |\mathcal{M}_4^t| + \sum_{i \neq t, i=1}^{k-1}(|\mathcal{N}_4^i| - 1) = 4 + \sum_{i \neq t, i=1}^{k-1}(4-1) = 4 + 3(k-2) = 4 + 3k - 6 = 3k-2$. Therefore, $\{\mathcal{Y}, \mathcal{X}_1, \ldots, \mathcal{X}_{k-1}\}$ is a collection of maximal-

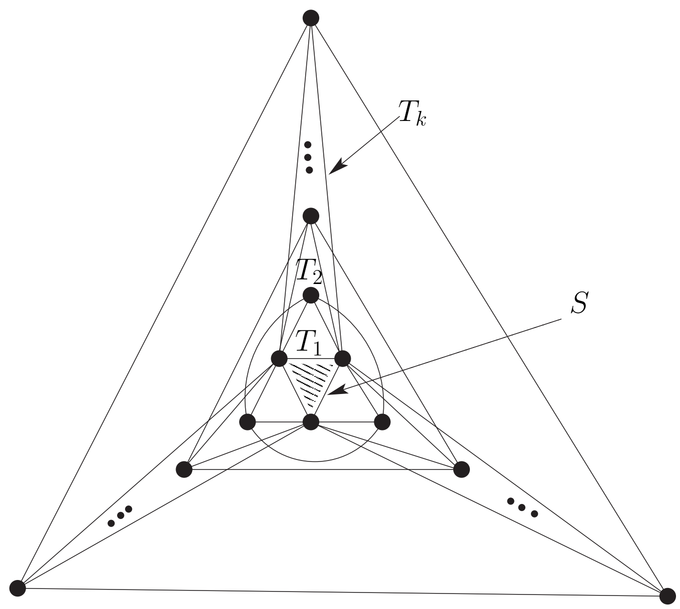


Figure 3.6: H_k

clique partitions of H_k, and each of them has size $3k-2$. Moreover, since $L(K_4)$ is clique-inseparable (by Lemma 2.3) and the glued graph of clique-inseparable graphs is clique-inseparable (by Theorem 3.9), we have H_k is clique-inseparable. Thus, we can conclude the following:

Theorem 3.10 (Theorem 3 in [32]) *For any positive integer k, there exists a graph with k maximal-clique partitions of the same size.*

Proof K_2 is a graph with one maximal-clique partition. When $k \geq 2$, then H_k is a graph with k maximal-clique partitions of the same size. ■

Corollary 3.2 *For any positive integer k, there exists a clique-inseparable graph with k maximal-clique partitions of the same size.*

Proof K_2 is clique-inseparable. When $k \geq 2$, by Theorem 2.3, Theorem 3.9, and Theorem 3.10, H_k is a clique-inseparable graph with k maximal-clique partitions of the same size. ■

3.3.2 Graphs with Maximal-clique Partitions of k Different Sizes

For $k \geq 2$, let V_k be the graph constructed in this section having k maximal-clique partitions of k different sizes. Similar to the construction of H_k, V_k will be obtained by gluing $k-1$ graphs. Since we need the target graph to have maximal-clique partitions of different sizes, we will glue graphs with maximal-clique partitions of different sizes. Recall from Section 2.3, for an integer $i \geq 5$, $L(K_i)$ has two maximal-clique partitions of two different sizes. This is the vital property to get our target graph.

The graph V_k will be constructed from $L(K_5), L(K_6), ...,$ and $L(K_{k+3})$, a total of $k-1$ graphs. Recall that $\mathcal{N}_i$ is the triangle partition of $L(K_i)$. For $i = 5, \ldots, k+3$, let R_i be any triangle in $\mathcal{N}_i$. We will glue the above $k-1$ graphs together at R_i. Since $L(K_n)$ is a vertex transitive graph, all isomorphisms between R_i and R_j, where $i \neq j$, give the isomorphic graphs. Thus,

$$V_k = \mathop{\lhd\!\!\!\mid\!\!\!\rhd}_{R_5 \cong R_6 \cong \ldots \cong R_{k+3}} (L(K_5), L(K_6), ..., L(K_{k+3}))$$

is well-defined. Rename the triangle representing all R_i in the glued graph V_k as R. So, $R = R_5 = \ldots = R_{k+3}$ in V_k.

Theorem 3.11 *For $k \geq 2$, V_k has k maximal-clique partitions of k different sizes.*

Proof It is easy to see that for each $i = 5, \ldots, k+3$, covering $L(K_i)\backslash R_i$ in V_k by $\mathcal{N}_i\backslash\{R_i\}$ yields a maximal-clique partition of $V_k\backslash R$. Let $\mathcal{Y} = \{R\} \cup \bigcup_{i=5}^{k+3}(\mathcal{N}_i \backslash \{R_i\})$, then $\mathcal{Y}$ is a maximal-clique partition of V_k, and by Lemma 2.2, $|\mathcal{Y}| = |\{R\}| + \sum_{i=5}^{k+3}(|\mathcal{N}_i| - 1) = 1 + \sum_{i=5}^{k+3}(\frac{i(i-1)(i-2)}{6} - 1) = \sum_{i=5}^{k+3} \frac{i(i-1)(i-2)}{6} - (k-2)$.

Next we will obtain other maximal-clique partitions of V_k. Fix $t \in \{5, \ldots, k+3\}$. Cover $L(K_t)$ in V_k by its vertex-induced partition $\mathcal{M}_t$, then R in V_k has already been covered by $\mathcal{M}_t$. Since $L(K_i)$ where $i \neq t$ in V_k share R with $L(K_t)$, it follows that the remaining edges in $L(K_i) \setminus \{R_i\}$ are partitioned by the unique maximal-clique partition which is the collection of triangles in $\mathcal{N}_i \setminus \{R_i\}$. That is, for $t = 5, \ldots, k+3$, let $\mathcal{X}_t = \mathcal{M}_t \cup \bigcup_{i \neq t, i=5}^{k+3}(\mathcal{N}_i \setminus \{R_i\})$; then $\mathcal{X}_t$ is a maximal-clique partition of V_k and by Lemma 2.2, $|\mathcal{X}_t| = |\mathcal{M}_t| + \sum_{i \neq t, i=5}^{k+3}(|\mathcal{N}_i| - 1) = t + \sum_{i \neq t, i=5}^{k+3}(\frac{i(i-1)(i-2)}{6} - 1) = t + \sum_{i=5, i\neq t}^{k+3} \frac{i(i-1)(i-2)}{6} - (k-2)$. Furthermore, there are only two ways to partition edges in $L(K_t) \setminus S_t$ by its maximal cliques. One way uses cliques in $\mathcal{M}_t$ and the other uses cliques in $\mathcal{N}_t$, and they induce maximal-clique partitions of V_k as mentioned above. Therefore, $\{\mathcal{Y}, \mathcal{X}_5, \ldots, \mathcal{X}_{k+3}\}$ is the collection of all k maximal-clique partitions of V_k.

We have $|\mathcal{Y}| \neq |\mathcal{X}_i|$ because for $i \geq 5$, $i \neq \frac{i(i-1)(i-2)}{6}$, and $|\mathcal{X}_i| \neq |\mathcal{X}_j|$ because for $i \neq j \in \{5, \ldots, k+3\}$, $\frac{i(i-1)(i-2)}{6} \neq \frac{j(j-1)(j-2)}{6}$. Hence, V_k has k different maximal-clique partitions, and they all have different sizes. ∎

Theorem 3.12 *For any positive integer k, there exists a clique-inseparable graph with k maximal-clique partitions, and they all have different sizes.*

Proof K_2 is a clique-inseparable graph with one maximal-clique partition. Since for each natural number n, $L(K_n)$ is clique-inseparable (by Lemma 2.3), and the glued graph of clique-inseparable graphs is clique-inseparable (by Theorem 3.9), we have V_k is clique-inseparable for $k \geq 2$. Therefore, by Theorem 3.11, for $k \geq 2$, V_k is a clique-inseparable graph with k maximal-clique partitions of k different sizes. ■

4. Graphs with $cc = cp < mcp$

The rest of the dissertation emphasizes the three clique parameters cc, cp and mcp. Since a clique partition is a clique covering, $cc \leq cp$. Since a maximal-clique partition is a clique partition, $cp \leq mcp$. Hence, it is always true that the three clique parameters of a graph satisfy $cc \leq cp \leq mcp$. These clique parameters of a graph are not always equal. However, all graphs of at most nine vertices have the same three clique parameter values. This fact was proved by Monson in her Ph.D. Dissertation in 1995 [22]. Thus, the three parameters could be different when a graph has at least ten vertices. From Chapter 2, if a line graph has a maximal-clique partition, it has $cc = cp = mcp$. Interval graphs [20] and graphs with norm at most 4 [32] are also examples of graphs having $cc = cp = mcp$.

As $cc \leq cp \leq mcp$, there are three more relationships possible among the three clique parameters, which are $cc < cp = mcp$, $cc = cp < mcp$ and $cc < cp < mcp$. Monson found the graph of ten vertices with $cc < cp = mcp$ shown in Figure 4.1. This graph has $cc = 8 < cp = mcp = 9$. Since ten vertices is the minimum number of vertices needed to get different values of clique parameters, Monson's graph is a graph with the minimum number of vertices satisfying $cc < cp = mcp$.

The question of the existence of graphs with $cc = cp < mcp$ has been investigated previously to no avail. In this chapter we solve this open problem by constructing an infinite family of graphs with $cc = cp < mcp$. We will

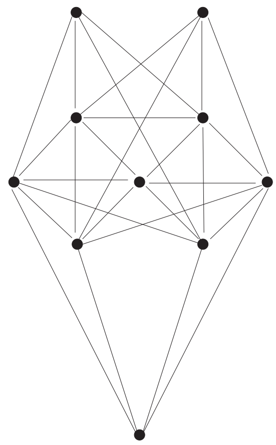


Figure 4.1: Monson's graph

start by constructing our first target graph U^* in Section 4.1. Using the glue operator we generalize this construction to obtain a family of graphs satisfying $cc = cp < mcp$. It will be shown in Section 4.3 that the smallest graph among those constructed in this fashion has 68 vertices and 234 edges.

4.1 Our First Target Graph

In this chapter we refer to graphs satisfying $cc = cp < mcp$ as target graphs. Let U^* be our first target graph. It will be constructed by gluing 12 copies of $L(K_5)$ to a graph called B. The graph B is an extension of another graph L. Graphs L and B are constructed to have some special properties so that B, glued with a number of $L(K_5)$, gives the target graph U^* with the desired property $cc = cp < mcp$.

4.1.1 The Construction

Definition 4.1 The notation $K_{1,3}(a : b, c, d)$ represents the complete bipartite graph with bipartition $\{a\}$ and $\{b, c, d\}$.

4.1.1.1 The Graph L

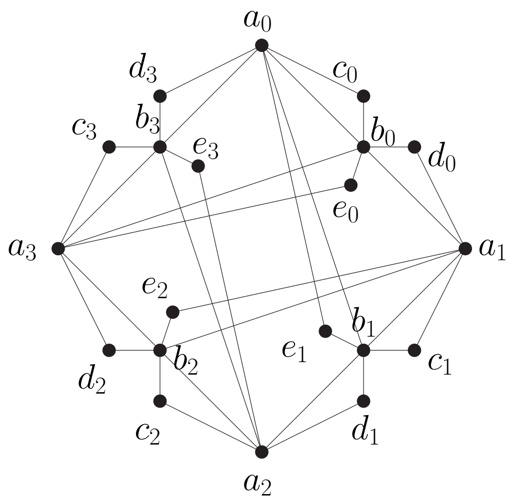


Figure 4.2: The link graph L

Let L be the graph with 20 vertices and 36 edges as in Figure 4.2. Two important characteristics of L are stated in Lemma 4.1. They are keys to getting the target graph U^*.

Lemma 4.1 (Characteristics of L)

(i) $E(L)$ can be partitioned into 12 triangles.

(ii) $E(L)$ can be partitioned into 12 induced subgraphs $K_{1,3}$, named $A_0, ..., A_{11}$ where for all $0 \leq i, j \leq 11$, A_i and A_j have at most one vertex in common.

Proof (i) Here is the list of 12 triangles partitioning $E(L)$:

$$
\begin{array}{lll}
Y_0 = \triangle(a_0, b_0, c_0) & Y_4 = \triangle(a_0, b_3, d_3) & Y_8 = \triangle(a_0, b_1, e_1) \\
Y_1 = \triangle(a_1, b_1, c_1) & Y_5 = \triangle(a_1, b_0, d_0) & Y_9 = \triangle(a_1, b_2, e_2) \\
Y_2 = \triangle(a_2, b_2, c_2) & Y_6 = \triangle(a_2, b_1, d_1) & Y_{10} = \triangle(a_2, b_3, e_3) \\
Y_3 = \triangle(a_3, b_3, c_3) & Y_7 = \triangle(a_3, b_2, d_2) & Y_{11} = \triangle(a_3, b_0, e_0).
\end{array}
$$

Let $\mathcal{Y} = \{Y_i : i = 0, \ldots, 11\}$ be the set composed of 12 triangles partitioning $E(L)$. Hence, $\mathcal{Y}$ is a clique partition of L. Figure 4.3 shows the partition of $E(L)$ into 12 triangles.

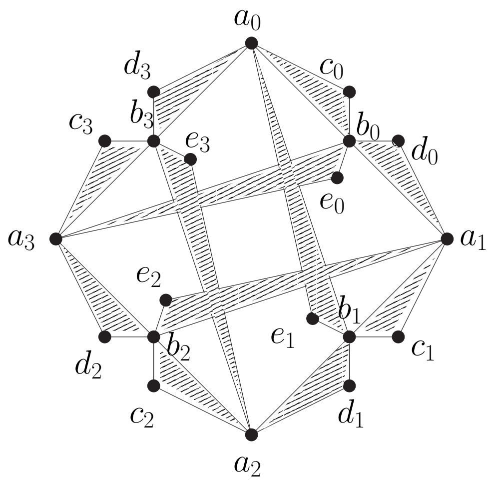


Figure 4.3: The partition of $E(L)$ into 12 triangles

(ii) The edge set of L can be partitioned into 12 copies of $K_{1,3}$ as follows:

$A_0 = K_{1,3}(b_0 : c_0, d_0, e_0)$ $A_4 = K_{1,3}(a_0 : b_1, c_0, d_3)$ $A_8 = K_{1,3}(a_0 : b_3, b_0, e_1)$

$A_1 = K_{1,3}(b_1 : c_1, d_1, e_1)$ $A_5 = K_{1,3}(a_1 : b_2, c_1, d_0)$ $A_9 = K_{1,3}(a_1 : b_0, b_1, e_2)$

$A_2 = K_{1,3}(b_2 : c_2, d_2, e_2)$ $A_6 = K_{1,3}(a_2 : b_3, c_2, d_1)$ $A_{10} = K_{1,3}(a_2 : b_1, b_2, e_3)$

$A_3 = K_{1,3}(b_3 : c_3, d_3, e_3)$ $A_7 = K_{1,3}(a_3 : b_0, c_3, d_2)$ $A_{11} = K_{1,3}(a_3 : b_2, b_3, e_0)$

Figure 4.4 illustrates $\mathcal{A} = \{A_i : i = 0, \ldots, 11\}$ partitioning $E(L)$. Furthermore, any two members of $\mathcal{A}$ have at most one vertex in common. ■

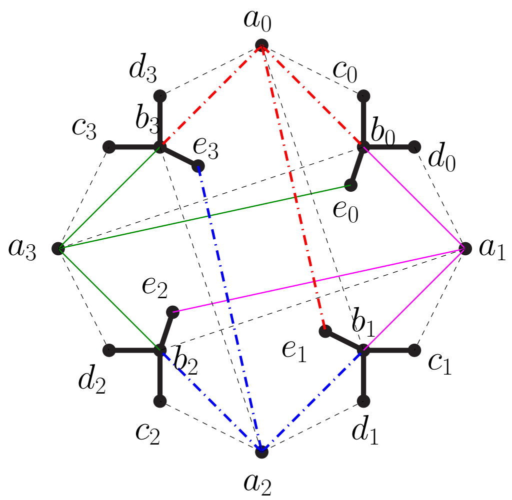


Figure 4.4: The partition of $E(L)$ into 12 copies of $K_{1,3}$

4.1.1.2 The Graph B

For $i = 0, \ldots 11$, let T_i be the graph that is the complement of A_i on the vertex set of A_i. They are:

$$\begin{array}{lll} T_0 = \triangle(c_0, d_0, e_0) & T_4 = \triangle(b_1, c_0, d_3) & T_8 = \triangle(b_3, b_0, e_1) \\ T_1 = \triangle(c_1, d_1, e_1) & T_5 = \triangle(b_2, c_1, d_0) & T_9 = \triangle(b_0, b_1, e_2) \\ T_2 = \triangle(c_2, d_2, e_2) & T_6 = \triangle(b_3, c_2, d_1) & T_{10} = \triangle(b_1, b_2, e_3) \\ T_3 = \triangle(c_3, d_3, e_3) & T_7 = \triangle(b_0, c_3, d_2) & T_{11} = \triangle(b_2, b_3, e_0) \end{array}$$

Since A_i is an induced subgraph $K_{1,3}$ of L, the triangle T_i does not contain any edges of L. Since for any $i \neq j \in \{0, \ldots, 11\}$, A_i and A_j share at most one vertex, T_i and T_j cannot share an edge.

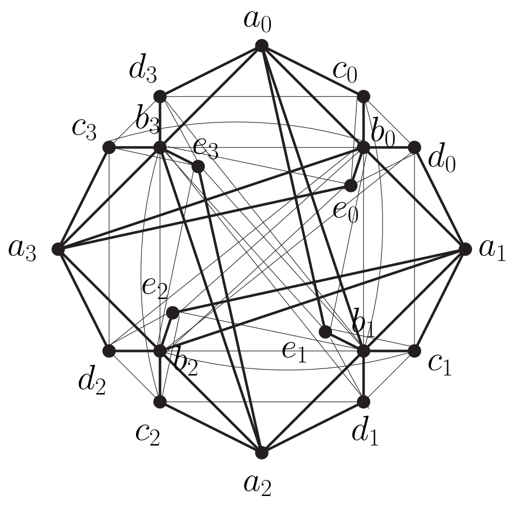


Figure 4.5: $B = L \cup T_0 \cup T_1 \cup \ldots \cup T_{11} = X_0 \cup \ldots \cup X_{11}$

Let $\mathcal{T} = \{T_i : i = 0, \dots, 11\}$. Now, B is constructed by adding all triangles T_i in $\mathcal{T}$ to L. Then B has the same vertex set as L which has 20 vertices, and since B contains 36 edges from L and 36 edges from the 12 triangles in $\mathcal{T}$, then B has a total of 72 edges. B is illustrated in Figure 4.5.

Now, for $i = 0, \dots, 11$, define $X_i = A_i \cup T_i$. Let $\mathcal{X} = \{X_i : i = 0, \dots, 11\}$, then $\mathcal{X}$ is a collection of 4-cliques that is a clique partition of B.

4.1.1.3 A Target Graph U^*

For each $i = 0, \dots, 11$, let $L(K_5)^i$ be the ith copy of $L(K_5)$. Recall from Section 2.3 that $L(K_5)$ has two maximal-clique partitions, the vertex-induced partition and the triangle partition. Let S_i be a triangle in the triangle partition of $L(K_5)^i$. For each i, $L(K_5)^i$ is glued to B at triangles S_i and T_i. Let U^* be the resulting graph. That is,

$$U^* = B \begin{cases} \underset{T_0 \cong S_0}{\lhd\!\rhd} L(K_5)^0 \\ \underset{T_1 \cong S_1}{\lhd\!\rhd} L(K_5)^1 \\ \cdots \\ \underset{T_{11} \cong S_{11}}{\lhd\!\rhd} L(K_5)^{11} \end{cases}$$

An example of gluing $L(K_5)^i$ to B at S_i and T_i is illustrated in Figure 4.6.

