BICLIQUE COVERS AND PARTITIONS OF BIPARTITE GRAPHS AND DIGRAPHS

AND RELATED MATRIX RANKS OF {0,1}-MATRICES

by
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Biclique Covers and Partitions of Bipartite Graphs and Digraphs

and Related Matrix Ranks of {0,1}-Matrices

Thesis directed by Professor J. Richard Lundgren

ABSTRACT

During the past twenty years there has been considerable research on the biclique cover and partition numbers of bipartite graphs and digraphs and several related matrix ranks. These include the boolean rank, nonnegative integer rank, term rank, and real rank. The main goal of the work in this thesis is to find classes of graphs with equal biclique cover and partition numbers or classes of {0,1}-matrices with at least two of the matrix ranks equal.

In 1991, Lundgren and Maybee showed that all four matrix ranks of nearly reducible matrices are equal. In 1977, Lovász and Plummer gave bipartite graph representations of nearly decomposable and fully indecomposable matrices. New results in this thesis complement this work. Three of the matrix ranks for nearly decomposable matrices are determined and bipartite graph representations of nearly reducible and irreducible matrices will be given.

In 1991, de Caen proved that the real rank of an n-tournament matrix is at least $n-1$. This lower bound implies that the term rank of an n-tournament matrix is also at least $n-1$. A new result in this thesis characterizes those tournaments with term rank n. The classification of singular tournaments remains an open problem, but some results are known for specific subclasses of tournaments. In 1990, Shader characterized singular upset tournament matrices and proved that the nonnegative integer rank of upset tournament matrices is equal to the real rank.

New results concerning the boolean rank and term rank of upset tournament matrices are discussed in this thesis. Specifically, a characterization of upset tournament matrices with respect to their boolean rank and a best possible lower bound for the boolean rank is given. In addition, it is shown that the number of nonisomorphic upset tournaments with equal biclique cover and partition numbers can be given in terms of convolutions of the Fibonacci sequence. These results, together with Shader's work, give a complete characterization of upset tournament matrices with respect to each rank and with respect to their biclique cover and partition numbers.

This abstract accurately represents the content of the candidate's thesis. I recommend its publication.

Signed ______________________________

J. Richard Lundgren

DEDICATION

This thesis is dedicated to my parents and my fiancée Lynette.

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It has been a privilege to do research and complete this thesis under the direction of my advisor, J. Richard Lundgren. I thank him for his encouragement, his guidance, and his generosity. The many trips to Rockies, Avalanche, and Nuggets games were much appreciated. I will strive to follow his example of excellence in both teaching and research. I also want to thank my thesis committee for their time and their advice.

Much of the work in this thesis was presented at the Discrete Mathematics Seminar at the University of Colorado at Denver. I would like to thank the discrete math group, faculty and students, for listening to my talks and offering advice. I greatly appreciated this opportunity to present and refine this material.

I would like extend my appreciation to the Graduate Committee at the University of Colorado at Denver for offering me teaching assistantships each semester. This type of financial support not only allowed me to pursue my degree, but gave me significant teaching experience that proved invaluable during the job search process.

Over the past ten years of undergraduate and graduate school, faculty, friends, and family have provided immeasurable amounts of encouragement. I will name just a few of them here. Brian Wick, Hilary Davies, Roi Miranda, and Marty Calderone provided the inspiration to get me to graduate school. My office-mates and friends, Faun Doherty, Janine Kennedy, Cary Miller, Dave Brown and Chris Mehl were always helpful and supportive. My brother and his wife, Dalynn and Tammy Siewert, and my parents, Manley and Arlene Siewert, were always quick with words of encouragement. Finally, I want to thank my finacée, Lynette Buss, for her unconditional love and unfailing support as I wrote this thesis. I will always be indebted to you.

Daluss. J. Siewert
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1. Introduction

1.1 Background

The covering of bipartite graphs and digraphs with bicliques has applications in many areas. Amilhastre et al. [AVJ98] gave references regarding applications in automata and language theories, graph compression, partial orders, artificial intelligence, and biology [NMWA78]. During the past twenty years there has been considerable research on the biclique cover and partition numbers of bipartite graphs and digraphs and several related matrix ranks including the boolean rank, nonnegative integer rank, term rank, and the real rank. See Alexe et al. [AAF+], Brualdi et al. [BHM80], de Caen et al. [dCGP81], Fishburn and Hammer [FH93], Gregory et al. [GJLP91], Monson et al. [MPR95], Orlin [Orl77], and Shader [Sha90]. Biclique covering and partitioning numbers and related matrix ranks for some individual classes of graphs have been determined. For example, for domino-free bipartite graphs, see Amilhastre et al. [AVJ98]; for tournaments, see Bain et al. [BLM92] and Shader [Sha90]; for unipathic digraphs, see Hefner et al. [HLM93]; and for directed cockades, see Lundgren and Maybee [LM91]. The main goal of the work in this thesis is to find classes of graphs with equal biclique cover and partition numbers or classes of {0,1}-matrices with at least two of the matrix ranks equal.

In 1991, Lundgren and Maybee showed that all four matrix ranks of nearly reducible matrices are equal [LM91]. In 1977, Lovász and Plummer gave bipartite graph representations of nearly decomposable and fully indecomposable matrices [LP77]. New results in this thesis will complement this work. Three of the matrix ranks for nearly decomposable matrices are determined and bipartite graph representations of nearly reducible and irreducible matrices will be given.

In 1991, de Caen proved that the real rank of an n-tournament matrix is at least $n-1$ [dC91]. This lower bound implies that the term rank of an n-tournament matrix is also at least $n-1$. A new result in this thesis characterizes those tournaments with term rank n. The classification of singular tournaments remains an open problem, but some results are known for specific subclasses of tournaments. In 1990, Shader characterized singular upset tournament matrices and proved that the nonnegative integer rank of upset tournament matrices is equal to the real rank [Sha90]. New results concerning the boolean rank and term rank of upset tournament matrices are discussed in this thesis. Specifically, a characterization of upset tournament matrices with respect to their boolean rank and a best possible lower bound for the boolean rank is given. In addition, it is shown that the number of nonisomorphic upset tournaments with equal biclique cover and partition numbers can be given in terms of convolutions of the Fibonacci sequence. These results, together with Shader's work, give a complete characterization of upset tournament matrices with respect to each rank and with respect to their biclique cover and partition numbers.

Chapter 1 will give the necessary background and preliminary information concerning biclique covers and partitions and related matrix ranks. Chapter 2 will focus on classes of graphs where at least two of the matrix ranks are equal for the corresponding adjacency matrix. Chapter 3 will give the ranks of nearly reducible and nearly decomposable matrices and will discuss a bipartite graph representation of each of these classes. Chapter 4 will give rank results for tournaments in general and for some specific classes of tournaments. Chapter 5 gives a complete classification of upset tournaments with respect to each rank and shows that the number of upset tournaments with equal biclique cover and partition numbers can be given in terms of convolutions of the Fibonacci sequence. Chapter 6 summarizes the known results, both new and old, and gives some open problems.

1.2 Preliminaries

Throughout this thesis, $A(D)$ will be the adjacency matrix corresponding to the digraph $D(A)$ and $A(B)$ will be the reduced adjacency matrix corresponding to the bipartite graph $B(A)$. The reduced adjacency matrix of a bipartite graph is the submatrix of the adjacency matrix that corresponds to the arcs from one set of the bipartition to the other set of the bipartition. That is, if A is the adjacency matrix of a bipartite graph B, then

$$A = \left[\begin{array}{c|c} 0's & A(B) \\ \hline A(B)^T & 0's \end{array}\right].$$

1.2.1 Biclique Cover and Partition Numbers

The *biclique cover number* of a graph G, denoted by $bc(G)$, is the smallest number of complete bipartite subgraphs that cover the edges of G. For example, the digraph D, given in Figure 1.1, can be covered with two bicliques. See Figure 1.2. The bicliques in the covering must have the same orientation as in the digraph D and all the arcs my be from one bipartition set to the other bipartition set.

The *biclique partition number* of a graph G, denoted by $bp(G)$, is the smallest number of complete bipartite subgraphs that partition the edges of G. For example, the digraph D, given in Figure 1.1, can be partitioned into three bicliques. See Figure 1.3.