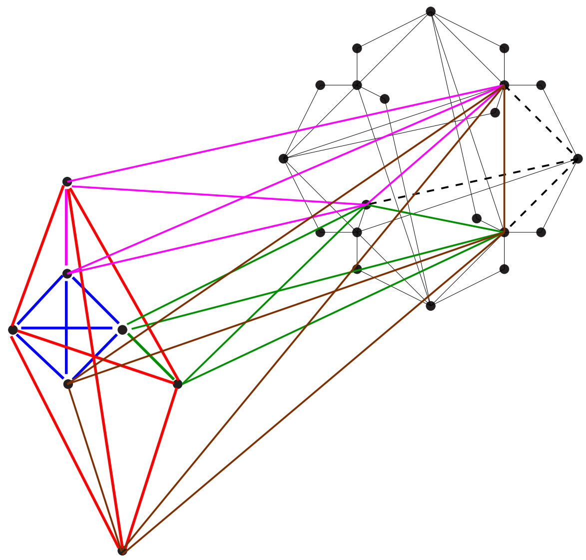


Figure 4.6: A copy of $L(K_5)$ glued to B at a triangle

Since for each $i = 0, \ldots, 11$, $L(K_5)^i$ shares exactly three vertices with B, the number of vertices of U^* is $|V(B)| + 12 \cdot |V(L(K_5))| - 12 \cdot |V(K_3)| = 20 + 12 \cdot 10 - 12 \cdot 3 = 104$. Also, U^* can be considered to be the edge-disjoint union of 12 copies of $L(K_5)$ and L. Hence, the number of edges in U^* is

$$|E(U^*)| = 12 \cdot |E(L(K_5))| + |E(L)| = 12 \cdot 30 + 36 = 396.$$

4.1.2 The Clique Numbers of U^*

In this subsection, all three clique numbers $cc(U^*), cp(U^*)$ and $mcp(U^*)$ will be determined. The vertex-induced partition of each copy of $L(K_5)$ and $\mathcal{Y}$ composed of triangles Y_i partitioning L will determine $cc(U^*)$ and $cp(U^*)$, while $mcp(U^*)$ can be deduced from the triangle partition of each copy of $L(K_5)$ together with $\mathcal{A} = \{A_i : i = 0, \ldots, 11\}$ in B.

Lemma 4.2 $cp(U^*) \leq 72$.

Proof Since U^* is an edge-disjoint union of subgraphs $L(K_5)^0, L(K_5)^1, \ldots, L(K_5)^{11}$ and L, the union of clique partitions of these subgraphs is a clique partition of U^*. For each $i = 0, \ldots, 11$, let $\mathcal{M}_i$ be the vertex-induced partition of $L(K_5)^i$. Recall that $\mathcal{Y} = \{Y_i : i = 0, \ldots, 11\}$ composed of 12 triangles partitioning L (in Lemma 4.1(i)) is a clique partition of L.

Hence, let $\mathcal{P} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \ldots \cup \mathcal{M}_{11} \cup \mathcal{Y}$. Then $\mathcal{P}$ is a clique partition of U^* with $|\mathcal{P}| = |\mathcal{M}_0| + |\mathcal{M}_1| + \ldots + |\mathcal{M}_{11}| + |\mathcal{Y}| = \underbrace{5 + 5 + \ldots + 5}_{12} + 12 = 72$.

Thus, U^* has a clique partition $\mathcal{P}$ of size 72. Therefore, $cp(U^*) \leq 72$. ■

Lemma 4.3 $cc(U^*) \geq 72$.

Proof Let $\mathcal{C}$ be any clique covering of U^*. First note that $\{L(K_5)^0 \setminus B$, $L(K_5)^1 \setminus B, \ldots, L(K_5)^{11} \setminus B$, $L\}$ is a collection of pairwise edge-disjoint subgraphs of U^*, and there is no clique in U^* covering two edges from different subgraphs. Hence,

$$|\mathcal{C}| \geq n_0 + n_1 + \ldots + n_{11} + l$$

where for i=0,...,11, n_i is the minimum number of cliques in U^* covering edges in $L(K_5)^i \setminus B$, and l is the minimum number of cliques in U^* covering edges in L.

Since the largest clique in $L(K_5)^i \setminus B$ has order 4, each clique can cover at most six edges of $L(K_5)^i \setminus B$. Because $L(K_5)^i \setminus B$ has 27 edges, it needs at least five cliques to cover its edges, i.e., $n_i \geq 5$ for all $i = 0, \ldots, 11$. Thus, $|\mathcal{C}| \geq \underbrace{5 + \ldots + 5}_{12} + l$.

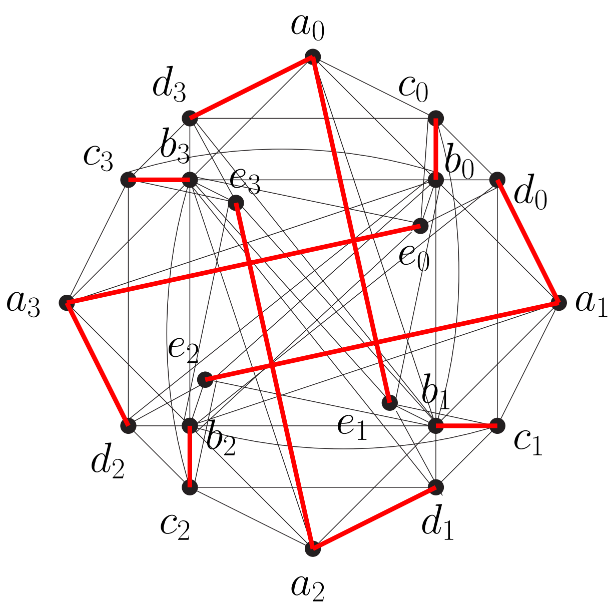


Figure 4.7: 12 cliques needed to cover the 12 thick edges

Note that no two of the 12 edges in Figure 4.7 are contained in the same clique in U^*. In other words, there is no clique that covers any two of these 12 edges at the same time. Hence, L needs at least 12 cliques to cover all its edges, and $l \geq 12$. Hence, $|\mathcal{C}| \geq \underbrace{5 + 5 + \ldots + 5}_{12} + 12 = 60 + 12 = 72$ cliques. Since $\mathcal{C}$ is arbitrary, $cc(U^*) \geq 72$. ■

Lemma 4.4 $mcp(U^*) = 120$.

Proof First note that not every clique in the partition $\mathcal{P}$ defined in Lemma 4.2 is maximal in U^*. For example, $Y_8 \in \mathcal{P}$, but $Y_8 = \triangle(a_0, b_1, e_1) \subseteq \boxtimes(a_0, b_1, e_1, b_0)$ in U^*. Since $\mathcal{P}$ uses the vertex-induced partition to cover $L(K_5)^i$, now we will cover $L(K_5)^i$ by using the triangle partition. For $i = 0, \ldots, 11$, let $\mathcal{N}_i$ be the triangle partition of $L(K_5)^i$. Then $\mathcal{N}_i$ is composed of ten triangles. Recall that $S_i \in \mathcal{N}_i$. Because for each i, B is glued with $L(K_5)^i$ at T_i and S_i, we have that U^* is a disjoint union of B and $L(K_5)^i \setminus S_i : i = 0, \ldots, 11$.

Let $\mathcal{N}_i' = \mathcal{N}_i \setminus \{S_i\}$. Then $\mathcal{N}_i'$ is a clique partition of $L(K_5)^i \setminus S_i$ and $|\mathcal{N}_i'| = 9$. From Subsection 4.1.1.2, $\mathcal{X} = \{X_i : i = 0, \ldots, 11\}$ is a collection of 4-cliques in U^* that partitions B. Let $\mathcal{M} = \mathcal{X} \cup \mathcal{N}_0' \cup \mathcal{N}_1' \cup \ldots \cup \mathcal{N}_{11}'$. Since X_i and nine triangles in $\mathcal{N}_i \setminus \{S_i\}$ are maximal in U^*, $\mathcal{M}$ is a maximal-clique partition of U^*. In fact $\mathcal{M}$ is the only maximal-clique partition of U^*. Thus, $mcp(U^*) = |\mathcal{M}| = |\mathcal{X}| + |\mathcal{N}_0'| + |\mathcal{N}_1'| + \ldots + |\mathcal{N}_{11}'| = 12 + \underbrace{9 + \ldots + 9}_{12} = 12 + 108 = 120$. ■

Theorem 4.1 *There exists a graph U^* with $cc(U^*) = cp(U^*) < mcp(U^*)$.*

Proof Consider U^* that has been constructed with 104 vertices and 396 edges. By Lemma 4.2 and Lemma 4.3, $cp(U^*) \leq 72 \leq cc(U^*)$. However, because a clique partition is always a clique covering, $cc(U^*)$ is always at most $cp(U^*)$. Hence, $cc(U^*) = cp(U^*) = 72$. By Lemma 4.4, $mcp(U^*) = 120$. Therefore, $cc(U^*) = cp(U^*) < mcp(U^*)$. ■

4.2 A Family of Target Graphs

Two characteristics of the graph L in Lemma 4.1 and the property of $L(K_5)$ that it has two maximal-clique partitions of different sizes are crucial to obtaining U^* with $cc = cp < mcp$. Hence, other graphs with $cc = cp < mcp$ can possibly be constructed in the same fashion as in the last section by generalizing U^* using two approaches. The first approach is replacing $L(K_5)$ by any graph with two maximal-clique partitions of different sizes, for example $L(K_n)$ when $n \geq 5$. The other one is generalizing L in a way that preserves the two characteristics in Lemma 4.1.

4.2.1 Link Graphs L_k and Base Graphs B_k

For $k \geq 3$, let L_k be the graph defined as follows:

$V(L_k) = \{v_i : v = a, b, c, d, e, \text{ and } i = 0, 1, \ldots, k-1\}$.

$E(L_k)= \{a_ic_i,\ a_id_{i-1(\text{mod k})},\ a_ib_{i+1(\text{mod k})},\ a_ib_i,\ a_ib_{i-1(\text{mod k})},\ a_ie_{i+1(\text{mod k})},\ b_ic_i,\ b_id_i,\ b_ie_i : i = 0, 1, \ldots, k-1\}$.

Therefore, L_k has $5k$ vertices and $9k$ edges. L_k is illustrated in Figure 4.8. We call L_k a *link graph.*

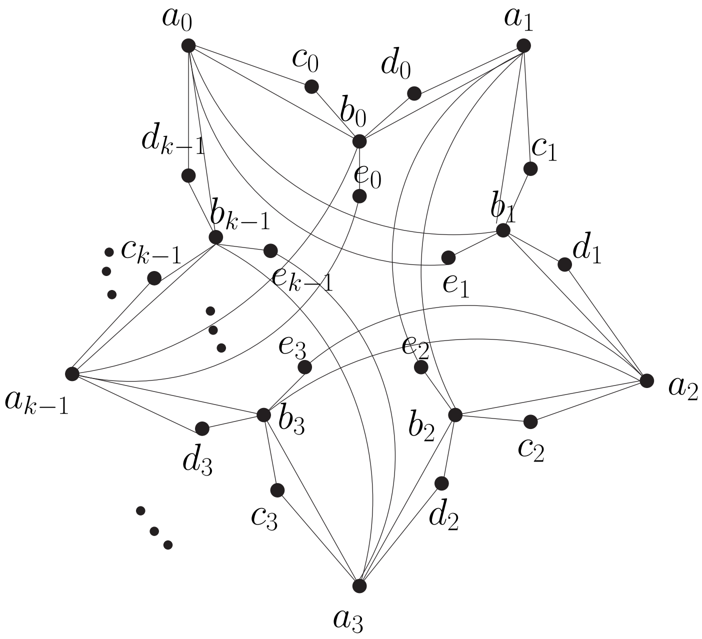


Figure 4.8: The link graph L_k

When $k = 4$, $L_4 = L$, i.e., L_k is a generalization of L. The following two lemmas give characteristics of L_k analogous to those of L in Lemma 4.1.

Lemma 4.5 *For $k \geq 3$, $E(L_k)$ can be partitioned into $3k$ copies of 3-cliques which are $Y_{k,i} : i = 0, 1, \ldots, 3k-1$, where for $i = 0, 1, \ldots, k-1$,*

$$\begin{aligned} Y_{k,i} &= \triangle(a_i, b_i, c_i) \\ Y_{k,k+i} &= \triangle(a_i, b_{i-1(mod\ k)}, d_{i-1(mod\ k)}) \\ Y_{k,2k+i} &= \triangle(a_i, b_{i+1(mod\ k)}, e_{i+1(mod\ k)}) \end{aligned}$$

Let $\mathcal{Y}_k = \{Y_{k,i} : i = 0, \ldots, 3k-1\}$. Then $\mathcal{Y}_k$ is a clique partition of L_k.

Lemma 4.6 *For $k \geq 3$, $E(L_k)$ can be partitioned into $3k$ copies of $K_{1,3}$ which are $A_{k,i}, i = 0, 1, \ldots, 3k-1$, where for $i = 0, 1, \ldots, k-1$,*

$$\begin{aligned} A_{k,i} &= K_{1,3}(b_i : c_i, d_i, e_i) \\ A_{k,k+i} &= K_{1,3}(a_i : b_{i+1(mod\ k)}, c_i, d_{i-1(mod\ k)}) \\ A_{k,2k+i} &= K_{1,3}(a_i : b_{i-1(mod\ k)}, b_i, e_{i+1(mod\ k)}) \end{aligned}$$

Moreover, $\mathcal{A}_k = \{A_{k,i} : i = 0, 1, \ldots, 3k-1\}$ has the property that any two members of $\mathcal{A}_k$ have at most one vertex in common.

Next, the link graph L_k will be extended to a *base graph* B_k. For $k \geq 3$, $i = 0, \ldots, 3k-1$, let $T_{k,i}$ be the graph complement of $A_{k,i}$ on the vertex set of $A_{k,i}$. That is, for $i = 0, 1, \ldots, k-1$,

$$\begin{aligned} T_{k,i} &= \triangle(c_i, d_i, e_i) \\ T_{k,k+i} &= \triangle(b_{i+1(\text{mod k})}, c_i, d_{i-1(\text{mod k})}) \\ T_{k,2k+i} &= \triangle(b_{i-1(\text{mod k})}, b_i, e_{i+1(\text{mod k})}). \end{aligned}$$

For each i, $A_{k,i}$ is an induced $K_{1,3}$ of L_k. So, $T_{k,i}$ is a triangle that does not contain any edges in L_k. Since for any $i \neq j \in \{0, \ldots, 3k-1\}$, $A_{k,i}$ and $A_{k,j}$ share at most one vertex, $T_{k,i}$ and $T_{k,j}$ do not share an edge. For $k \geq 3$, let $\mathcal{T}_k = \{T_{k,i} : i = 0, \ldots, 3k-1\}$. B_k is constructed by adding all triangles in $\mathcal{T}_k$ to L_k. Then B_k has the same vertex set as L_k which contains $5k$ vertices, and B_k contains $9k$ edges from L_k and $9k$ more edges from the $3k$ triangles in $\mathcal{T}_k$; so, B_k has a total of $18k$ edges. For future reference, the numbers of vertices and edges in L_k and B_k are summarized in the next Lemma.

Lemma 4.7 For $k \geq 3$,

(i) L_k has $5k$ vertices and $9k$ edges.

(ii) B_k has $5k$ vertices and $18k$ edges.

As in the previous section, For $k \geq 3$, $i = 0, \ldots, 3k-1$, define $X_{k,i} = A_{k,i} \cup T_{k,i}$. Then $X_{k,i}$ is a 4-clique. Let $\mathcal{X}_k = \{X_{k,i} : i = 0, \ldots, 3k-1\}$. Thus, $\mathcal{X}_k$ is a clique partition of B_k. When $k \geq 4$, for $i = 0, \ldots, k-1$, $X_{k,i}$ is maximal in B_k. This fact will be shown in the next Lemma.

Lemma 4.8 *For $k \geq 4$, $\mathcal{X}_k = \{X_{k,i} : i = 0, \ldots, 3k-1\}$ is a maximal-clique partition of B_k.*

Proof Let $k \geq 4$. Let $t \in \{0, \ldots, 3k-1\}$. Consider $X_{k,t}$ in B_k. Let $v \in V(B_k) \setminus V(X_{k,t})$.

Case 1 $v \in \{b_0, \ldots, b_{k-1}\}$. Say $v = b_i$. First note that b_i is adjacent to b_j if and only if $j \equiv i-1 \pmod{k}$ or $j \equiv i+1 \pmod{k}$. If $X_{k,t}$ contains b_j where $j \not\equiv i-1 \pmod{k}$ and $j \not\equiv i+1 \pmod{k}$, then v is not adjacent to b_j. Thus, $X_{k,t} \cup \{v\}$ is not a clique. Otherwise, $X_{k,t}$ does not contain b_j where $j \not\equiv i-1 \pmod{k}$ or $j \not\equiv i+1 \pmod{k}$. From the definition of $X_{k,t}$ in B_k where $k \geq 4$, the only possible form for $X_{k,t}$ is $\boxtimes(b_j, c_j, d_j, e_j)$ where $j \equiv i+1 \pmod{k}$ or $j \equiv i-1 \pmod{k}$. However, b_i is not adjacent to either $c_{i+1 \pmod{k}}$ or $c_{i-1 \pmod{k}}$. Hence, $v = b_i$ is not adjacent to all vertices of $X_{k,t}$. Hence, the induced subgraph on $V(X_{k,t}) \cup \{v\}$ is not a clique.

Case 2 $v \notin \{b_0, \ldots, b_{k-1}\}$. Then v is contained in exactly two members of $\mathcal{A}_k = \{A_{k,i} : i = 0, \ldots, 3k-1\}$, say $A_{k,p}$ and $A_{k,q}$ where $p \neq q$. Suppose v is adjacent to all vertices in $V(X_{k,t})$. By the Pigeon Hole Principle, at least one of $\{A_{k,p}, A_{k,q}\}$ contains at least two vertices in $V(X_{k,t})$. Without loss of generality say it is $A_{k,p}$. This yields that $A_{k,p}$ and A_t share at least two vertices, contradicting the property of $\mathcal{A}_k$ in Lemma 4.6. Hence, v cannot be adjacent to all vertices in $V(X_{k,t})$.

Hence, for any vertex v in $B_k \setminus X_{k,t}$, vertex v cannot be combined with $X_{k,t}$ to get a larger clique containing $X_{k,t}$. Therefore, $X_{k,t}$ is maximal in B_k. Since $\mathcal{X}_k = \{X_{k,i} : i = 0, \ldots, 3k-1\}$ is a clique partition of B_k, we have proved $\mathcal{X}_k$ is a maximal-clique partition of B_k. ■

4.2.2 Target Graphs U_k

For any integer $k \geq 3$, let U_k be the target graph resulting from gluing B_k to $3k$ copies of $L(K_5)$. For $i = 0, 1, \ldots, 3k-1$, let $L(K_5)^i$ be the ith copy of $L(K_5)$. Let S_i be any triangle in the triangle partition of $L(K_5)^i$. Recall that $T_{k,i}$ is a triangle added to L_k to get B_k. Define

$$U_k = B_k \begin{cases} \underset{T_{k,0} \cong S_0}{\lhd\!\rhd} L(K_5)^0 \\ \underset{T_{k,1} \cong S_1}{\lhd\!\rhd} L(K_5)^1 \\ \cdots \\ \underset{T_{k,3k-1} \cong S_{3k-1}}{\lhd\!\rhd} L(K_5)^{3k-1} \end{cases}$$

Lemma 4.9 *For $k \geq 3$, U_k has $26k$ vertices and $99k$ edges.*

Proof Since U_k is the graph that results from gluing copies of $L(K_5)$ at triangles of B_k, by Lemma 4.7 the number of vertices in U_k is

$$|V(B_k)| + 3k \cdot |V(L(K_5))| - 3k \cdot |V(K_3)| = 5k + 3k \cdot 10 - 3k \cdot 3 = 26k.$$

Since U_k can be considered to be the edge-disjoint union of L and $3k$ copies of $L(K_5)$, the number of edges in U_k is

$$|E(L_k)| + 3k \cdot |E(L(K_5))| = 9k + 3k \cdot 30 = 99k.$$

■

Lemma 4.10 *For $k \geq 3$, $cp(U_k) \leq 18k$.*

Proof Similar to the proof in Lemma 4.2, for each $i = 0, \ldots, 3k-1$, let $\mathcal{M}_i$ be the vertex-induced partition of $L(K_5)^i$. Let $\mathcal{T}_k = \{T_{k,i} : i = 0, \ldots, 3k-1\}$ partitioning $E(L_k)$. Let $\mathcal{P}_k = \mathcal{T}_k \cup \mathcal{M}_0 \cup \ldots \cup \mathcal{M}_{3k-1}$. Then $\mathcal{P}_k$ is a clique partition of U_k. Moreover, $|\mathcal{P}_k| = |\mathcal{T}_k| + |\mathcal{M}_0| + \ldots + |\mathcal{M}_{3k-1}| = 3k + \underbrace{5 + 5 + \ldots + 5}_{3k} = 3k + 15k = 18k$. Hence, $cp(U_k) \leq 18k$. ■

If n is a number of edges in a graph G so that no pair of edges is contained in the same clique of G, then the clique covering number of G is at least n. Hence, finding the maximum such n is the dual problem for finding the minimum size of clique coverings of G.

Definition 4.2 Let e and f be any two edges in a graph G. If there is no clique in G containing both e and f, then e and f are *clique-independent with respect to* G.

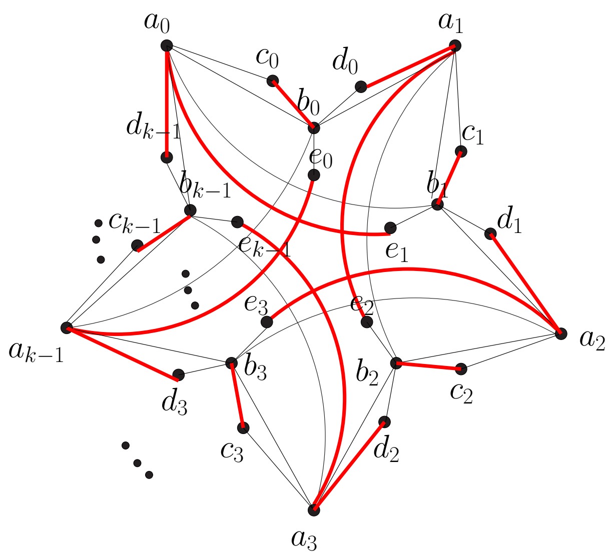


Figure 4.9: $3k$ pairwise clique-independent edges

Although the graph in Figure 4.9 does not show all edges in B_k, the set of $3k$ thick edges in the figure demonstrates $3k$ pairwise clique-independent edges with respect to B_k, i.e., there is no clique in B_k that can cover any two thick edges in Figure 4.9.

Lemma 4.11 *For $k \geq 3$, L_k contains a set of $3k$ pairwise clique-independent edges with respect to B_k.*

Proof Let $I_k = \{c_i b_i, a_i d_{i-1(\text{mod k})}, a_i e_{i+1(\text{mod k})} : i = 0, \ldots, k-1\}$. Let $e \neq f$ be in I_k.

Case 1 $\exists i \in \{0, \ldots, k\}$, $a_i \in e$ and $a_i \in f$. Since $d_{i-1(\text{mod k})}$ is not adjacent to $e_{i+1(\text{mod k})}$, we have e and f are clique-independent with respect to B_k.

Case 2 $\exists i\ \exists j \in \{0, \ldots, k\}$, $a_i \in e$ and $a_j \in f$. Since when $i \neq j$, a_i is not adjacent to a_j, then e and f are clique-independent with respect to B_k.

Case 3 $\exists i \in \{0, \ldots, k\}\ \forall j \in \{0, \ldots, k\}$, $a_i \in e$ but $a_j \notin f$. Note that a_i is not adjacent to c_j when $j \neq i$, $d_{i-1(\text{mod k})}$ is not adjacent to $b_{i+1(\text{mod k})}$, and c_i is not adjacent to $e_{i+1(\text{mod k})}$. Hence, e and f are clique-independent with respect to B_k.

Case 4 $\forall i \in \{0, \ldots, k\}$, $a_i \notin e$ and $a_i \notin f$. Since c_i is not adjacent to c_j when $i \neq j$, then e and f are clique-independent with respect to B_k.

Since the above four cases include all possibilities, the lemma is proved. ■

Lemma 4.12 *For $k \geq 3$, $cc(U_k) \geq 18k$.*

Proof Let $\mathcal{C}_k$ be a clique covering of U_k. First note that $\{L(K_5)^0 \setminus B_k$, $L(K_5)^1 \setminus B_k, \ldots, L(K_5)^{3k-1} \setminus B_k, L_k\}$ is a collection of pairwise edge-disjoint subgraphs of U_k, and there is no clique in U_k covering any two edges from different ones of these subgraphs. Hence,

$$|\mathcal{C}_k| \geq n_0 + n_1 + \ldots + n_{3k-1} + l$$

where for $i = 0, \ldots, 3k-1$, n_i is the minimum number of cliques in U_k covering edges in $L(K_5)^i \setminus B_k$, and l is the minimum number of cliques in U_k covering edges in L_k.

Since for each $i = 0, \ldots, 3k-1$, the largest clique in $L(K_5)^i \setminus B_k$ has order 4, each clique can cover at most six edges of $L(K_5)^i \setminus B_k$. Because $L(K_5)^i \setminus B_k$ has 27 edges, it needs at least five cliques to cover its edges, i.e., $n_i \geq 5$ for all $i = 0, \ldots, 11$. Thus, $|\mathcal{C}_k| \geq \underbrace{5 + \ldots + 5}_{3k} + l$.

By Lemma 4.11, L_k contains $3k$ pairwise clique-independent edges with respect to B_k. Besides, since there is no clique in U_k containing edges in both B_k and in $U_k \backslash B_k$, these $3k$ edges also form a set of $3k$ pairwise clique-independent edges with respect to U_k. Thus, $l \geq 3k$. Hence, $|\mathcal{C}_k| \geq \underbrace{5 + 5 + \ldots + 5}_{3k} + 3k = 15k + 3k = 18k$. Since $\mathcal{C}_k$ is arbitrary, $cc(U_k) \geq 18k$. ■

By Lemma 4.8, for $k \geq 4$, B_k has a maximal-clique partition which induces a maximal-clique partition of U_k. The maximal-clique partition number of U_k will be discussed in the next Lemma.

Lemma 4.13 *For $k \geq 4$, $mcp(U_k) = 30k$.*

Proof First note that $\mathcal{P}_k$ in Lemma 4.10 is not a maximal-clique partition of U_k because $Y_{2k} = \triangle(a_0, b_1, e_1) \subseteq \boxtimes(a_0, b_1, e_1, b_0)$ in U_k, but $Y_8 \in \mathcal{P}_k$. We can get a maximal-clique partition of U_k similarly to Lemma 4.4. For $i = 0, \ldots, 3k-1$, let $\mathcal{N}_i$ be the triangle partition of $L(K_5)^i$. Then $\mathcal{N}_i$ is composed of ten triangles. Recall that $S_i \in \mathcal{N}_i$. Since for each i, B_k is glued with $L(K_5)^i$ at $T_{k,i}$ and S_i, we have U_k is a disjoint union of B_k and $L(K_5)^i \setminus S_i : i = 0, \ldots, 3k-1$.

For $i = 0, \ldots, 3k-1$, let $\mathcal{N}_i' = \mathcal{N}_i \setminus \{S_i\}$. Then $\mathcal{N}_i'$ is a clique partition of $L(K_5)^i \setminus S_i$ and $|\mathcal{N}_i'| = 9$. By Lemma 4.8, $\mathcal{X}_k = \{X_{k,i} : i = 0, \ldots, 3k-1\}$ is a maximal-clique partition of B_k. Let $\mathcal{M}_k = \mathcal{X}_k \cup \mathcal{N}_0' \cup \mathcal{N}_1' \cup \ldots \cup \mathcal{N}_{3k-1}'$. Since $X_{k,i}$ and nine triangles in $\mathcal{N}_i \setminus \{S_i\}$ are maximal in U_k, we have that $\mathcal{M}_k$ is a maximal-clique partition of U_k, and in fact $\mathcal{M}_k$ is the only maximal-clique partition of U_k. Thus, $mcp(U_k) = |\mathcal{M}_k| = |\mathcal{X}_k| + |\mathcal{N}_0'| + |\mathcal{N}_1'| + \ldots + |\mathcal{N}_{3k-1}'| = 3k + \underbrace{9 + \ldots + 9}_{3k} = 3k + 27k = 30k$. ■

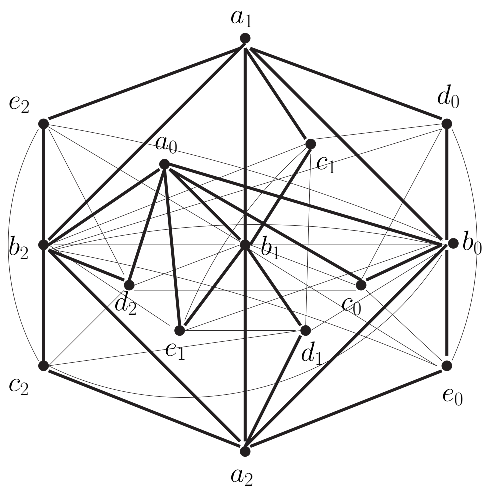


Figure 4.10: The base graph B_3 and its subgraph L_3 represented by the thick edges

The graphs L_3 and B_3 are shown in Figure 4.10. Consider $A_{3,6} = K_{1,3}(a_0 : b_2, b_0, e_1)$ and $T_{3,6} = \triangle(b_2, b_0, e_1)$ in B_3. Hence, $X_{3,6} = \boxtimes(a_0, b_2, b_0, e_1)$. However, b_1 is adjacent to all vertices in $X_{3,6}$; so, $X_{3,6}$ is contained in a 5-clique. Hence, $X_{3,6}$ is not maximal in B_3. It follows that B_3 does not have a maximal-clique partition and neither does U_3. Hence, U_3 does not have the property $cc = cp <$

mcp. Therefore, U_4, which is our first target graph U^* described in Section 4.1, is the smallest graph in this family.

Theorem 4.2 *There exist infinitely many graphs with $cc = cp < mcp$.*

Proof Consider U_k, for $k \geq 4$. By Lemma 4.10 and Lemma 4.12, $cp(U_k) \leq 18k \leq cc(U_k)$. However, because a clique partition is always a clique covering, $cc(U_k)$ is always at most $cp(U_k)$. Thus, $cc(U_k) = cp(U_k)$. By Lemma 4.13, $mcp(U_k) = 30k$. Hence, for $k \geq 4$, $cc(U_k) = cp(U_k) < mcp(U_k)$. Let $\mathcal{F} = \{U_k : k \geq 4\}$. Then $\mathcal{F}$ is a family of infinitely many graphs with $cc = cp < mcp$. ■

4.3 The Smaller Target Graphs

The fact that $L(K_5)$ has two maximal-clique partitions of different sizes is an essential property for our construction of target graphs in this fashion. By Lemma 2.2, $L(K_n)$, for all $n \geq 5$, has two maximal-clique partitions. So, they can be used to replace $L(K_5)$ in the construction. Hence, the graph constructed by gluing B_k with any $3k$ graphs in $\{L(K_n) : n \geq 5\}$ always has the property of $cc = cp < mcp$. Now we are interested in finding a graph smaller than U_4 (or U^*) that results from gluing B_4 with 12 copies of $L(K_5)$. Because $L(K_4)$ has two maximal-clique partitions, all copies of $L(K_5)$ could be replaced by $L(K_4)$ in the construction to obtain a graph with two clique partitions; one is maximal but the other is not. However, unlike $L(K_5)$, which has two maximal-clique partitions of the same size, replacing all copies of $L(K_5)$ by $L(K_4)$ yields a graph with $cc = cp = mcp$. However, it is not necessary to use all copies of $L(K_5)$ to get $cc = cp < mcp$. Thus, the sizes of the target graphs can be reduced by replacing some copies of $L(K_5)$ by $L(K_4)$ but not all. For $k \geq 4$ and $0 \leq d \leq 3k$, let $U_{k,d}$ be the graph resulted by gluing B_k to d copies of $L(K_5)$ and $3k-d$ copies of $L(K_4)$. For $i = 0, \ldots, d-1$, let $L(K_5)^i$ be the ith copy of $L(K_5)$, and S_i be a triangle in the triangle partition of $L(K_5)^i$. For $i = d, \ldots, 3k-1$, let $L(K_4)^i$ be the $(i-d)$th copy of $L(K_4)$, and S_i be a triangle in the triangle partition of $L(K_4)^i$. Define

$$U_{k,d} = B_k \begin{cases} \underset{T_{k,0} \cong S_0}{\lhd\!\rhd} L(K_5)^0 \\ \cdots \\ \underset{T_{k,d-1} \cong S_{d-1}}{\lhd\!\rhd} L(K_5)^{d-1} \\ \underset{T_{k,d} \cong S_d}{\lhd\!\rhd} L(K_4)^d \\ \cdots \\ \underset{T_{k,3k-1} \cong S_{3k-1}}{\lhd\!\rhd} L(K_4)^{3k-1} \end{cases}$$

Note that if $d = 3k$, $U_{k,3k} = U_k$.

Lemma 4.14 *For $k \geq 4$ and $0 \leq d \leq 3k$, $U_{k,d}$ has $14k + 4d$ vertices and $45k + 18d$ edges.*

Proof Since $U_{k,d}$ is the glued graph of B_k with each copy of $L(K_5)$ and $L(K_4)$ at a triangle, then by Lemma 4.7, the number of vertices in $U_{k,d}$ is

$$\begin{aligned} |V(U_{k,d})| &= |V(B_k)| + d \cdot |V(L(K_5))| + (3k-d)|V(L(K_4))| - 3k \cdot |V(K_3)| \\ &= 5k + d \cdot 10 + (3k-d) \cdot 6 - 3k \cdot 3 \\ &= 5k + 10d + 18k - 6d - 9k \\ &= 14k + 4d. \end{aligned}$$

The graph $U_{k,d}$ can be considered to be an edge-disjoint union of d copies of $L(K_5)$, $3k-d$ copies of $L(K_4)$ and L. Hence, the number of edges in $U_{k,d}$ is

$$\begin{aligned} |E(U_{k,d})| &= |E(L_k)| + d \cdot |E(L(K_5))| + (3k-d) \cdot |E(L(K_4))| \\ &= 9k + d \cdot 30 + (3k-d) \cdot 12 \\ &= 9k + 30d + 36k - 12d \\ &= 45k + 18d. \end{aligned}$$

■

Lemma 4.15 *For $k \geq 4$ and $0 \leq d \leq 3k$, let $\mathcal{M}_{k,d}$ be a clique partition of $U_{k,d}$ induced by the vertex-induced partitions of $L(K_5)^i$, $i = 0, \ldots, d-1$ and $L(K_4)^j$, $j = d, \ldots, 3k-1$. Let $\mathcal{N}_{k,d}$ be a clique partition of $U_{k,d}$ induced by the triangle partitions of $L(K_5)^i$, $i = 0, \ldots, d-1$ and $L(K_4)^j$, $j = d, \ldots, 3k-1$. Then $|\mathcal{M}_{k,d}| = 15k + d$ and $|\mathcal{N}_{k,d}| = 12k + 6d$.*

Proof For $i = 0, \ldots, d-1$ and for $j = d, \ldots, 3k-1$, let $\mathcal{M}_5^i$ and $\mathcal{M}_4^j$ be the vertex-induced partition of $L(K_5)^i$ and $L(K_4)^j$, respectively. Recall that $\mathcal{T}_k =$

$\{T_{k,i} : i = 0, \ldots, 3k-1\}$. Let $\mathcal{M}_{k,d} = \mathcal{T}_k \cup \mathcal{M}_5^0 \cup \ldots \cup \mathcal{M}_5^{d-1} \cup \mathcal{M}_4^d \cup \ldots \cup \mathcal{M}_4^{3k-1}$. Then

$$\begin{aligned} |\mathcal{M}_{k,d}| &= |\mathcal{T}_k| + |\mathcal{M}_5^0| + \ldots |\mathcal{M}_5^{d-1}| + |\mathcal{M}_4^d| + \ldots + |\mathcal{M}_4^{3k-1}| \\ &= 3k + \underbrace{5 + \ldots + 5}_{d} + \underbrace{4 + \ldots + 4}_{3k-d} \\ &= 3k + 5d + 4(3k-d) \\ &= 3k + 5d + 12k - 4d \\ &= 15k + d. \end{aligned}$$

Next, for $i = 0, \ldots, d-1$ and for $j = d, \ldots, 3k-1$, let $\mathcal{N}_5^i$ and $\mathcal{N}_4^j$ be the triangle partition of $L(K_5)^i$ and $L(K_4)^j$, respectively. Recall that $\mathcal{X}_k = \{X_{k,i} : i = 0, \ldots, 3k-1\}$, $S_i \in \mathcal{N}_5^i$, and $S_j \in \mathcal{N}_4^j$. Let $(\mathcal{N}_5^i)' = \mathcal{N}_5^i \setminus \{S_i\}$ and $(\mathcal{N}_4^j)' = \mathcal{N}_4^j \setminus \{S_j\}$. Let $\mathcal{N}_{k,d} = \mathcal{X}_k \cup (\mathcal{N}_5^0)' \cup \ldots \cup (\mathcal{N}_5^{d-1})' \cup (\mathcal{N}_4^d)' \cup \ldots \cup (\mathcal{N}_4^{3k-1})'$. Then

$$\begin{aligned} |\mathcal{N}_{k,d}| &= |\mathcal{X}_k| + |(\mathcal{N}_5^0)'| + \ldots |(\mathcal{N}_5^{d-1})'| + |(\mathcal{N}_4^d)'| + \ldots + |(\mathcal{N}_4^{3k-1})'| \\ &= 3k + \underbrace{9 + \ldots + 9}_{d} + \underbrace{3 + \ldots + 3}_{3k-d} \\ &= 3k + 9d + 3(3k-d) \\ &= 3k + 9d + 9k - 3d \\ &= 12k + 6d. \end{aligned}$$

■

Theorem 4.3 *For $k \geq 4$, if $0 \leq d \leq \frac{3k}{5}$, then $cc(U_{k,d}) = cp(U_{k,d}) = mcp(U_{k,d}) = 12k + 6d$. If $\frac{3k}{5} < d \leq 3k$, then $cc(U_{k,d}) = cp(U_{k,d}) < mcp(U_{k,d})$.*

Proof For integers k and d where $k \geq 4$ and $0 \leq d \leq 3k$, similar to the proof of Lemma 4.13, we can prove that $\mathcal{N}_{k,d}$ in Lemma 4.15 is the only maximal-clique partition of $U_{k,d}$. So, $mcp(U_{k,d}) = |\mathcal{N}_{k,d}| = 12k + 6d$.