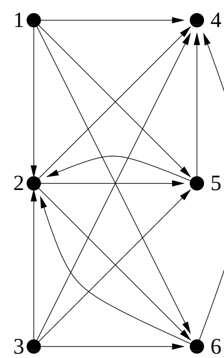

Figure 1.1. Digraph D.

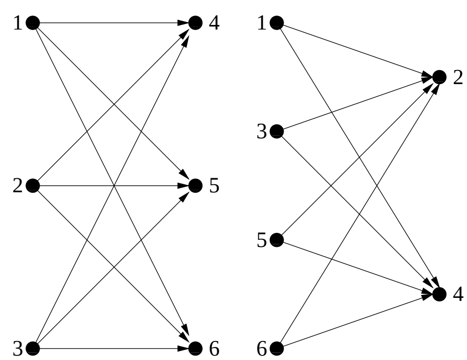

Figure 1.2. The digraph D, given in Figure 1.1, can be covered with two bicliques.

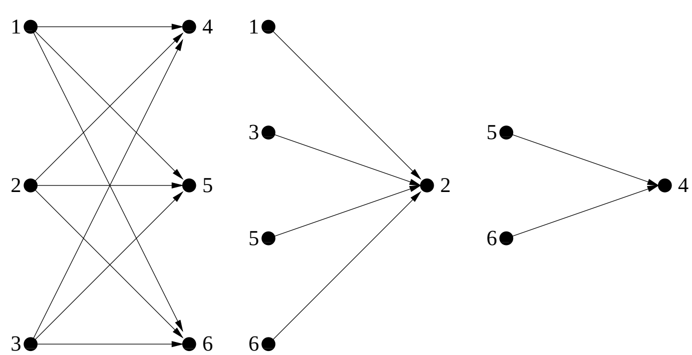

Figure 1.3. The digraph D, given in Figure 1.1, can be partitioned into three bicliques.

1.2.2 Boolean Rank

The *boolean rank* of an $m \times n$ {0,1}-matrix A, denoted by $b(A)$, is defined to be the smallest k for which there is an $m \times k$ {0,1}-matrix B and a $k \times n$ {0,1}-matrix C such that $A = BC$ where boolean arithmetic is used $(1 + 1 = 1)$. The boolean rank of a {0,1}-matrix A is equivalent to the minimum number of {0,1}-matrices of rank one whose boolean sum is A. For example, let $A = A(D)$ be the adjacency matrix of the digraph given in Figure 1.1. Then

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}_{6\times 6} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}_{6\times 2} \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}_{2\times 6} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The two rank one matrices correspond to the two bicliques used to cover the digraph $D = D(A)$. See Figure 1.2. Every factorization

$$A_{n\times m} = B_{n\times k}C_{k\times m}$$

of A into {0,1}-matrices corresponds to a biclique covering of size k. Conversely, every biclique covering of size k, corresponds to a factorization

$$A_{n\times m} = B_{n\times k}C_{k\times m}$$

of A into {0,1}-matrices.

The same result holds for bipartite graphs. Thus, we have Lemma 1.2.1.

Lemma 1.2.1 (Gregory et al. [GJLP91]) *If D is a digraph and B is a bipartite graph, then*

$$b(A(D)) = bc(D) \quad \textit{and} \quad b(A(B)) = bc(B).$$

1.2.3 Nonnegative Integer Rank

The *nonnegative integer rank* of an $m \times n$ matrix A, denoted by $r_{z^+}(A)$, is defined to be the smallest k for which there is an $m \times k$ matrix B and a $k \times n$ matrix C, over the nonnegative integers, such that $A = BC$. If A is a {0,1}-matrix, then B and C are {0,1}-matrices. The nonnegative integer rank of a matrix A is equivalent to the minimum number of rank one matrices, over the nonnegative integers, whose sum is A. For example, let $A = A(D)$ be the adjacency matrix of the digraph given in Figure 1.1. Then