However, both $\mathcal{M}_{k,d}$ and $\mathcal{N}_{k,d}$ in Lemma 4.15 are clique coverings and clique partitions of $U_{k,d}$. The one with smaller size will determine $cc(U_{k,d})$ and $cp(U_{k,d})$.

If $0 \leq d \leq \frac{3k}{5}$, then $15k + d \geq 12k + 6d$. Thus, $|\mathcal{M}_{k,d}| \geq |\mathcal{N}_{k,d}|$. Hence, $\mathcal{N}_{k,d}$ is a minimum clique covering and a minimum clique partition of $U_{k,d}$. Hence, $cc(U_{k,d}) = cp(U_{k,d}) = mcp(U_{k,d}) = |\mathcal{N}_{k,d}| = 12k + 6d$.

If $\frac{3k}{5} < d \leq 3k$, then $15k + d < 12k + 6d$. Thus, $|\mathcal{M}_{k,d}| < |\mathcal{N}_{k,d}|$. Hence, $\mathcal{M}_{k,d}$ is a minimum clique covering and a minimum clique partition of $U_{k,d}$. Therefore, $cc(U_{k,d}) = cp(U_{k,d}) = |\mathcal{M}_{k,d}| = 15k + d$. Since $15k + d < 12k + 6d$, $cc(U_{k,d}) = cp(U_{k,d}) < mcp(U_{k,d})$. ■

Theorem 4.4 *$\mathcal{F}' = \{U_{k,d} : k \geq 4$ and $\frac{3k}{5} < d \leq 3k\}$ is a family of infinitely many graphs with $cc = cp < mcp$.*

Proof Immediate from Theorem 4.3. ■

Note that $U_{k,3k} = U_k$. Recall that $\mathcal{F} = \{U_k : k \geq 4\}$ is defined in Theorem 4.2 and $\mathcal{F}' = \{U_{k,d} : k \geq 4$ and $\frac{3k}{5} < d \leq 3k\}$ is defined in Theorem 4.4. Hence, $\mathcal{F} \subseteq \mathcal{F}'$ and $U_{4,3}$ is the smallest graph in $\mathcal{F}'$. Thus, $U_{4,3}$ is the smallest possible graph with $cc = cp < mcp$ among graphs constructed in this fashion.

Theorem 4.5 *There exists a graph of 68 vertices with $cc = cp < mcp$.*

Proof By Lemma 4.14, $U_{4,3}$ has $14\cdot 4+4\cdot 3 = 68$ vertices and $45\cdot 4+18\cdot 3 = 180 + 54 = 234$ edges. By Theorem 4.3, $cc(U_{4,3}) = cp(U_{4,3}) = 15 \cdot 4 + 3 = 63$, and $mcp(U_{4,3}) = 12 \cdot 4 + 6 \cdot 3 = 66$. Hence, $cc(U_{4,3}) = cp(U_{4,3}) < mcp(U_{4,3})$. ■

We have solved the existence problem. However, there is no guarantee that $U_{4,3}$ is the smallest possible graph with $cc = cp < mcp$. It remains an open question whether there exists a graph with $cc = cp < mcp$ having fewer vertices than $U_{4,3}$.

5. Clique Partitions of Graphs with Ten Vertices

This chapter continues the investigation of the three clique parameters cc, cp and mcp. We know that many graphs do not have a maximal-clique partition. Throughout this chapter, we will explore only graphs with a maximal-clique partition. From Monson's Ph.D. Dissertation [22], we know that for any graph of at most nine vertices with a maximal-clique partition, all three clique parameters are the same, i.e., $cc = cp = mcp$. Hence, ten is the smallest number that the three parameters could have different values. Monson [22] found an example of a graph of ten vertices with $cp = mcp$ but different from cc. Thus, graphs with $cc < cp = mcp$ have a minimum number of ten vertices. For any graph G with a maximal-clique partition, if G has $cp < mcp$, then G must have at least two clique partitions of different sizes and the partitions with the smallest cardinality cannot be maximal-clique partitions. In Chapter 4, we found a family of graphs with $cc = cp < mcp$. The smallest graph in the family has 68 vertices, which is somewhat large. However, we do not know whether or not it has the minimum number of vertices among graphs with $cc = cp < mcp$. On the other hand, there is an example of a graph on as few as 11 vertices with $cc < cp < mcp$. Pullman, Shank and Wallis [32] asked whether or not there is such a graph with fewer than eleven vertices. However, since Monson showed that any graph of fewer than ten vertices has $cc = cp = mcp$ [22], it is sufficient to answer the open problem of Pullman, Shank and Wallis by investigating only graphs with ten vertices. This chapter will confirm that there is no graph of ten vertices with $cp < mcp$. Consequently, graphs with $cc < cp < mcp$ have a minimum of 11 vertices. From this we can conclude that any graph of ten vertices has a minimum clique partition whose cliques are maximal or it has no maximal-clique partition. In Section 5.1 we study characteristics of maximal-clique partitions of graphs on ten vertices having clique partitions satisfying various properties. Finally, the nonexistence of graphs having ten vertices and $cp < mcp$ will be investigated.

5.1 Ten Vertex Graphs with a Maximal-Clique Partition

The maximal-clique partitions of graphs of ten vertices will be explored in this section. These results will be used to prove the main theorem in the next section.

Lemma 5.1 *Every clique partition of a graph on ten vertices contains at most five 4-cliques. Furthermore, if a clique partition of a graph G on ten vertices is composed of exactly five 4-cliques, then $G = L(K_5)$.*

Proof Let $\mathcal{P}$ be a clique partition of a graph G of ten vertices. Consider any vertex v of G. Since G has ten vertices, v can be contained in at most three 4-cliques in $\mathcal{P}$. If v is contained in three 4-cliques in $\mathcal{P}$, then all ten vertices of G are covered by these 4-cliques. Let K be a clique in $\mathcal{P}$ other than those three 4-cliques. Since every two cliques in $\mathcal{P}$ can share at most one vertex, K can contain at most one vertex from each of those three 4-cliques. Thus, K contains at most three vertices, that is, $\mathcal{P}$ cannot have other 4-cliques. Otherwise, every vertex in G appears in at most two 4-cliques in $\mathcal{P}$. Hence, ten vertices can be used at most $10 \cdot 2 = 20$ times. Since one 4-clique contains four vertices, the number of 4-cliques in $\mathcal{P}$ is at most $\frac{20}{4} = 5$.

Furthermore, if $\mathcal{P}$ has five 4-cliques, then each vertex of G belongs to exactly two 4-cliques. Since every two cliques in $\mathcal{P}$ can share at most one vertex, no two vertices of G can belong to the same pair of 4-cliques in $\mathcal{P}$. Labelling vertices of G according to the 4-cliques containing them gives a graph isomorphic to $L(K_5)$. ■

Lemma 5.2 *If a graph has a vertex adjacent to all other vertices, then such a vertex must be in every clique of each maximal-clique partition of the graph.*

Proof Let G be a graph with a vertex v adjacent to all other vertices. Suppose the statement is not true for a maximal-clique partition $\mathcal{M}$ of G, i.e., there exists a clique C in $\mathcal{M}$ such that $v \notin C$. Since, v is connected to every vertex in C, then $C \vee \{v\}$ (the join graph of C and $\{v\}$) is a larger clique containing C, which contradicts the maximality of C. ■

Theorem 5.1 *Let $\mathcal{M}$ be a maximal-clique partition of a graph with ten vertices. If $\mathcal{M}$ does not contain a 2-clique, then $|\mathcal{M}| \leq 10$.*

Proof Let G be a graph of ten vertices with a maximal-clique partition $\mathcal{M}$. Then $|E(G)| \leq \binom{10}{2} = 45$. Assume there is no 2-clique in $\mathcal{M}$. If there is no 3-clique in $\mathcal{M}$, then $|\mathcal{M}| \leq \frac{45}{|E(K_4)|} = \frac{45}{6} < 10$ as desired.

Now, assume that there exists a 3-clique in $\mathcal{M}$, say $\triangle(a, b, c)$. Since $\triangle(a, b, c)$ is maximal, none of the remaining seven vertices can be adjacent to all three vertices in $\triangle(a,b,c)$. This yields $|E(G)| \leq 45 - 7 = 38$.

Suppose $|\mathcal{M}| \geq 11$. If $\mathcal{M}$ contains a clique of order 5 or more, the total number of edges of G is minimized when the remaining ten cliques are 3-cliques. Hence, $38 \geq |E(G)| \geq |E(K_5)| + 10 \cdot |E(K_3)| = 10 + 10 \cdot 3 = 40$, a contradiction. Thus, there is no clique of order 5 or more in $\mathcal{M}$. If $\mathcal{M}$ contains two 4-cliques, $38 \geq |E(G)| \geq 2 \cdot |E(K_4)| + 9 \cdot |E(K_3)| = 2 \cdot 6 + 9 \cdot 3 = 12 + 27 = 39$, another contradiction. Therefore, $\mathcal{M}$ contains at most one 4-clique, and the rest of its cliques are 3-cliques.

Now if each of the seven vertices not in $\triangle(a,b,c)$ is adjacent to at most one vertex of $\triangle(a,b,c)$, then $|E(G)| \leq 45 - 2 \cdot 7 = 45 - 14 = 31$. Since the smallest clique in $\mathcal{M}$ is a 3-clique, $|\mathcal{M}| \leq 10$ as desired. Otherwise, there exists a vertex d adjacent to exactly two vertices of $\triangle(a,b,c)$. Without loss of generality, say $d \leftrightarrow a, d \leftrightarrow b$ but $d \not\leftrightarrow c$. Since $\triangle(a,b,d)$ shares an edge with $\triangle(a,b,c)$ and $\triangle(a,b,c)$ is in $\mathcal{M}$, it follows that $\triangle(a,b,d)$ cannot be in $\mathcal{M}$. So, there must be two different maximal cliques C_1 and C_2 containing edges ad and bd, respectively.

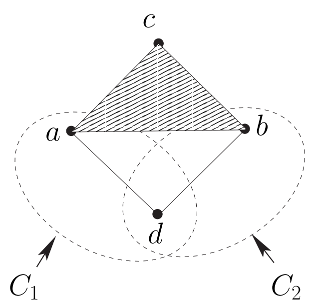


Figure 5.1: $ad \in C_1$ and $bd \in C_2$

Because there is at most one 4-clique in $\mathcal{M}$, then either of C_1, C_2 is the

4-clique or neither of them is.

Case 1 Either C_1 or C_2 is the 4-clique. Without loss of generality, say C_1 is the 4-clique and C_2 is a 3-clique. Let C_1 be $\boxtimes(a, d, e, f)$ and C_2 be $\triangle(d, b, g)$. We note that no two vertices in $\{a, b, c, d, e, f, g\}$ can be the same. Next, consider the subgraph of G consisting of these three cliques in $\mathcal{M}$. It has seven vertices as shown in Figure 5.2. Let $W = \{a, b, c, d, e, f, g\} \subseteq V(G)$. Then we have three more vertices not in W. We will consider the maximum possible degree of each vertex in $V(G)$. Note that there is no vertex contained in all of three subgraphs C_1, C_2 and $\triangle(a, b, c)$. Thus, by Lemma 5.2 no vertex of G can be adjacent to every other vertex of G. Hence, each vertex in $V(G)$ has degree at most 8.

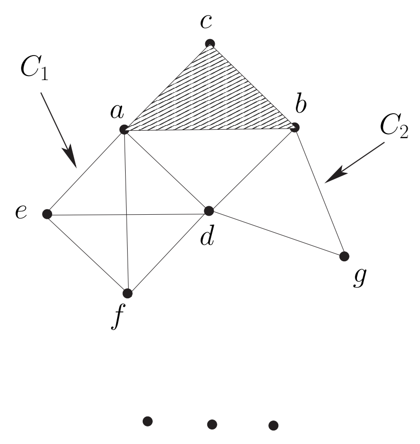


Figure 5.2: C_1 is a 4-clique and C_2 is a triangle.

However, the degrees of some vertices can be even less than 8. Let x be a vertex in C_1. Since C_1 is the 4-clique, the set of edges incident to x is partitioned into one 4-clique and a number of 3-cliques. This forces x to have odd degree. Hence, every vertex in C_1 has degree at most 7.

Next, consider vertices not in W. Let $y \in V(G) \setminus W$. Since $\triangle(a, b, c), C_1$ and C_2 are maximal, y cannot be adjacent to all vertices in any one of them. This forces y to be nonadjacent to at least two vertices in W and $d(y) \leq 7$. Since y is not in the 4-clique in $\mathcal{M}$, the edges incident to y must be covered by 3-cliques. Hence, $d(y)$ is even and $d(y) \leq 6$.

Therefore, the three vertices outside W are of degree at most 6. For vertices in W, four vertices in C_1 are of degree at most 7, and the remaining three vertices are of degree at most 8. Hence, the degree sum of G is at most $3 \cdot 6 + 4 \cdot 7 + 3 \cdot 8 = 18 + 28 + 24 = 70$, yielding $|E(G)| \leq \frac{70}{2} = 35$. But ten 3-cliques and one 4-clique require $3 \cdot 10 + 6 = 36$ edges to be disjoint. This is a contradiction. So, Case 1 cannot occur.

Case 2 Neither C_1 nor C_2 is a 4-clique. Then both C_1 and C_2 are 3-cliques. Let $C_1 = \triangle(a, d, e)$ and $C_2 = \triangle(b, d, f)$. As in Figure 5.3, the subgraph composed of $\triangle(a, b, c), C_1$ and C_2 has six vertices. Let $S = \{a, b, c, d, e, f\} \subseteq V(G)$. Since G has ten vertices, the degree sum of the six vertices in S is at most $9 \cdot 6 = 54$. But since $\triangle(a, b, c), C_1$ and C_2 are maximal, $a \not\leftrightarrow f$, $b \not\leftrightarrow e$ and $c \not\leftrightarrow d$. Then the degree sum of S is at most $54 - 6 = 48$. Moreover, to preserve the maximality of $\triangle(a, b, c), C_1$ and C_2, each of the remaining four vertices must be nonadjacent to at least two vertices in S, this reduces the maximum possible degree sum of S to $48 - 2 \cdot 4 = 40$. Next, we will consider the degrees of vertices outside S. Since $\mathcal{M}$ contains at most one 4-clique, there are two possibilities.

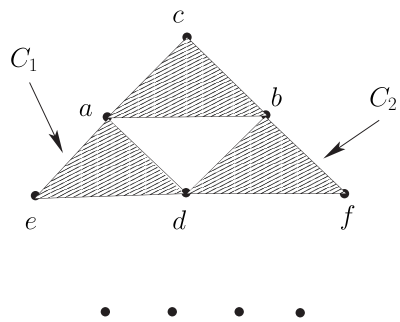


Figure 5.3: Both C_1 and C_2 are triangles

Case 2.1 Suppose there exists a 4-clique in $\mathcal{M}$. Let w be any vertex outside S. Again, to preserve the maximality of $\triangle(a, b, c), C_1$ and C_2, vertex w must be nonadjacent to at least two vertices in S making $d(w) \leq 7$. Counting the degree sum of vertices in S, the degree sum of G is at most $7 \cdot 4 + 40 = 28 + 40 = 68$. Hence, $|E(G)| \leq \frac{68}{2} = 34$. Since ten 3-cliques and one 4-clique with disjoint edges cover 36 edges, this is a contradiction.

Case 2.2 Suppose there is no 4-clique in $\mathcal{M}$. Similar to Case 2.1, vertices not in S have degree at most 7. But because cliques in $\mathcal{M}$ are 3-cliques, each vertex of G must have even degree. Hence, each vertex not in S is of degree at most 6. Hence, the degree sum of G is at most $6 \cdot 4 + 40 = 64$. This yields $|E(G)| \leq \frac{64}{2} = 32$, which is insufficient to obtain 11 copies of the 3-clique, a contradiction.

Thus, there is a contradiction in every possible case, and $|\mathcal{M}| \leq 10$. ■

Note that the conclusion of Theorem 5.1 may not be true if a maximal-clique partition contains a 2-clique. For example, the graph in Figure 5.4 has ten vertices and maximal-clique partition number 13.

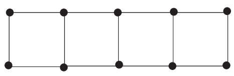
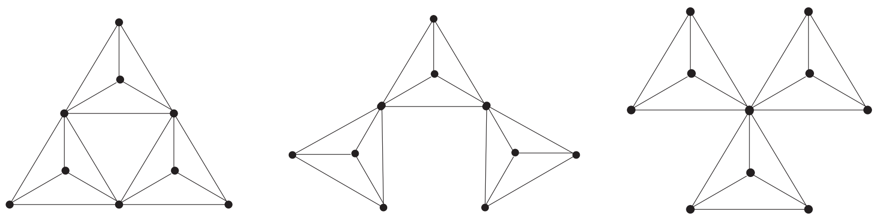


Figure 5.4: A graph of ten vertices with a maximal-clique partition containing a 2-clique.

Remark 5.1 Let G be a graph of ten vertices with a clique partition $\mathcal{P}$. Let H be the subgraph of G composed of any three 4-cliques H_1, H_2 and H_3 in $\mathcal{P}$. Then, H has 18 edges with either nine or ten vertices. There are three possible configurations for H as follows:



Configuration 1 Configuration 2 Configuration 3

Figure 5.5: Three configurations of H

Let I be the subgraph of G composed of any four 4-cliques in $\mathcal{P}$. Then there is only one possible configuration of I as follows:

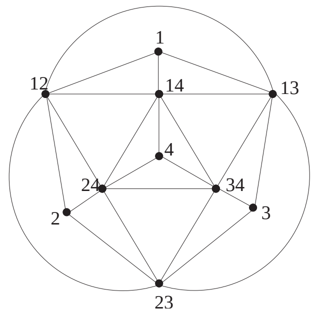


Figure 5.6: The configuration of I

Let J be the subgraph of G composed of any five 5-cliques in $\mathcal{P}$. By Lemma 5.1, there is one possible configuration of J and it is isomorphic to $L(K_5)$.

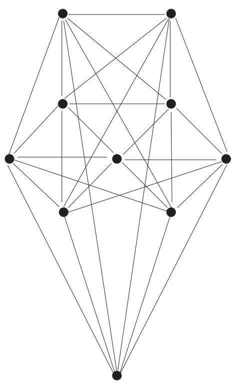


Figure 5.7: The configuration of J

Definition 5.1 Let H be a graph. Then e is a *type-1 edge of* H if every maximal triangle of any supergraph of H containing e does not contain other edges of H.

The next two theorems will show characteristics of maximal-clique partitions of a graph having a clique partition containing at least three 4-cliques. As Configuration 1 of H will play a central role in the proofs, let us investigate some of its properties here.

Definition 5.2 The following terminology is used when G is a graph of ten vertices containing Configuration 1 of H as a subgraph. Recall that H consists of three 4-cliques H_1,H_2 and H_3 in $\mathcal{P}$. Since Configuration 1 of H has nine vertices, let z be the vertex in $G \setminus H$. Let T be the triangle composed of the three edges in the middle of Configuration 1 of H. It is simple to verify that there are three type-1 edges in H as illustrated in Figure 5.8. For the remaining 15 edges in H, let edges outside T be called *type-2 edges* and edges in T be called *type-3 edges*. Thus, the edge set of H is partitioned into three groups: three type-1 edges, 12 type-2 edges and three type-3 edges.

Remark 5.2 Let uv be a type-2 edge in H. Then uv is in a 4-clique in $\{H_1, H_2, H_3\}$, say $u \in H_i$; and one of vertices $\{u, v\}$ is in $H_i \cap T$ and the other one is in $H_i \setminus T$, say $u \in H_i \cap T$, $v \in H_i \setminus T$. Moreover, $u \in H_i \cap H_j$ where $i \neq j \in \{1, 2, 3\}$.

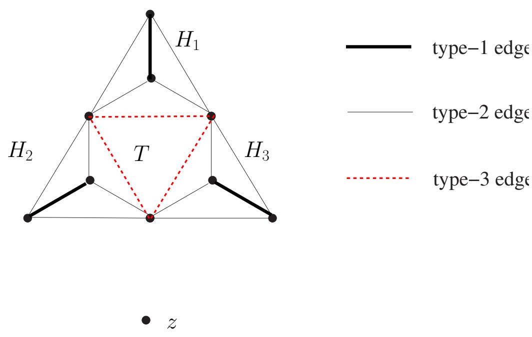


Figure 5.8: The edge set of H is partitioned into three groups

Theorem 5.2 *For any graph with ten vertices, if one of its clique partitions contains at least three 4-cliques, then any of its maximal-clique partitions composed of 2-cliques and 3-cliques has cardinality at least 10.*

Proof Let G be a graph of ten vertices with a maximal-clique partition. Let $\mathcal{P}$ and $\mathcal{M}$ be a clique partition and a maximal-clique partition of G, respectively. Assume $\mathcal{P}$ has at least three 4-cliques and $\mathcal{M}$ is composed of 2-cliques and 3-cliques. Let H be the subgraph of G composed of any three 4-cliques in $\mathcal{P}$. Thus, as in Remark 5.1, H has 18 edges with either nine or ten vertices, and there are three possible configurations for H. Since H is a subgraph of G, we can obtain the lower bound on $|\mathcal{M}|$ by determining the minimum number possible of maximal cliques needed to cover H. We first locate type-1 edges in H. Then we will deal with the remaining edges of H.

Configuration 1 In Figure 5.9, a, b, and c are three type-1 edges of H; so, they need to be covered by three cliques in $\mathcal{M}$. Next, because $\mathcal{M}$ is composed of 3-cliques and 2-cliques, at most three edges of H can share the same clique in

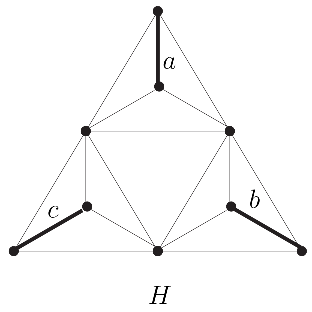


Figure 5.9: Edges a, b, and c are the type-1 edges in Configuration 1 of H

$\mathcal{M}$. Indeed, there is only one way to get three edges sharing the same maximal triangle, which are three edges in T. (Recall from Definition 5.2 that T is the middle triangle in Figure 5.8.) Otherwise, at most two of the edges in H can share the same maximal clique in $\mathcal{M}$.

If T is not in $\mathcal{M}$, then 15 edges of H in addition to a, b, and c need to be covered by at least $\lceil \frac{15}{2} \rceil$=8 maximal cliques. Counting three cliques covering the type-1 edges, $\mathcal{M}$ has at least $3+8 = 11$ cliques. If T is in $\mathcal{M}$, then the remaining 12 edges of H in addition to edges a, b, c or edges in T need to be covered by at least $\frac{12}{2} = 6$ maximal cliques in $\mathcal{M}$. Counting three cliques covering the type-1 edges and the triangle T, there are at least $3+1+6 = 10$ maximal cliques in $\mathcal{M}$.

Configuration 2 In Figure 5.10, the seven edges a, b, c, d, e, f and g are type-1 edges. Hence, they need to be covered by seven maximal cliques in $\mathcal{M}$. Again, for the remaining $18 - 7 = 11$ edges, we have at least $\lceil \frac{11}{2} \rceil = 6$ more cliques in $\mathcal{M}$. Hence, $|\mathcal{M}| \geq 7 + 6 = 13$.

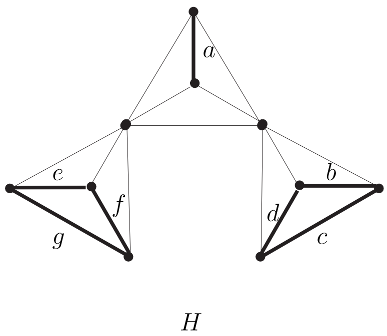


Figure 5.10: The seven type-1 edges in Configuration 2 of H

Configuration 3 As before, we need nine maximal cliques in $\mathcal{M}$ to cover the nine type-1 edges in Figure 5.11. And again to cover the remaining $18-9=9$ edges, $\mathcal{M}$ must contain at least $\lceil \frac{9}{2} \rceil = 5$ more cliques. Hence, $|\mathcal{M}| \geq 9+5=14$.

Hence, for every configuration of H, we have shown that $\mathcal{M}$ contains at least ten cliques. ■

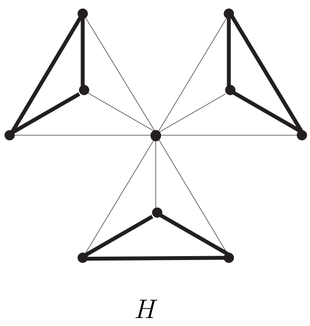


Figure 5.11: Nine type-1 edges in Configuration 3 of H

The next lemma continues to examine graphs of ten vertices containing H as a subgraph. We determine the possible maximal triangles containing edges of each type in Configuration 1 of H.

Lemma 5.3 *Let G be a graph of ten vertices containing Configuration 1 of H as a subgraph. (Configuration 1 of H is defined in Remark 5.1 and the triangle T and vertex z are defined in Definition 5.2.) Let uv be an edge of H. Let S be a maximal triangle of G containing uv. Then we have the following:*
(i) If uv is a type-1 edge, then $S = \triangle(u, v, z)$.
(ii) If uv is a type-2 edge where we can assume as in Remark 5.2 that $u \in H_i \cap H_j$ and $v \in H_i \setminus T$, then $S = \triangle(u, v, z)$ or $S = \triangle(u, v, w)$ where w is a vertex in $H_j \setminus T$.
(iii) If uv is a type-3 edge, then $S = T$ or $S = \triangle(u, v, z)$.

Proof (i) Assume uv is a type-1 edge in H. Let w be a vertex such that $\triangle(u, v, w)$ is maximal in G. Let uv be in H_i for some $i = 1, 2, 3$. (Recall that H_1, H_2 and H_3 are 4-cliques in H.) Then $w \notin H_i$, otherwise $\triangle(u, v, w) \subseteq H_i$, a contradiction. Suppose $w \in H_j, j \neq i \in \{1, 2, 3\}$. Let $c \in V(H_i) \cap V(H_j)$. Then c is adjacent to all vertices in H_i and H_j. Hence, $\triangle(u, v, w) \subseteq \boxtimes(u, v, w, c)$, a contradiction. Therefore, $w = z$.

(ii) Assume uv is a type-2 edge where $u \in H_i \cap H_j$ and $v \in H_i \setminus T$. Let w be a vertex such that $\triangle(u, v, w)$ is maximal in G. Then $w \notin H_i$; otherwise, $S \subseteq H_i$, a contradiction. Let H_k be the other 4-clique among the three 4-cliques, i.e., $k \in \{1, 2, 3\}, k \neq i, j$. Let c be the vertex shared by H_i and H_k. Then $w \notin H_k$, otherwise, $S \subseteq \boxtimes(u, v, w, c)$, a contradiction. Hence, w must be a vertex outside $H_i \cup H_k$. Indeed, there are three possible choices for w: z and two vertices in $H_j \setminus T$.

(iii) Assume uv is a type-3 edge. Then, uv is in T. Let w be a vertex such that $\triangle(u, v, w)$ is maximal in G. Let c be the third vertex in T. Suppose $w \neq c$ or z. Let uv be in H_i. Then $w \notin H_i$, otherwise $\triangle(u, v, w) \subseteq H_i$, contradicting the maximality of $\triangle(u, v, w)$. And $w \notin H_j, j \neq i$, otherwise, $\triangle(u, v, w) \subseteq \boxtimes(u, v, w, c)$, a contradiction. Hence, w can only be c or z. ■

Theorem 5.3 *For any graph with ten vertices, if one of its clique partitions contains at least three 4-cliques and one of its maximal-clique partitions is composed of ten triangles, then* $G = L(K_5)$.

Proof Let G be a graph of ten vertices with a maximal-clique partition. Let $\mathcal{P}$ and $\mathcal{M}$ be a clique partition and a maximal-clique partition of G, respectively. Assume $\mathcal{P}$ has at least three 4-cliques and $\mathcal{M}$ is composed of ten 3-cliques. Let H be the subgraph of G composed of any three 4-cliques in $\mathcal{P}$. As in Remark 5.1, there are three possible configurations for H. As we observed in the proof of Theorem 5.2, if H has Configuration 2 or 3, then the cardinality of $\mathcal{M}$ is more than ten. Since $|\mathcal{M}| = 10$, H must be Configuration 1. Then recall from Definition 5.2 that z is the vertex of G not in H. Now, since $\mathcal{M}$ is composed of triangles, by Lemma 5.3(i), there is a unique maximal triangle in G containing each type-1 edge in H. Thus, each maximal triangle in $\mathcal{M}$ containing a type-1 edge of H contains z. It follows that each maximal triangle in G containing a type-2 edge cannot contain z because each type-2 shares an endpoint with a type-1 edge. Hence, by Lemma 5.3(ii), there are two possible maximal triangles that can contain each type-2 edge. However, both possibilities yield the same graph $L(K_5)$. Therefore, $G = L(K_5)$. ■

Lemma 5.4 (Properties of a maximal 4-clique of G)
Let G be a graph of ten vertices containing Configuration 1 of H (see Figure 5.8) as a subgraph. Let X be a maximal 4-clique of G, then
(i) X contains at most one type-1 edge of H.
(ii) If X is in a maximal-clique partition $\mathcal{M}$ of G, then X cannot contain exactly three vertices of H_i, where $i = 1, 2,$ or 3.

Proof (i) Recall that the three type-1 edges of H are illustrated in Figure 5.8. Suppose X contains two type-1 edges. Because any two 4-cliques of H containing these two type-1 edges share a common vertex, such a common vertex is adjacent to every vertex in X. Thus, there is a 5-clique containing X. This contradicts the maximality of X.

(ii) Assume X is in a maximal-clique partition $\mathcal{M}$ of G. Suppose X contains three vertices of H_i where $i = 1, 2$, or 3. Without loss of generality, say X contains three vertices of H_1. Then X contains at least one vertex of $H_1 \cap T$, say vertex v. There are two type-2 edges vp and vq in H_1 incident to v. Since X contains three vertices of H_1, we have that X contains either one or both of the type-2 edges of H_1 incident to v, i.e., at most one of them is not contained in X.

Now, consider another 4-clique in H that shares v with H_1, say H_2. Consider the two type-2 edges vr and vs in H_2 incident to v. Let M_1 and M_2 be two maximal cliques in $\mathcal{M}$ containing vr and vs, respectively. By Lemma 5.3(ii), M_1 is a triangle in $\{\triangle(v,r,z), \triangle(v,r,p), \triangle(v,r,q)\}$ and M_2 is a triangle in $\{\triangle(v,s,z), \triangle(v,s,p), \triangle(v,s,q)\}$. However, vp is in X; so, $M_1 \neq \triangle(v,r,p)$ and $M_2 \neq \triangle(v,s,p)$. Moreover, the edge vq cannot belong to both M_1 and M_2. Hence, one of them must contain vertex z; without loss of generality, $M_1 = \triangle(v,r,z)$. However, rs is a type-1 edge and by Lemma 5.3(i), the maximal triangle M_3 in $\mathcal{M}$ containing rs is $\triangle(r,s,z)$. This yields edge rz in both M_1 and M_3 contradicting that M_1 and M_3 are in a clique partition of G. Therefore, X cannot contain exactly three vertices of H_1. Since H_1, H_2 and H_3 are symmetric, the lemma has been proved. ■

Theorem 5.4 *For any graph of ten vertices, if one of its clique partitions contains at least three 4-cliques, then none of its maximal-clique partitions can be composed of 3-cliques and exactly one 4-clique.*

Proof Let G be a graph of ten vertices with a maximal-clique partition. Let $\mathcal{P}$ and $\mathcal{M}$ be a clique partition and a maximal-clique partition of G, respectively. Assume $\mathcal{P}$ contains at least three 4-cliques. Let H_1, H_2, and H_3 be any three 4-cliques in $\mathcal{P}$. Let H be the subset of G composed of H_1, H_2 and H_3. As Theorem 5.2, we will examine separately the three configurations of H.

Suppose $\mathcal{M}$ is composed of 3-cliques and exactly one 4-clique. Let X be the

4-clique in $\mathcal{M}$. We will show a contradiction for each configuration of H.

Configuration 1 Recall from Definition 5.2 that T is the middle triangle in Figure 5.8 and z is the vertex in $G \setminus H$. Then either T is in $\mathcal{M}$ or T is not in $\mathcal{M}$.

Case 1 Suppose $T \notin \mathcal{M}$. Then either $T \subseteq X$ or each edge of T is in its own maximal clique. If the latter case, since $\mathcal{M}$ is composed of triangles and X, at least two of the maximal cliques containing the three edges of T are triangles. By Lemma 5.3 (*iii*), both triangles contain z; so, they share an edge, a contradiction. Hence, $T \subseteq X$. Now, let w be the vertex of $X \setminus T$. Then w cannot be in H_i where $i = 1, 2$, or 3; otherwise, there are three vertices of X in H_i ($i = 1, 2$ or 3), contradicting Lemma 5.4 (*ii*). Hence, $w = z$. Then z is adjacent to every vertex in T. Because every vertex in H outside T is a type-1 edge, by Lemma 5.3 (*i*), z is adjacent to every vertex outside T. Hence, z joins all vertices in H. By Lemma 5.2, z is contained in every triangle in $\mathcal{M}$. This yields that every edge outside X must be in a triangle containing z. Then the maximal triangles of any two edges outside X that share a vertex will share an edge, a contradiction.

Case 2 Suppose $T \in \mathcal{M}$. Then either $z \in X$ or $z \notin X$.

Case 2.1 Assume $z \in X$. *First, we claim that both endpoints of a type-1 edge are either in X or neither endpoint is in X.* Suppose not. Let uv be a type-1 edge where $u \in X$ but $v \notin X$. Then edge uv is not contained in X. By Lemma 5.3 (*i*), the maximal triangle in $\mathcal{M}$ containing uv is $\triangle(u, v, z)$. Because $z \in X$

and $u \in X$, the edge uz is contained in X. Hence, edge uz is in both X and $\triangle(u, v, z)$ which are disjoint cliques in a clique partition of G, a contradiction. Hence, the claim is proved.

Next, if none of the type-1 edges is in X, by the claim above, the six vertices in type-1 edges in H cannot be in X; so, X needs to contain all three vertices of T, which contradicts the maximality of T. Now, by Lemma 5.4 (i), at most one type-1 edge can be in X. Hence, exactly one type-1 edge is in X, say the type-1 edge is in H_1. So, X contains z and two vertices in the type-1 edge in H_1. Let y be the other vertex of X. Then $y \notin H_1$; otherwise, three vertices of H_1 are in X, contradicting Lemma 5.4 (ii). Besides, $y \notin T \setminus H_1$; otherwise, T is not maximal, a contradiction. However, if y is one of the remaining vertices, y is in a type-1 edge e, while the other vertex of e is not in X, which contradicts the claim. Hence, it is not possible to find y. Therefore, Case 2.1 cannot occur.

Case 2.2 Assume $z \notin X$. Suppose one vertex s of T is in X. Without loss of generality, say $s \in H_1 \cap H_2$. Since T is maximal, the 4-clique X in $\mathcal{M}$ cannot contain any vertices in H_3. Hence, three more vertices of X must come from H_1 and H_2. In any case, it yields three vertices in a clique in H_1 or H_2, contradicting Lemma 5.4 (ii). Hence, X cannot contain any vertices in T. Thus, all four vertices of X come from the six vertices of $V(G) \setminus (V(T) \cup \{z\})$. This set of six vertices is partitioned into three groups according to their membership in the three type-1 edges. Thus, by the Pigeon Hole Principle, consider the four vertices of X as the four pigeons and the three type-1 edges as the three pigeon holes, we have that at least two vertices of X are in the same type-1 edge, i.e., at least one type-1 edge is in X.

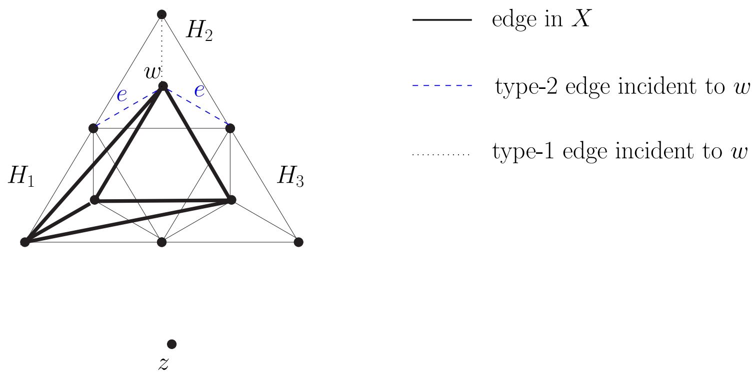


Figure 5.12: The possible X when $T \in \mathcal{M}$ and $z \notin X$

However, by Lemma 5.4 (i), at most one type-1 edge can be in X; hence, exactly one type-1 edge is in X, without loss of generality, say the type-1 edge is in H_1. Furthermore, for the other two vertices of X, one of them must be in the type-1 edge in H_2 and the other must be in the type-1 edge in H_3. Let w be the vertex of X in the type-1 edge in H_2. Consider a type-2 edge e that is incident to w in Figure 5.12.

Examining Figure 5.12 it is easy to see that the maximal triangle containing e must contain z because the other possibilities in Lemma 5.3(ii) do not work. However, the maximal triangle of the type-1 edge incident to w also contains z. Hence, there exist two maximal triangles in $\mathcal{M}$ sharing edge zw, a contradiction.

Therefore, when G contains Configuration 1, we cannot have a maximal-clique partition composed of 3-cliques and exactly one 4-clique. Next we examine G when H has Configuration 2.