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}_{6\times 6} \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}_{6\times 3} \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}_{3\times 6} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&\qquad + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

$$= \begin{bmatrix} 0&0&0&1&1&1\\ 0&0&0&1&1&1\\ 0&0&0&1&1&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix} + \begin{bmatrix} 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&1&0&0&0&0 \end{bmatrix} + \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&1&0&0 \end{bmatrix}.$$

The three rank one matrices correspond to the three bicliques used to partition the digraph $D = D(A)$. See Figure 1.3. Every factorization

$$A_{n\times m} = B_{n\times k}C_{k\times m}$$

of A into matrices B and C over the nonnegative integers corresponds to a biclique partition of size k. Conversely, every biclique partition of size k, corresponds to a factorization

$$A_{n\times m} = B_{n\times k}C_{k\times m}$$

of A into matrices B and C over the nonnegative integers.

The same result holds for bipartite graphs. Thus, we have Lemma 1.2.2.

Lemma 1.2.2 (Gregory et al. [GJLP91]) *If D is a digraph and B is a bipartite graph, then*

$$r_{z^+}(A(D)) = bp(D) \quad \textit{and} \quad r_{z^+}(A(B)) = bp(B).$$

1.2.4 Real Rank The *real rank* (usual rank) of the matrix A, denoted by $r(A)$, can be defined as the smallest k for which there is an $m \times k$ matrix B and a $k \times n$ matrix C, over the real numbers, such that $A = BC$. The real rank of a matrix A is equivalent to the minimum number of rank one matrices, over the real numbers, whose sum is A. This factorization into rank one matrices can be found using the LU-decomposition of A. For example,

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}_{6\times 6} \\
&= LU \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}_{6\times 6} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 6} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{6\times 4} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}_{4\times 6} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix}
\end{aligned}
$$

$$+\begin{bmatrix}0\\0\\1\\1\\1\\0\end{bmatrix}\begin{bmatrix}0 & 0 & 1 & 0 & 0 & -1\end{bmatrix}+\begin{bmatrix}0\\0\\0\\1\\1\\1\end{bmatrix}\begin{bmatrix}0 & 0 & 0 & 1 & 1 & 1\end{bmatrix}$$

1.2.5 Term Rank The *term rank* of a matrix A, denoted by $t(A)$, is the smallest number of rows and columns containing all the nonzero entries of A. If A is a $\{0,1\}$-matrix, a set of 1's of A is called an *independent set* if no two 1's in this set are in the same row or column of A. Hence, $t(A)$ is the maximum number of independent 1's in A. Also, if A is the adjacency matrix of a bipartite graph B, then $t(A)$ is the size of a maximum matching in B. For example, let

$$A=\begin{bmatrix}\mathbf{1} & 0 & 1 & 0\\0 & 0 & \mathbf{1} & 0\\1 & 1 & 0 & \mathbf{1}\\0 & 0 & 1 & 0\end{bmatrix}.$$

Then rows one and three and column three form a minimum line cover of all the 1's in A and the three bold **1**'s give a maximum set of independent 1's. These independent 1's correspond to a maximum matching in the bipartite graph $B(A)$. See Figure 1.4.

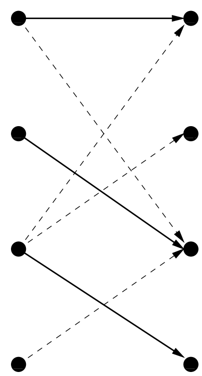

Figure 1.4. Maximum matching in the bipartite graph $B(A)$.

1.2.6 Lower Bounds Lemma 1.2.3, Lemma 1.2.4, and Corollary 1.2.5 will be useful in finding lower bounds for the boolean rank and the biclique covering number.

Lemma 1.2.3 (Gregory et al. [GJLP91]) *If a bipartite graph B contains a matching S, no two edges of which are in a four-cycle of B, then $bc(B) \geq |S|$, the number of edges in S.*

A set S of independent 1's of a {0,1}-matrix is said to be *isolated* if no two 1's of S are in a submatrix of the form

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Lemma 1.2.4 is a matrix formulation of Lemma 1.2.3.