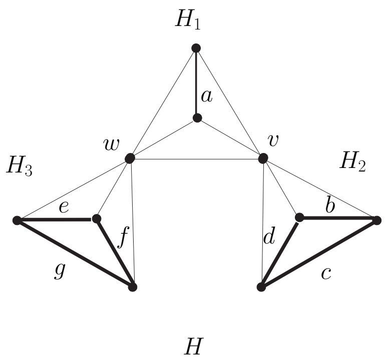


Figure 5.13: Seven type-1 edges in Configuration 2 of H

Configuration 2 There are seven type-1 edges a, b, c, d, e, f and g in Configuration 2 of H as indicated in Figure 5.13. Consider edge a. Let S be a maximal triangle of G containing a, and let u be the vertex in S that is not an endpoint of a. Then $u \notin H_1$; otherwise, S is not maximal, a contradiction. Thus, $u \neq v$ or w where v and w are vertices as indicated in Figure 5.13. Also, $u \notin H_2$; otherwise, $S \vee \{v\}$ (the join graph of S and $\{v\}$) is a 4-clique containing S, contradicting the maximality of S. Similarly, $u \notin H_3$. Hence, a cannot be in any maximal triangle of G. This means a belongs to the 4-clique X. Then b, c and d cannot be in X; otherwise, $X \vee \{v\}$ (the join graph of X and $\{v\}$) is a 5-clique containing X, contradicting its maximality. Similarly, none of e, f and g are in X. Moreover, because e,f and g are type-1 edges, they are all in distinct maximal triangles in $\mathcal{M}$. However, because v is adjacent to all vertices in H_1 and H_2 as in Figure 5.13, edges b, c and d cannot be in a maximal triangle containing any vertices in H_1. Similarly, w prevents e, f and g from being in the same maximal triangle with any vertices in H_1. Hence, maximal triangles containing b, c, d, e, f and g contain edges between endpoints of $\{b, c, d\}$ and $\{e, f, g\}$. However, we need 12 more edges besides b, c, d, e, f and g to be in the six maximal triangles while only nine edges between vertices of $\{b, c, d\}$ and $\{e, f, g\}$ are available, this is a contradiction. Therefore, if G contains Configuration 2, then G cannot have a maximal-clique partition composed of 3-cliques and exactly one 4-clique.

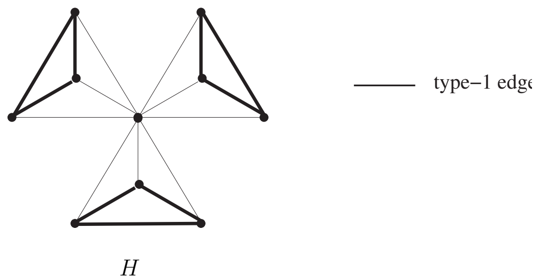


Figure 5.14: Nine type-1 edges in Configuration 3 of H

Configuration 3 There are nine type-1 edges in Configuration 3 of H as indicated by the thick edges in Figure 5.14. Since the middle vertex of H in Figure 5.14 is incident to all other vertices, no type-1 edge of Configuration 3 of H can be in any maximal triangle of G. Hence, all of them need to be in X. However, there are nine type-1 edges while X contains six edges, a contradiction. Therefore, if G contains Configuration 3, then G cannot have a maximal-clique partition composed of 3-cliques and exactly one 4-clique.

We encounter a contradiction for all possible cases of every configuration of

H. Therefore, no graph of ten vertices with a clique partition composed of at least three 4-cliques can have a maximal-clique partition composed of 3-cliques and exactly one 4-clique. ■

Theorem 5.5 *For any graph of ten vertices, if one of its clique partitions contains at least four 4-cliques, then none of its maximal-clique partitions can be composed of 3-cliques and exactly one 5-clique.*

Proof Let G be a graph of ten vertices with a maximal-clique partition $\mathcal{M}$ and a clique partition $\mathcal{P}$. Assume $\mathcal{P}$ contains at least four 4-cliques. Let H be the subgraph of G composed of any four 4-cliques in $\mathcal{P}$, called H_1, H_2, H_3 and H_4. There is only one possible configuration as illustrated in Figure 5.15. Since H has ten vertices, $V(G) = V(H)$. Label vertex v in H with all subscripts of cliques in $\{H_1, H_2, H_3, H_4\}$ containing v. For example, if v is in cliques H_1 and H_2, then v is labelled as 12. Hence, $V(G) = V(H) = \{1, 2, 3, 4, 12, 13, 14, 23, 24, 34\}$. Note that for any two vertices v and w in $V(G)$, we have that v is adjacent to w in H if and only if their labels share a digit.

Now let $\mathcal{S} = \{\triangle(12, 14, 24), \triangle(13, 14, 34), \triangle(23, 24, 34), \triangle(12, 13, 23)\}$ be a set of four triangles in H. Note that each triangle in $\mathcal{S}$ is maximal in H.

Let e be any edge of a triangle in $\mathcal{S}$, say $e = (xy, yz)$, where $x, y, z \in \{1, 2, 3, 4\}$, $x \neq y \neq z$. Let T be a maximal triangle of G containing e. Let u be the third vertex of any maximal triangle T of G containing e, i.e., $T = \triangle(xy, yz, u)$. Note that $\triangle(xy, yz, xz)$ (in $\mathcal{S}$) contains e. Suppose that $u \neq xz$. If $u \in H_y$, then $\triangle(xy, yz, u) \subseteq H_y$, contradicting the maximality of T. So, $u \notin H_y$.

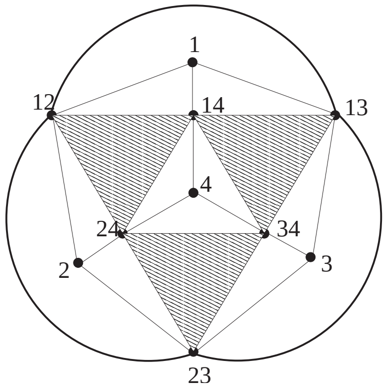


Figure 5.15: The subset H of G composed of four 4-cliques in $\mathcal{P}$

If $u \in H_x$ or H_z, then $u \leftrightarrow xz$. Hence, $\boxtimes(xz, xy, yz, u)$ is a 4-clique containing T. Thus, $u \notin H_x$ and $u \notin H_z$. If $u \in H_w$ where $w \in \{1, 2, 3, 4\}, w \neq x, y, z$, then $u \leftrightarrow yw$. Since $yw \in H_y$, $yw \leftrightarrow yx$ and $yw \leftrightarrow yz$. Hence, yw is adjacent to all vertices in $\triangle(yx, yz, u)$. Thus, $T \subseteq \boxtimes(yw, yx, yz, u)$, a contradiction. Hence, $u = xz$ and $T = \triangle(xy, yz, xz)$ in $\mathcal{S}$ is the unique maximal triangle in G containing e.

Now, by way of contradiction suppose $\mathcal{M}$ is composed of 3-cliques and one 5-clique. Let X be the 5-clique in $\mathcal{M}$. If $T \notin \mathcal{M}$, then $e \in X$ because there is no other maximal triangle containing e. It follows that all of the other edges in the triangle T are also in X, i.e., $T \subseteq X$. Hence, either $T \in \mathcal{M}$ or $T \subseteq X$. Since e is arbitrary, each triangle in $\mathcal{S}$ is either in $\mathcal{M}$ or a subset of X, i.e., the whole triangle is either inside X or outside. On the other hand, it is straight forward to verify that if X contains two triangles in $\mathcal{S}$, then there would exist a triangle in $\mathcal{S}$ having some edges in X and some edges outside X. Since we showed this cannot happen, at most one triangle in $\mathcal{S}$ can be a subset of X and the rest must be in $\mathcal{M}$. Then there are two possible cases as follows:

Case 1 Assume every triangle in $\mathcal{S}$ is in $\mathcal{M}$. Consider vertex 12 in Figure 5.15. It is incident to six edges in H where four of the edges incident to 12 are covered by two triangles in $\mathcal{S}$. Only the remaining two of them can be in X. Moreover, to preserve the maximality of triangles in $\mathcal{S}$, we have $12 \not\leftrightarrow 3, 12 \not\leftrightarrow 4$, and $12 \not\leftrightarrow 34$. Hence, there are no more edges in $G \setminus H$ incident to 12. Thus, 12 has only two incident edges that could be in X. However, X is a 5-clique; so, vertices in X have degree four. Thus, 12 cannot be a vertex in X. Similarly, neither 13, 14, 23, 24, or 34 can be vertices in X. Therefore, there are at most four vertices that can possibly be in X. Since X contains five vertices, this is a contradiction.

Case 2 Assume exactly one triangle in $\mathcal{S}$ is contained in X. Because of symmetry, assume without loss of generality that $\triangle(12, 23, 13) \subseteq X$. Then $\triangle(13, 14, 34)$, $\triangle(12, 14, 24)$ and $\triangle(23, 24, 34)$ are in $\mathcal{M}$. Moreover, it follows that $14, 24, 34 \notin V(X)$; otherwise, X shares an edge with a triangle in $\mathcal{S} \setminus \{\triangle(12, 23, 13)\}$.

Next, consider vertex 12 in X. To preserve the maximality of $\triangle(12, 14, 24)$, we have $12 \not\leftrightarrow 34$ and $12 \not\leftrightarrow 4$. Thus, 12 has either degree six or seven in G depending on whether or not vertex 12 is adjacent to vertex 3. If $12 \leftrightarrow 3$, then 12 has degree seven where four of them are covered by X and two of them are covered by a triangle in $\mathcal{S}$. Hence, there is only one edge incident to 12 left and it must be in $\mathcal{M}$. However, this 2-clique is not maximal, a contradiction. Therefore, $12 \not\leftrightarrow 3$. Now, since $12 \not\leftrightarrow 3$, $12 \not\leftrightarrow 4$ and $12 \in V(X)$, we have $3, 4 \notin V(X)$; and it is shown above that $14, 24, 34 \notin V(X)$; therefore, $V(X) = \{12, 13, 23, 1, 2\}$. Next, consider vertex 13. It has degree six in H, and $13 \leftrightarrow 2$ in G because they are vertices in X. However, $13 \not\leftrightarrow 4$ and $13 \not\leftrightarrow 24$; otherwise this contradicts the maximality of $\triangle(13, 14, 34)$. Therefore, 13 has degree seven in G. Again, four edges incident to 13 in G are in X and two more edges are in a triangle in $\mathcal{S}$. Hence, there is one edge incident to 13 in G which must be in $\mathcal{M}$. But, it is not maximal, a contradiction.

Therefore, $\mathcal{M}$ cannot be composed of one 5-clique and a number of triangles.

■

5.2 The Nonexistence of Graphs having Ten Vertices and $cp < mcp$

Now we further explore graphs of ten vertices under the restriction that $cp < mcp$. Our objective is to confirm that there is no such graph. We first suppose that it exists, then hope to encounter a contradiction. We start by inspecting the numbers of k-cliques in a clique partition and a maximal-clique partition of a desired graph. At the end, it will be combined with the results in earlier sections to yield the conclusion that any graph of ten vertices with $cp < mcp$ must be $L(K_5)$, which is a contradiction because $cp(L(K_5)) = mcp(L(K_5))$. Details will be shown in this section.

Throughout this section, let G be a graph having the minimum number of edges among graphs of ten vertices with a maximal-clique partition satisfying

$cp(G) < mcp(G)$. Let $\mathcal{P}$ be any clique partition of G with $|\mathcal{P}| = cp(G)$ and $\mathcal{M}$ be any maximal-clique partition of G with $|\mathcal{M}| = mcp(G)$.

If $\mathcal{P} \cap \mathcal{M} \neq \emptyset$, remove the common cliques from G. This preserves the property $cp < mcp$ of the resulting graph. By Monson's Ph.D. Dissertation [22], all graphs of fewer than ten vertices have $cp = mcp$; thus, removing common cliques from G cannot reduce the number of vertices. That is, the resulting graph still has ten vertices and satisfies $cp < mcp$. Since the resulting graph has fewer edges than G, this contradicts the minimality of G. Hence, $\mathcal{P} \cap \mathcal{M} = \emptyset$. Also note that $G \neq K_{10}$ because $cp(K_{10}) = mcp(K_{10}) = 1$. So, the maximum order of cliques in G is 9.

5.2.1 Definitions and Values of p_i and m_i

For each $i = 2, \ldots, 9$, let $\mathcal{P}_i$ and $\mathcal{M}_i$ be the set of all cliques of order i in $\mathcal{P}$ and $\mathcal{M}$, respectively. Then $\mathcal{P} = \bigcup_{i=2}^{9} \mathcal{P}_i$ and $\mathcal{M} = \bigcup_{i=2}^{9} \mathcal{M}_i$. Because both $\mathcal{P}$ and $\mathcal{M}$ partition the edge set in G, the total number of edges in $\mathcal{P}$ and in $\mathcal{M}$ must be the same. That is, $\sum_{i=2}^{9} \binom{i}{2} |\mathcal{P}_i| = \sum_{i=2}^{9} \binom{i}{2} |\mathcal{M}_i|$. Moreover, this equation can be simplified by counting the difference of the cardinalities of P_i and M_i $(2 \leq i \leq 9)$.

Let $p_i = max\{|\mathcal{P}_i| - |\mathcal{M}_i|, 0\}$ and $m_i = max\{|\mathcal{M}_i| - |\mathcal{P}_i|, 0\}$, for $2 \leq i \leq 9$. Lemma 5.5 collects trivial properties directly from the definition of p_i and m_i:

Lemma 5.5 *(i) For each integer* $2 \leq i \leq 9$, $p_i \geq 0$ *and* $m_i \geq 0$

(ii) $\sum_{i=2}^{9} \frac{i(i-1)}{2} p_i = \sum_{i=2}^{9} \frac{i(i-1)}{2} m_i$.

(iii) If $p_i > 0$ *then* $m_i = 0$ *and if* $m_i > 0$ *then* $p_i = 0$.

(iv) If $|\mathcal{P}_i| \leq |\mathcal{M}_i|$, then $p_i = 0$. Similarly, if $|\mathcal{M}_i| \leq |\mathcal{P}_i|$ then $m_i = 0$.

Lemma 5.6 (The values of p_i and m_i)

Let G be a graph of ten vertices with $cp(G) < mcp(G)$. Let $\mathcal{P}$ and $\mathcal{M}$ be a minimum clique partition and a minimum maximal-clique partition of G, respectively, (such that $\mathcal{P} \cap \mathcal{M} = \emptyset$). Let p_i and m_i be defined as above, then

(i) $m_2 = 0$ and there is no 2-clique in $\mathcal{M}$.

(ii) $\forall i \geq 6$, $p_i = 0$ and each maximal clique of order at least 6 must be in $\mathcal{M}$.

(iii) $\forall i \geq 8$, $m_i = 0$.

(iv) $p_5 = 0$ and there is a one to one relation from $\mathcal{P}_5$ into $\mathcal{M}_5$.

(v) $p_3 = 0$ and $m_3 > 0$.

(vi) $p_4 > 0$ and $m_4 = 0$.

(vii) $m_7 = 0$.

(viii) $m_5 \equiv p_2 (mod\ 3)$.

(ix) If $p_4 \geq 5$, then $G = L(K_5)$.

(x) $m_6 = 0$.

(xi) $m_5 = 0$.

(xii) So, the only nonzero parameters possible are p_2, p_4 and m_3.

Proof (i) If a 2-clique is in $\mathcal{M}$, it is a maximal clique. Then it is not contained in any other cliques. Thus, it must be in $\mathcal{P}$. This contradicts $\mathcal{P} \cap \mathcal{M} = \emptyset$. Hence, *there is no 2-clique in $\mathcal{M}$*. Therefore, $m_2 = 0$.

(ii) Let C be a maximal clique of order i in G where i is at least 6. Suppose

$C \notin \mathcal{M}$. By Theorem 3.2, there are at least six vertices of G not in C. Thus, G has at least $i + 6 \geq 12 > 10$ vertices, a contradiction. Therefore, *all cliques of order at least 6 are in* $\mathcal{M}$. It follows that for $i \geq 6$, $|\mathcal{P}_i| \leq |\mathcal{M}_i|$. By Lemma 5.5 (iv), $\forall i \geq 6, p_i = 0$.

(iii) Suppose there exists a clique of order at least 8 in $\mathcal{M}$. If there is another clique in $\mathcal{M}$, by (i), it has at least three vertices. Because two cliques in $\mathcal{M}$ can share at most one vertex and G has ten vertices, $\mathcal{M}$ can contain at most two cliques. Thus, $\mathcal{M}$ is also a minimum clique partition of G. Hence, $cp(G) = mcp(G)$, a contradiction.

(iv) Let C be any clique in $\mathcal{P}_5 \setminus \mathcal{M}_5$. Since C is a 5-clique, G has five vertices not in C. By Theorem 3.2, C needs to be covered by five maximal cliques in $\mathcal{M}$, say $M_1, \ldots, M_5$. By Corollary 3.1, $\{C \cap M_i : i = 1, \ldots, 5\}$ is composed of one 4-clique and four 2-cliques. By Theorem 3.2 (i), each M_i contains at least one vertex outside the other cliques $M_j, 1 \leq j \leq 5, i \neq j$. Since G has ten vertices, for each $i = 1, \ldots, 5$, M_i contains exactly one vertex outside C. Hence, $\{M_1, \ldots, M_5\}$ is composed of one 5-clique and four 3-cliques. Therefore, a 5-clique C in $\mathcal{P}$ shares a 4-clique with a 5-clique in $\mathcal{M}$. Moreover, two different 5-cliques in $\mathcal{P}$ cannot share a 4-clique with the same 5-clique in $\mathcal{M}$ and two different 5-cliques in $\mathcal{M}$ cannot share a 4-clique with the same 5-clique in $\mathcal{P}$. Hence, we have a one to one relation from $\mathcal{P}_5$ into $\mathcal{M}_5$. Hence, $|\mathcal{P}_5| \leq |\mathcal{M}_5|$. By Lemma 5.5 (iv), $p_5 = 0$.

(v) By $(i) - (iv)$ and Lemma 5.5 (ii),

$$p_2 + 3p_3 + 6p_4 = 3m_3 + 6m_4 + 10m_5 + 15m_6 + 21m_7. \tag{5.1}$$

Suppose $m_3 = 0$. Since $cp(G) < mcp(G)$, $p_2 + p_3 + p_4 < m_4 + m_5 + m_6 + m_7$. Hence, $1p_2 + 3p_3 + 6p_4 \leq 6p_2 + 6p_3 + 6p_4 < 6m_4 + 6m_5 + 6m_6 + 6m_7 \leq 6m_4 + 10m_5 + 15m_6 + 21m_7$. Thus, $1p_2 + 3p_3 + 6p_4 < 6m_4 + 10m_5 + 15m_6 + 21m_7$, contradicting (5.1). Therefore, $m_3 = 0$, and by Lemma 5.5 (iii), $p_3 = 0$.

(vi) By $(i) - (v)$ and Lemma 5.5(ii),

$$p_2 + 6p_4 = 3m_3 + 6m_4 + 10m_5 + 15m_6 + 21m_7. \tag{5.2}$$

Suppose $p_4 = 0$. Since $cp(G) < mcp(G)$, $p_2 < m_3 + m_4 + m_5 + m_6 + m_7 < 3m_3 + 6m_4 + 10m_5 + 15m_6 + 21m_7$. This contradicts (5.2). Therefore, $p_4 > 0$, and by Lemma 5.5 (iii), $m_4 = 0$.

(vii) Since G has ten vertices, $\mathcal{M}$ cannot contain two 7-cliques. Then $m_7 = 0$ or 1. Suppose $m_7 = 1$, i.e., $\mathcal{M}$ has a 7-clique. Then G does not have enough vertices to also have a 5-clique or 6-clique in $\mathcal{M}$. So, $m_5 = 0$ and $m_6 = 0$. Together with $(i) - (vi)$ and by Lemma 5.5 (ii), we have $p_2 + 6p_4 = 3m_3 + 21m_7$. Since $m_7 = 1$, $p_2 + 6p_4 = 3m_3 + 21$. Then p_2 is divisible by 3 because all other terms are divisible by 3. Let $p_2 = 3k$, where k is a non-negative integer. Then, $3k + 6p_4 = 3m_3 + 21$, which yields $k + 2p_4 = m_3 + 7$; so that $m_3 = k + 2p_4 - 7$. Because $cp(G) < mcp(G)$, we have $p_2 + p_4 < m_3 + m_7$, i.e., $p_2 + p_4 < m_3 + 1$.