Lemma 1.2.4 (Gregory et al. [GJLP91]) *If the adjacency matrix A of a bipartite graph B has an isolated set of k 1's, then $b(A) = bc(B) \geq k$.*

Corollary 1.2.5 *If the adjacency matrix A of a digraph D has an isolated set of k 1's, then $b(A) = bc(D) \geq k$.*

Corollary 1.2.6 *If a {0,1}-matrix A has an isolated set of k 1's, then*

$$b(A) \geq k.$$

For example, the bold **1**'s in the adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

of the digraph D, given in Figure 1.1, give a set of two isolated 1's. By Corollary 1.2.5, $b(A) = bc(D) \geq 2$. Thus, $b(A) = bc(D) = 2$, since we previously exhibited a biclique covering of size two.

1.2.7 Basic Rank Relationships From the definitions of nonnegative integer rank and real rank, it is clear that $r(A) \leq r_{z^+}(A)$. Since every biclique partition is a biclique covering, Lemmas 1.2.1 and 1.2.2 imply that $b(A) \leq r_{z^+}(A)$, for any {0,1}-matrix A. Also, for any {0,1}-matrix A, $r_{z^+}(A) \leq t(A)$.

There is no relationship between $b(A)$ and $r(A)$ for a {0,1}-matrix A. To illustrate this, consider

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

which has $r(A) = 4$. The 1's on the main diagonal of A give a set of six isolated 1's. By Corollary 1.2.5, $b(A) \geq 6$. Thus,

$$4 = r(A) < b(A) = 6.$$

But, for the adjacency matrix A of the digraph D, given in Figure 1.1, we have

$$3 = r(A) > b(A) = 2.$$

We can summarize the basic relationships between the ranks in the following way. If A is a {0,1}-matrix, then

$$b(A) \leq r_{z^+}(A) \leq t(A) \qquad \text{and} \qquad r(A) \leq r_{z^+}(A) \leq t(A).$$

2. Classes With Equal Ranks

2.1 Bipartite graphs

A *domino* is a 6-cycle with exactly one chord and this chord creates two four-cycles. See figure 2.1. Bipartite *domino-free* graphs are bipartite graphs without any induced subgraphs isomorphic to a domino [AVJ98].

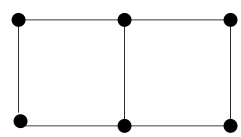

Figure 2.1. Domino.

A recent breakthrough concerning the problem of determining what bipartite graphs B have the property that $bc(B) = bp(B)$ is the following result.

Theorem 2.1.1 (Amilhastre et al. [AVJ98]) *If B is a bipartite domino-free graph, then $bc(B) = bp(B)$.*

A bipartite *C4-free* graph is a bipartite graph without any induced subgraphs isomorphic to a four-cycle. By definition, a bipartite C4-free graph is necessarily domino-free. Corollary 2.1.2 follows from the fact that every set of independent 1's is isolated in an adjacency matrix corresponding to a C4-free bipartite graph.

Corollary 2.1.2 *If A is the reduced adjacency matrix of a C4-free bipartite graph B, then*

$$r(A) \leq b(A) = r_{z^+}(A) = t(A).$$

The following example shows that we can have $r(A) < t(A)$ if A is the reduced adjacency matrix of a C4-free bipartite graph. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

$B(A)$ is C4-free and $3 = r(A) < b(A) = r_{z^+}(A) = t(A) = 4$.

Open problem: If $B(A)$ is a C4-free bipartite graph, find a lower bound for

$$\frac{r(A)}{b(A)}.$$

Recall, if A is the reduced adjacency matrix of a domino-free bipartite graph, then

$$r(A) \leq b(A) = r_{z^+}(A) \leq t(A).$$

The following two examples show there exists a class of domino-free bipartite graphs with

$$\frac{r(A)}{b(A)} = \frac{r(A)}{t(A)} = \frac{3}{4}$$

and a class with

$$\frac{b(A)}{t(A)} = \frac{r(A)}{t(A)} = \frac{3}{4},$$

where A is the reduced adjacency matrix of any graph in the particular class.