Next, since $m_3 = k + 2p_4 - 7$, we have $p_2 + p_4 < k + 2p_4 - 7 + 1$. Hence, $p_4 > 2k + 6$. Now since $k \geq 0$, $p_4 > 6$, which contradicts $p_4 \leq 5$(by Lemma 5.1). Hence, $m_7 = 0$.

$(viii)$ By $(i) - (vii)$ and Lemma 5.5(ii), $p_2 + 6p_4 = 3m_3 + 10m_5 + 15m_6$. Then $3(2p_4 - m_3 - 5m_6) = 10m_5 - p_2$. Hence, 3 divides $(10m_5 - p_2)$, i.e., $10m_5 \equiv p_2 (mod\ 3)$ or $m_5 \equiv p_2(mod\ 3)$.

(ix) Assume $p_4 \geq 5$. Since $p_4 \leq 5$ (by Lemma 5.1) and $m_4 = 0$ (by (vi)), $\mathcal{P}$ contains five 4-cliques and $\mathcal{M}$ does not contain 4-cliques. Consider the subgraph J composed of the five 4-cliques in $\mathcal{P}$ of G. As in Remark 5.1, J is $L(K_5)$. Let v be any vertex in J. We examine edges in $G \setminus J$ incident to v. Since $J = L(K_5)$, v is nonadjacent to three vertices in J. But any two vertices among those three vertices are adjacent in J; so, any edges in $G \setminus J$ incident to v are not contained in a larger clique in $\mathcal{P}$. Since $L(K_5)$ is vertex-transitive, we can conclude that $\mathcal{P}$ is composed of 2-cliques and exactly five 4-cliques. Now since $|\mathcal{P}| < |\mathcal{M}|$ and $|\mathcal{M}| \leq 10$ (by Theorem 5.1), we have $|\mathcal{P}| \leq 9$. It follows that $\mathcal{P}$ contains at most four 2-cliques which are edges in $G \setminus J$. Since $m_2 = 0$ (by (i)), p_2 is the number of 2-cliques in $\mathcal{P}$. By $(viii)$, $m_5 \equiv p_2(mod\ 3)$ and note that $m_5 \leq 2$ because G has ten vertices. If $p_2 = 3$, then it follows that $|\mathcal{P}| = 8$, G has 33 edges, and $m_5 = 0$. By $(i) - (viii)$, $\mathcal{M}$ can contain 3-cliques and 6-cliques. Since G has ten vertices, $\mathcal{M}$ contains at most one 6-clique. If there is a 6-clique in $\mathcal{M}$, since G has 33 edges, $\mathcal{M}$ is composed of one 6-clique and six triangles. This yields $|\mathcal{M}| = 7 < |\mathcal{P}|$, a contradiction. Otherwise, there is no 6-clique in $\mathcal{M}$. That means $\mathcal{M}$ is composed of 11 triangles, contradicting Theorem 5.1. Hence $p_2 \neq 3$.

Similarly, if $p_2 = 4$, then G has 34 edges. Since $m_5 \equiv p_2 (mod\ 3)$ and $m_5 \leq 2$, we have $m_5 = 1$. If there exists one 6-clique in $\mathcal{M}$, then $\mathcal{M}$ is composed of one 5-clique, one 6-clique and three triangles. This means $|\mathcal{M}| = 5 < 9 = |\mathcal{P}|$, a contradiction. Otherwise, there is no 6-clique in $\mathcal{M}$. Hence $\mathcal{M}$ is composed of one 5-clique and eight triangles which yields $|\mathcal{M}| = 9 = |\mathcal{P}|$, a contradiction. Hence, $p_2 \neq 4$.

If $p_2 = 2$, then $m_5 = 2$ and G has two edges outside J. Note that adding any two edges in $L(K_5)$ cannot yield a 6-clique. Hence, G contains no 6-cliques. Thus, $\mathcal{M}$ is composed of 5-cliques and triangles. Since $m_5 = 2$ and G has 32 edges, $\mathcal{M}$ is composed of two 5-cliques and four triangles which yields $|\mathcal{M}| = 6 < 7 = |\mathcal{P}|$, a contradiction. Hence, $p_2 \neq 2$.

Suppose $p_2 = 1$. Note that adding one edge in $L(K_5)$ cannot yield a 5-clique. Hence G does not have any 5-cliques, contradicting $m_5 \equiv p_2 \equiv 1 (mod\ 3)$. Hence, $p_2 \neq 1$.

Therefore, $p_2 = 0$ which follows that $G = J = L(K_5)$.

(x) Since G has ten vertices, $\mathcal{M}$ cannot contain two 6-cliques. Then $m_6 = 0$ or 1. Suppose $m_6 = 1$. If $m_5 = 1$, then since G has ten vertices, G is composed of one 6-clique and one 5-clique. This yields $cp(G) = mcp(G)$, a contradiction. Hence, $m_5 = 0$. Together with $(i) - (vii)$ and by Lemma 5.5(ii), we have

$p_2 + 6p_4 = 3m_3 + 15m_6$. Since $m_6 = 1, p_2 + 6p_4 = 3m_3 + 15$. Since each of other terms is divisible by 3, so is p_2. Let $p_2 = 3k$, where k is a non-negative integer. Hence, $3k + 6p_4 = 3m_3 + 15$. Therefore, $m_3 = k + 2p_4 - 5$. Next, because $cp(G) < mcp(G)$, $p_2 + p_4 < m_3 + m_6$. Then plug in $m_6 = 1$ and $m_3 = k + 2p_4 - 5$ to get $p_2 + p_4 < k + 2p_4 - 5 + 1$. Hence, $p_4 > 2k + 4$, i.e., $p_4 \geq 5$. By (ix), $G = L(K_5)$; so, $cp(G) = mcp(G)$, a contradiction. Hence, $m_6 = 0$.

(xi) By $(i) - (x)$ and Lemma 5.5 (ii),

$$p_2 + 6p_4 = 3m_3 + 10m_5. \tag{5.3}$$

Since G has ten vertices, $\mathcal{M}$ cannot contain more than two 5-cliques. Then $m_5 = 0, 1$ or 2. Suppose $m_5 = 2$. By $(viii)$, $p_2 = 2+3k$, where k is a non-negative integer. Replace $p_2 = 2 + 3k$ and $m_5 = 2$ in (5.3); so, $2 + 3k + 6p_4 = 3m_3 + 20$. Hence, $m_3 = k + 2p_4 - 6$. Since $cp(G) < mcp(G)$, $p_2 + p_4 < m_3 + m_5$. Rewrite p_2 and m_3 in terms of k; so, $p_4 > 2k + 6 \geq 6$ contradicting Lemma 5.1. Hence, $m_5 \neq 2$.

Suppose $m_5 = 1$. By $(viii)$, $p_2 = 1 + 3k$, where k is a non-negative integer. Rewrite $p_2 = 1 + 3k$ and $m_5 = 1$ in (5.3); so, $1 + 3k + 6p_4 = 3m_3 + 10$. Hence, $m_3 = k+2p_4-3$. Since $cp(G) < mcp(G)$, $p_2+p_4 < m_3+m_5$. Rewrite p_2 and m_3 in terms of k; so, $1+3k+p_4 < k+2p_4-3+1$. Hence, $p_4 > 2k+3 \geq 3$, i.e., there are at least four 4-cliques in $\mathcal{P}$, and because of $\mathcal{M}$ being composed of triangles and one 5-clique, this contradicts Theorem 5.5. Hence, $m_5 \neq 1$. Therefore, we have $m_5 = 0$ as desired.

(xii) Immediate. ■

Lemma 5.7 *Let G be a graph of ten vertices with $cp(G) < mcp(G)$. Then there is no clique of order 6 or more in G.*

Proof By Lemma 5.6 (ii), each maximal clique of order at least 6 must be in $\mathcal{M}$. However, by Lemma 5.6(iii), (vii) and (x), we have $m_i = 0$ for i at least 6. Hence, each clique of order at least 6 must also be in $\mathcal{P}$. But $\mathcal{P} \cap M = \emptyset$. This implies that there is no clique of order at least 6 in G. ■

From Lemma 5.6, we finally have that all members of $\{p_i, m_i : 2 \leq i \leq 9\} \setminus \{p_2, p_4, m_3\}$ have zero values. Next we will show that actually p_2 is also zero. Consequently, since $\mathcal{P} \neq \mathcal{M}$, we have that p_4 and m_3 are positive.

Lemma 5.8 *For any graph of ten vertices with a minimum clique partition $\mathcal{P}$ and a minimum maximal-clique partition $\mathcal{M}$ where $\mathcal{P} \cap \mathcal{M} = \emptyset$ and $|\mathcal{P}| < |\mathcal{M}|$, then $\mathcal{P}$ contains at most four 4-cliques.*

Proof Let G be a graph of ten vertices with a maximal-clique partition. Let $\mathcal{P}$ and $\mathcal{M}$ be a minimum clique partition and a minimum maximal-clique partition of G, respectively, satisfying $\mathcal{P} \cap \mathcal{M} = \emptyset$ and $|\mathcal{P}| < |\mathcal{M}|$. Assume $\mathcal{P}$ contains at least five 4-cliques. By Lemma 5.1, $\mathcal{P}$ contains exactly five 4-cliques. Let H be the subgraph of G composed of the five 4-cliques in $\mathcal{P}$. Hence, $H = L(K_5)$.

Since $|\mathcal{P}| < |\mathcal{M}|$ and $|\mathcal{M}| \leq 10$ (by Theorem 5.1), we have $|\mathcal{P}| \leq 9$. Let v

be any vertex in H. We examine edges in $G \setminus H$ incident to v. Since $H = L(K_5)$, the vertex v is nonadjacent to three vertices in H. But every edge between these three vertices is in H. So, any edges in $G \setminus H$ incident to v must be in $\mathcal{P}$. Hence, $\mathcal{P}$ is composed of 2-cliques and exactly five 4-cliques. Now since $|\mathcal{P}| \leq 9$, $\mathcal{P}$ contains at most four 2-cliques, the four edges in $G \setminus H$.

By Lemma 5.6, $m_i = 0$ for all $i \neq 3$. Since $m_3 > 0$, we have $\mathcal{M}$ contains 3-cliques. Moreover, it is possible that $\mathcal{M}$ contains cliques of order i where $i \neq 3$ as they are discarded to get the values of m_i. Since $\mathcal{P}$ contains 2-cliques and 4-cliques, $\mathcal{M}$ could contain 2-cliques and 4-cliques. However, any edges in H and 2-cliques in $\mathcal{P}$ are not maximal; so, $\mathcal{M}$ does not contain 2-cliques. Hence, we have that $\mathcal{M}$ is composed of 3-cliques and 4-cliques. Let a be the number of 2-cliques in $\mathcal{P}$. Let b and c be the number of 3-cliques and 4-cliques in $\mathcal{M}$. Recall that $\mathcal{P}$ has five 4-cliques. Since both $\mathcal{P}$ and $\mathcal{M}$ are clique partitions of G, we have

$$\binom{4}{2}5 + \binom{2}{2}a = \binom{3}{2}b + \binom{4}{2}c. \text{ , i.e., } 30 + a = 3b + 6c. \qquad (5.4)$$

Then a is divisible by 3. Because $\mathcal{P}$ contains at most four edges, $a = 0$ or 3.

If $a = 0$, G is $L(K_5)$, a contradiction. Otherwise, $a = 3$, i.e., G is the result of adding three edges to $L(K_5)$. Thus, from (5.4), $3b+6c = 33$, i.e., $b+2c = 11$. But $|\mathcal{M}| = b + c$; since $8 = |\mathcal{P}| < |\mathcal{M}|$, we have $b + c \geq 9$. However, $|\mathcal{M}| \leq 10$ (by Theorem 5.1); so, $b+c$ is either 9 or 10. Together with $b+2c = 11$, we have b is either 7 or 8. However, by Remark 2.1, adding three edges to $L(K_5)$ results in a graph with at most six maximal triangles contradicting $b = 7$ or 8.

Hence, $\mathcal{P}$ can contain at most four 4-cliques. ∎

Lemma 5.9 *No clique partition of a graph of ten vertices with $cp < mcp$ contains 2-cliques. In other words, $p_2 = 0$.*

Proof Because $m_5 \equiv p_2(mod\ 3)$ (by Lemma 5.6 $(viii)$) and $m_5 = 0$ (by Lemma 5.6 (xi)), we have $p_2 = 3k$, where k is a non-negative integer. By Lemma 5.6 $(i) - (xi)$ and Lemma 5.5 (ii), $p_2 + 6p_4 = 3m_3$. But $p_2 = 3k$; so, $3k + 6p_4 = 3m_3$. Hence, $m_3 = k + 2p_4$. Since $cp(G) < mcp(G)$, $p_2 + p_4 < m_3$. Rewrite p_2 and m_3 in terms of k; so, $p_4 > 2k$. It follows that $m_3 > 5k$. By Theorem 5.1, m_3 is at most 10. Thus, $k = 0$ or 1. We will prove $k = 0$ by showing that k cannot be 1.

Suppose $k = 1$. Then $p_2 = 3k = 3$ and $m_3 = k + 2p_4 = 1 + 2p_4$. Thus, m_3 is an odd number. Since $m_3 > 5k$, then $m_3 > 5$. Hence, in fact, m_3 is 7 or 9. Since $m_3 = 1 + 2p_4$, it follows that $p_4 = 3$ or 4.

Next we will consider all possibilities of $\mathcal{P}$ and $\mathcal{M}$. We now have only two possible solutions for p_i and m_i, for all i. Recall that p_i and m_i are the excess of i-cliques in $\mathcal{P}$ and $\mathcal{M}$. Then we need to consider k-cliques that are discarded from $\mathcal{P}$ and $\mathcal{M}$. Let $\mathcal{D}$ be the set of cliques discarded from $\mathcal{P}$ and $\mathcal{M}$. By Lemma 5.6(i), $\mathcal{M}$ cannot have 2-cliques. Since $p_4 \geq 3$, there are at least three 4-cliques in $\mathcal{P}$. Besides, G has ten vertices; so, $\mathcal{P}$ does not have enough vertices to contain three 4-cliques and a clique of order 5 or more. Therefore, $\mathcal{D}$ is composed of 3-cliques and 4-cliques. Let d_3 and d_4 be the number of 3-cliques and 4-cliques in $\mathcal{D}$, respectively. Hence $\mathcal{P}$ is composed of p_2 2-cliques, d_3 3-cliques, and $p_4 + d_4$ 4-cliques, and $\mathcal{M}$ is composed of $m_3 + d_3$ 3-cliques and d_4 4-cliques.

Case 1: Suppose $p_4 = 3$. Since $m_4 = 0$, all 4-cliques in $\mathcal{M}$ are in $\mathcal{D}$. Suppose $d_4 \neq 0$. Since $\mathcal{P}$ has at least three 4-cliques and $\mathcal{M}$ is composed of 3-cliques and 4-cliques, by Theorem 5.4, $d_4 \geq 2$. Hence, $\mathcal{P}$ is composed of $p_4 + d_4 \geq 3 + 2 = 5$ 4-cliques. By Lemma 5.8, this is a contradiction. Hence, $d_4 = 0$. Thus, D is composed of 3-cliques. Then $\mathcal{M}$ is composed of 3-cliques while $\mathcal{P}$ has at least three 4-cliques; hence, by Theorem 5.2, $|\mathcal{M}| \geq 10$. Since $|\mathcal{M}| \leq 10$ (by Theorem 5.1), it follows that $\mathcal{M}$ is composed of ten triangles. Moreover, G has at least three 4-cliques and by Theorem 5.3, $G = L(K_5)$. Thus, $cp(G) = mcp(G)$, a contradiction.

Case 2: Suppose $p_4 = 4$. If $d_4 \geq 1$, there are at least five 4-cliques in $\mathcal{P}$. By Lemma 5.8, $|\mathcal{P}| = |\mathcal{M}|$, a contradiction. Hence, D is composed of only 3-cliques and so is $\mathcal{M}$. Again by Theorem 5.2 and Theorem 5.1, $\mathcal{M}$ is composed of exactly ten triangles. Since G has at least three 4-cliques, by Theorem 5.3, $G = L(K_5)$. Thus, $cp(G) = mcp(G)$, a contradiction.

Hence, there is no graph of this type where $k = 1$. Hence $k = 0$. Therefore, $p_2 = 0$. ∎

Lemma 5.10 *Let G be a graph of ten vertices with $cp(G) < mcp(G)$. Let $\mathcal{P}$ and $\mathcal{M}$ be a minimum clique partition and a minimum maximal-clique partition of G, respectively. Then each 5-clique in $\mathcal{M}$ must share a 4-clique with exactly one 5-clique in $\mathcal{P}$.*

Proof Let $\mathcal{P}_5$ and $\mathcal{M}_5$ be the set of 5-cliques in $\mathcal{P}$ and the set of 5-cliques in $\mathcal{M}$, respectively. Let f be a relation from $\mathcal{P}_5$ to $\mathcal{M}_5$ defined as follows: for

$A \in \mathcal{P}_5$, $B \in \mathcal{M}_5$, $(A, B) \in f$ if and only if A shares a 4-cliques with B. By Lemma 5.6(iv), we have a one to one relation from $\mathcal{P}_5$ into $\mathcal{M}_5$. Furthermore, since $p_5 = 0$ (by Lemma 5.6(iv)) and $m_5 = 0$ (by Lemma 5.6(xi)), $|\mathcal{P}_5| = |\mathcal{M}_5|$. It follows that f is a one to one correspondence function between $\mathcal{M}_5$ and $\mathcal{P}_5$. Therefore, each 5-clique in $\mathcal{M}$ shares a 4-clique with exactly one 5-clique in $\mathcal{P}$. ■

5.2.2 The Target Theorem

The main theorem showing the nonexistence of graphs on ten vertices satisfying $cp < mcp$ will be proved in this subsection by using all earlier results in this chapter.

Lemma 5.11 *Let $\mathcal{M}$ be any maximal-clique partition of a graph. Let K be a 4-clique in the graph that does not belong to $\mathcal{M}$ and suppose that the edges of K are covered by six maximal cliques in $\mathcal{M}$. If there exist two maximal cliques among the six maximal cliques covering edges of K sharing a vertex not in K, then there are at least seven vertices not in K.*

Proof Let $K = \boxtimes(a, b, c, d)$. Since K has six edges and there are six maximal cliques in $\mathcal{M}$ covering edges of K, each of these maximal cliques covers one edge of K. For any $i, j \in \{a, b, c, d\}$, let M_{ij} be the maximal clique in $\mathcal{M}$ covering edge ij of K. By Lemma 3.1(i), each M_{ij} contains a vertex not in K. Assume there are two maximal cliques sharing the same vertex not in K. By Lemma 3.1(ii), they cannot share another vertex, without loss of generality say M_{ac} and M_{bd} share the same vertex v_1 outside K. It follows that vertices in $\{v_1, a, c, b, d\}$ form a 5-clique X containing the triangle $\triangle(v_1, a, c)$. Since M_{ac} is maximal, it cannot be $\triangle(v_1, a, c)$. Thus, M_{ac} must contain another vertex, say v_2. Similarly, the 5-clique X contains the triangle $\triangle(v_1, b, d)$. Hence, $M_{bd} \neq \triangle(v_1, b, d)$; so, it contains another vertex, say v_3. If v_3 is the same as v_2, by the same argument as above, M_{bd} must contain another vertex outside K, which must eventually be a different vertex from v_2. Hence, we can assume that v_3 is different from v_2. Now, let v_4 and v_5 be vertices of M_{ab} and M_{ad} not in K, respectively. Since edges ab and ad share a, by Lemma 3.1(ii), $v_4 \neq v_5$. Since edges ab and ad share a vertex with both ac and bd, $v_4, v_5 \notin \{v_1, v_2, v_3\}$. Hence, all vertices in $\{v_1, \ldots, v_5\}$ are different.

Next let v_6 and v_7 be vertices of M_{bc} and M_{cd} not in K, respectively. Since edge bc shares a vertex with all edges in $\{ac, bd, ab, cd\}$, vertex $v_6 \notin \{v_1, v_2, v_3, v_4\}$. If $v_6 = v_5$, vertices in $\{v_6, a, b, c, d\}$ form a 5-clique containing $\triangle(v_6, b, c)$. Since M_{bc} is maximal, it cannot be $\triangle(v_6, b, c)$. Thus, M_{bc} must contain another vertex other than v_5. Hence, we can assume without loss of generality $v_6 \neq v_5$. Similarly, vertex $v_7 \notin \{v_1, v_2, v_3, v_5, v_6\}$ and we can assume without loss of generality that $v_7 \neq v_4$. Hence, we have at least seven different vertices not in K. ∎

Theorem 5.6 *There is no graph of ten vertices with a maximal-clique partition and* $cp < mcp$.

Proof Suppose not. Let G be a graph having the smallest number of edges among graphs with ten vertices and $cp(G) < mcp(G)$. Let $\mathcal{P}$ be a minimum clique partition of G and $\mathcal{M}$ be a minimum maximal-clique partition of G.

Because of the minimality of G, we have $\mathcal{P} \cap \mathcal{M} = \emptyset$. Now since each 4-clique in $\mathcal{M}$ can share a triangle with only one 4-clique in $\mathcal{P}$ and by Lemma 5.6 (vi), $\mathcal{M}$ has fewer 4-cliques than $\mathcal{P}$, there exists a 4-clique X in $\mathcal{P}$, that does not share a triangle with any 4-clique in $\mathcal{M}$. Moreover, Lemma 5.10 says that each 5-clique in $\mathcal{M}$ must share a 4-clique with some 5-clique in $\mathcal{P}$; so a 5-clique in $\mathcal{M}$ cannot share a triangle with X. Now by Lemma 5.7, the largest order of cliques in G is 5. Hence, X does not share a triangle with any clique in $\mathcal{M}$. Therefore, each clique in $\mathcal{M}$ covers at most one edge of X.

Since X has six edges, X has nonempty intersection with six cliques in $\mathcal{M}$. Since each clique in $\mathcal{M}$ is maximal, it contains a vertex not in X. If there exist two of these maximal cliques sharing the same vertex not in X. By Lemma 5.11, there are at least seven vertices not in X. However, there are only six vertices not in X, a contradiction. Hence, each of these six maximal cliques contains a distinct vertex not in X and each clique can have exactly one vertex outside X. This yields that those cliques are triangles in $\mathcal{M}$.

Now, let $V(X) = \{a, b, c, d\}$. For $i, j \in V(X)$, let the other vertex of the triangle intersecting the edge joining vertices i and j be named ij. Then $V(G) = \{a, b, c, d, ab, ac, ad, bc, bd, cd\}$.

Let H be the subgraph of G composed of these six triangles $\{\triangle(i, j, ij) : i, j \in V(X)\}$ in $\mathcal{M}$. Note that $X \subset H \subseteq G$. Now the clique partition of H containing X as a member must be composed of X and the remaining 12 edges, which has cardinality 13 while the six triangles in $\{\triangle(i, j, ij) : i, j \in V(X)\}$ are also a clique partition of H of smaller cardinality. Hence, X is not an element

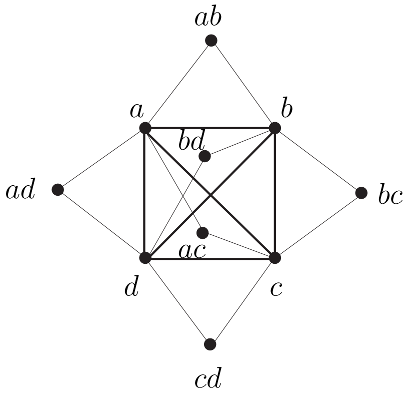


Figure 5.16: The subgraph H of G

of a minimum clique partition of H. But X is contained in $\mathcal{M}$ a minimum maximal-clique partition of G. This means $H \neq G$. Hence, there is an edge of G not in H.

Next, let $i, j, k \in \{a, b, c, d\} = V(X)$ where $i \neq j \neq k$. If $i \leftrightarrow jk$, then $\boxtimes(i, j, k, jk)$ is a 4-clique containing a maximal $\triangle(j, k, jk)$, a contradiction. Hence, $i \not\leftrightarrow jk$, i.e., there are no more edges of G outside H incident to any vertices of X. Now consider any vertex i of X. There are three edges of $H \setminus X$ incident to i. However, by Lemma 5.9, $\mathcal{P}$ cannot contain a 2-clique. Hence, all three edges must be in the same clique in $\mathcal{P}$, which induces a 4-clique in $\mathcal{P}$. Since X has four vertices, it induces four 4-cliques. Together with X, there are at least five 4-cliques in $\mathcal{P}$. Therefore, by Lemma 5.8, this is a contradiction. Hence, a graph of ten vertices with $cp < mcp$ does not exist. ■

Corollary 5.1 *Graphs of at most ten vertices with a maximal-clique partition have $cp = mcp$.*

Proof Monson [22] confirms this for graphs of at most nine vertices with a maximal-clique partition, and Theorem 5.6 confirms it for graphs of ten vertices with maximal-clique partitions. ■

Corollary 5.2 *Pullman, Shank and Wallis's graph in Figure 5.17 has the minimum number of vertices to satisfy the property $cc < cp < mcp$.*

Proof Because there is no graph of ten vertices with $cp < mcp$, the smallest graph with $cc < cp < mcp$ must have 11 vertices. ■

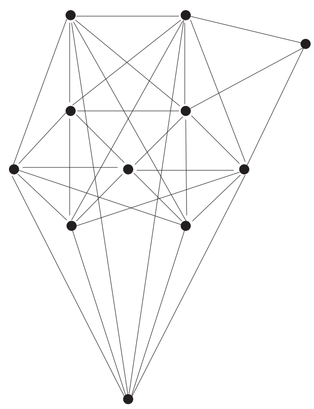


Figure 5.17: Pullman, Shank and Wallis's Graph

Corollary 5.3 *Graphs with $cc = cp < mcp$ have at least 11 vertices.*

Corollary 5.4 *For any graph of at most ten vertices, if each of its minimum clique partitions is not a maximal-clique partition, then the graph does not have a maximal-clique partition.*

Corollary 5.4 is another way to check the existence of a maximal-clique partition of a graph of ten vertices. In general, it is not necessary that a maximal-clique partition be a clique partition of minimum size. However, it is true for graphs satisfying $cp = mcp$.

6. Conclusion and Open Problems for Future Work

6.1 Conclusion

We have investigated maximal-clique partitions of graphs and relationships between maximal-clique partition numbers and other clique parameters. We have solved several open problems and found other new results as follows:

- A line graph that is not a triangle has a maximal-clique partition if and only if every triangle in its root graph contains no vertex of degree two. Moreover, if a line graph has a maximal-clique partition, then its three clique parameters satisfy $cc = cp = mcp$. The maximal-clique partition problem for line graphs can be solved by a linear time algorithm. (Theorem 2.1, Theorem 2.3)

- The minimum number of vertices of a graph having at least two maximal-clique partitions of different sizes is 10. $L(K_5)$ is a graph with ten vertices and two maximal-clique partitions. (Theorem 3.3)

- There exists a clique-inseparable graph V_n with n maximal-clique partitions of n different sizes where n is any natural number. (Theorem 3.11, Theorem 3.12)

- There exists a family $\mathcal{F}' = \{U_{k,d} : k \leq 4, \frac{3k}{5} \leq d \leq 3k\}$ of infinitely many graphs with $cc = cp < mcp$. The smallest known graph among graphs with $cc = cp < mcp$ is the smallest graph in the family $\mathcal{F}'$ which is $U_{4,3}$ with 68 vertices and 234 edges. (Theorem 4.4, Theorem 4.5)

- There is no graph of ten vertices with $cp = mcp$. Hence, graphs satisfying $cc = cp < mcp$ or $cc < cp < mcp$ have at least 11 vertices. It follows that PSW's graph is a graph with the smallest number of vertices having $cc < cp < mcp$. (Theorem 5.6, Corollary 5.2, Corollary 5.3)

6.2 Open Problems for Future Work

This dissertation motivates some open problems for future work:

- Confirm that $L(K_5)$ is the only graph of ten vertices having two maximal-clique partitions of different sizes.

- Find necessary and sufficient conditions for a line graph to have a maximal-clique partition without determining its root graph.

- Examine characteristics of other categories of graphs that guarantee that they have maximal-clique partitions.

- V_n has n maximal-clique partitions of n different sizes. Is there a clique-inseparable graph with fewer vertices than V_n having n maximal-clique partitions of n different sizes?

- Find a graph with $cc = cp < mcp$ having fewer vertices than $U_{4,3}$.

- Investigate characteristics of a minimum clique partition and a minimum maximal-clique partition of a graph.

- Investigate other properties of the glue operator and its applications.

- As many graphs have no maximal-clique partition, it is interesting to consider how much a graph deviates from having a maximal-clique partition. A way to examine it is by considering its smallest supergraphs or its largest subgraphs having a maximal-clique partition. We could determine its deviation as the difference between the numbers of edges in the graph and its smallest supergraphs or largest subgraphs with a maximal-clique partition. Let G be any graph. Let F and H be one of its largest subgraphs and one of its smallest supergraphs having a maximal-clique partition, respectively. Hence, $F \subseteq G \subseteq H$. Note that if G has a maximal-clique partition, then $F = G = H$. Define,

$$\mu^+(G) = |E(H)| - |E(G)| \text{ and } \mu^-(G) = |E(G)| - |E(F)|.$$

For example, in Figure 6.1, H and F are a smallest supergraph and a largest subgraph of G, respectively, where

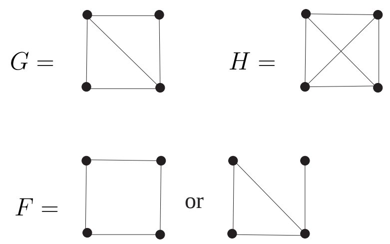


Figure 6.1: A graph illustrating μ^+ and μ^-

$\mu^+(G) = |E(H)| - |E(G)| = 1$ and $\mu^-(G) = |E(G)| - |E(F)| = 1.$

Now note that a 2-clique is a subgraph of any graph containing at least one edge, and an n-clique is a supergraph of any graph of n vertices. Let G be any graph of n vertices with at least one edge; then, we have

$$0 \leq \mu^-(G) \leq |E(G)| - 1 \text{ and } 0 \leq \mu^+(G) \leq |E(K_n)| - |E(G)|.$$

This motivates future work to investigate μ^+ and μ^- of a variety of graphs and related problems.

REFERENCES

[1] L. B. Beasley. Linear operators preserving partition numbers of graphs. In *Graphs, matrices, and designs*, volume 139 of *Lecture Notes in Pure and Appl. Math.*, pages 19–29. Dekker, New York, 1993.

[2] R. C. Brigham and R. D. Dutton. Changing and unchanging invariants: the edge clique cover number. In *Proceedings of the Twentieth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1989)*, volume 70, pages 145–152, 1990.

[3] L. Caccetta, P. Erdős, E. T. Ordman, and N. J. Pullman. The difference between the clique numbers of a graph. *Ars Combin.*, 19(A):97–106, 1985.

[4] L. Caccetta and N. J. Pullman. On clique covering numbers of cubic graphs. In *Combinatorial mathematics, X (Adelaide, 1982)*, volume 1036 of *Lecture Notes in Math.*, pages 121–127. Springer, Berlin, 1983.

[5] L. Caccetta and N. J. Pullman. On clique covering numbers of regular graphs. *Ars Combin.*, 15:201–230, 1983.

[6] N. G. de Bruijn and P. Erdős. On a combinatorial problem. *Nederl. Akad. Wetensch., Proc.*, 51:1277–1279, 1948.

[7] D. de Caen, P. Erdős, N. J. Pullmann, and N. C. Wormald. Extremal clique coverings of complementary graphs. *Combinatorica*, 6(4):309–314, 1986.

[8] D. de Caen, D. A. Gregory, and N. J. Pullman. Clique coverings of complements of paths and cycles. In *Cycles in graphs (Burnaby, B.C., 1982)*, volume 115 of *North-Holland Math. Stud.*, pages 257–267. North-Holland, Amsterdam, 1985.

[9] P. Erdős, R. Faudree, and E. T. Ordman. Clique partitions and clique coverings. In *Proceedings of the First Japan Conference on Graph Theory and Applications (Hakone, 1986)*, volume 72, pages 93–101, 1988.

[10] P. Erdős, A. W. Goodman, and L. Pósa. The representation of a graph by set intersections. *Canad. J. Math.*, 18:106–112, 1966.

[11] P. Erdős, E. T. Ordman, and Y. Zalcstein. Clique partitions of chordal graphs. *Combin. Probab. Comput.*, 2(4):409–415, 1993.

[12] D. A. Gregory, S. McGuinness, and W. Wallis. Clique partitions of the cocktail party graph. *Discrete Math.*, 59(3):267–273, 1986.

[13] K. F. Jones, J. R. Lundgren, N. J. Pullman, and R. Rees. A note on the biclique covering numbers of $K_n \setminus K_m$ and complete t-partite graphs. *Congr. Numer.*, 66:181–183, 1988. Nineteenth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Baton Rouge, LA, 1988).

[14] P. G. H. Lehot. An optimal algorithm to detect a line graph and output its root graph. *J. Assoc. Comput. Mach.*, 21:569–575, 1974.

[15] Z. Lonc. Clique packings and clique partitions of graphs without odd chordless cycles. *Discuss. Math. Graph Theory*, 16(2):143–149, 1996.

[16] L. Lovász. On covering of graphs. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 231–236. Academic Press, New York, 1968.

[17] S. Ma, W. D. Wallis, and J. Wu. Clique covering of chordal graphs. *Utilitas Math.*, 36:151–152, 1989.

[18] S. H. Ma and W. D. Wallis. Clique numbers of threshold graphs. *Caribbean J. Math.*, 5(1):29–45, 1986.

[19] S. H. Ma and W. D. Wallis. Clique partitions of triangulated graphs. In *Proceedings of the Seventeenth Southeastern International Conference on Combinatorics, Graph theory, and Computing (Boca Raton, Fla., 1986)*, volume 55, pages 93–101, 1986.

[20] S. H. Ma and W. D. Wallis. Maximal-clique partitions of interval graphs. *J. Austral. Math. Soc. Ser. A*, 45(2):227–232, 1988.

[21] S. McGuinness and R. Rees. On the number of distinct minimal clique partitions and clique covers of a line graph. *Discrete Math.*, 83(1):49–62, 1990.

[22] S. Monson. *Clique Numbers.* PhD thesis, Queens University, Canada, 1995.

[23] S. D. Monson. The effects of vertex deletion and edge deletion on the clique partition number. *Ars Combin.*, 42:89–96, 1996.

[24] S. D. Monson, N. J. Pullman, and R. Rees. A survey of clique and biclique coverings and factorizations of $(0, 1)$-matrices. *Bull. Inst. Combin. Appl.*, 14:17–86, 1995.

[25] S. D. Monson, K. N. Vander Meulen, and R. S. Rees. Clique coverings and maximal-clique partitions: an example. *J. Combin. Math. Combin. Comput.*, 16:61–63, 1994.

[26] J. Orlin. Contentment in graph theory: covering graphs with cliques. *Nederl. Akad. Wetensch. Proc. Ser. A*, 39(5):406–424, 1977.

[27] N. J. Pullman. Clique coverings of graphs—a survey. In *Combinatorial mathematics, X (Adelaide, 1982)*, volume 1036 of *Lecture Notes in Math.*, pages 72–85. Springer, Berlin, 1983.

[28] N. J. Pullman. Clique covering of graphs. IV. Algorithms. *SIAM J. Comput.*, 13(1):57–75, 1984.

[29] N. J. Pullman and D. de Caen. Clique coverings of graphs. III. Clique coverings of regular graphs. In *Proceedings of the Eleventh Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1980), Vol. II*, volume 29, pages 795–808, 1980.

[30] N. J. Pullman and D. de Caen. Clique coverings of graphs. I. Clique partitions of regular graphs. *Utilitas Math.*, 19:177–205, 1981.

[31] N. J. Pullman and A. Donald. Clique coverings of graphs. II. Complements of cliques. *Utilitas Math.*, 19:207–213, 1981.

[32] N. J. Pullman, H. Shank, and W. D. Wallis. Clique coverings of graphs. V. Maximal-clique partitions. *Bull. Austral. Math. Soc.*, 25(3):337–356, 1982.

[33] R. Rees. Minimal clique partitions and pairwise balanced designs. *Discrete Math.*, 61(2-3):269–280, 1986.

[34] R. Rees. Minimal clique partitions of norm three. In *Proceedings of the Seventeenth Southeastern International Conference on Combinatorics, Graph theory, and Computing (Boca Raton, Fla., 1986)*, volume 54, pages 129–136, 1986.

[35] F. S. Roberts. Applications of edge coverings by cliques. *Discrete Appl. Math.*, 10(1):93–109, 1985.

[36] H. J. Ryser. Intersection properties of finite sets. *J. Combinatorial Theory Ser. A*, 14:79–92, 1973.

[37] W. D. Wallis. The clique partition number of the complement of a cycle. In *Cycles in graphs (Burnaby, B.C., 1982)*, volume 115 of *North-Holland Math. Stud.*, pages 335–344. North-Holland, Amsterdam, 1985.

[38] W. D. Wallis. Clique partitions of the complement of a one-factor. In *Proceedings of the Fourteenth Manitoba Conference on Numerical Mathematics and Computing (Winnipeg, Man., 1984)*, volume 46, pages 317–319, 1985.

[39] W. D. Wallis. Finite planes and clique partitions. In *Finite geometries and combinatorial designs (Lincoln, NE, 1987)*, volume 111 of *Contemp. Math.*, pages 279–285. Amer. Math. Soc., Providence, RI, 1990.

[40] W. D. Wallis and J. Wu. On clique partitions of split graphs. *Discrete Math.*, 92(1-3):427–429, 1991.

[41] W. D. Wallis and J. L. Wu. Generalized clique coverings of chordal graphs. *Australas. J. Combin.*, 1:233–236, 1990. Combinatorial mathematics and combinatorial computing, Vol. 1 (Brisbane, 1989).

[42] W. D. Wallis and J. L. Wu. Clique partitions of squares of trees. In *Advances in Graph Theory*, pages 301–305. Vishwa, Gulbarga, 1991.

[43] W. D. Wallis and G. Zhang. On the clique partition number of the complement of a triangle. *Congr. Numer.*, 66:217–226, 1988. Nineteenth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Baton Rouge, LA, 1988).

[44] W. D. Wallis and G.-H. Zhang. On maximal clique irreducible graphs. *J. Combin. Math. Combin. Comput.*, 8:187–193, 1990.

[45] W. D. Wallis and G.-H. Zhang. On the partition and coloring of a graph by cliques. *Discrete Math.*, 120(1-3):191–203, 1993.