First, let

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, F_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad \text{and}$$

$$A_1 = \begin{bmatrix} E_1 & F_1 & & \cdots & 0's \\ & E_1 & F_1 & & \\ \vdots & & & & \vdots \\ & & & & F_1 \\ 0's & & & \cdots & E_1 \end{bmatrix}.$$

Then A_1 is of order $n = 4m$, for some positive integer m. Since the 1's on the main diagonal of A_1 are isolated, $b(A_1) \geq n$ and hence

$$b(A_1) = t(A_1) = 4m.$$

Furthermore, $r(A_1) = 3m$ since

$$r(E_1) = r \begin{bmatrix} F_1 \\ E_1 \end{bmatrix} = 3.$$

Thus,

$$\frac{r(A_1)}{b(A_1)} = \frac{r(A_1)}{t(A_1)} = \frac{3}{4}.$$

Second, let

$$E_2 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & \mathbf{1} & 0 & 1 \\ 0 & 1 & 1 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 \end{bmatrix}, F_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and}$$

$$A_2 = \begin{bmatrix} E_2 & F_2 & & \cdots & 0's \\ & E_2 & F_2 & & \\ \vdots & & & & \vdots \\ & & & & F_2 \\ 0's & & & \cdots & E_2 \end{bmatrix}.$$

Then A_2 is of order $n = 4m$, for some positive integer m. Since the bold **1**'s in each of the F_2's are isolated, $b(A_2) \geq 3m$. Clearly, there exists a biclique covering of

$$B\left(\begin{bmatrix} E_2 & F_2 \end{bmatrix}\right)$$

consisting of three bicliques. Thus,

$$b(A_2) = 3m.$$

Furthermore, $r(A_2) = 3m$ since

$$r(A_2) = r\begin{bmatrix} F_2 \\ E_2 \end{bmatrix} = 3.$$

Since the bold **1**'s and the upper left 1 in the E_2's give an independent set in A_2 of cardinality $4m$, $t(A_2) = 4m$. Thus,

$$\frac{b(A_2)}{t(A_2)} = \frac{r(A_2)}{t(A_2)} = \frac{3}{4}.$$

Open problems: If $B(A)$ is a domino-free bipartite graph, find lower bounds for

$$\frac{r(A)}{b(A)}, \frac{b(A)}{t(A)}, \quad \text{and} \quad \frac{r(A)}{t(A)}.$$

2.2 Digraphs

A digraph is *strongly connected* if there is a directed path between any two vertices. A digraph is *strongly unipathic* if it has exactly one directed path between any two vertices.

Theorem 2.2.1 (Hefner et al. [HLM93]) *If D is a strongly unipathic digraph, then*

$$b(A(D)) = r(A(D)) = r_{z^+}(A(D)) = t(A(D)).$$

Hefner et al. [HLM93] showed that the simultaneous value of the ranks of strongly unipathic digraphs of order n is any number in the range 2 though n. Each of these values can be realized for any $n > 2$. A strongly connected digraph is called *minimally strong* if the removal of any arc results in a digraph that is not strongly connected. Lundgren and Maybee [LM91] showed that minimally strong digraphs are a subclass of a class of digraphs, called directed cockades, for which all four matrix ranks are equal and every biclique is a claw. Theorem 2.2.2 and Lemma 2.2.3 follow from this work.

Theorem 2.2.2 (Lundgren and Maybee [LM91]) *If D is a minimally strong digraph, then*

$$b(A(D)) = r(A(D)) = r_{z^+}(A(D)) = t(A(D)).$$

Lemma 2.2.3 (Lundgren and Maybee [LM91]) *If A is the adjacency matrix of a minimally strong digraph, then $B(A)$ is a C4-free, hence domino-free, bipartite graph.*

Lemma 2.2.4 *If D is a strongly connected unipathic digraph, then D is minimally strong.*

Proof: Suppose D is a strongly unipathic digraph. Suppose e is an arc in D. Then e is in a directed path from some vertex v_i to some other vertex v_j. If e is removed, there is no directed path from v_i to v_j. Thus, D is minimally strong. ■

The digraph in figure 2.2 is minimally strong, but not unipathic, so the converse of Lemma 2.2.4 is false. Lemma 2.2.4, and the fact that the converse is not true, will be useful in section 3.1.

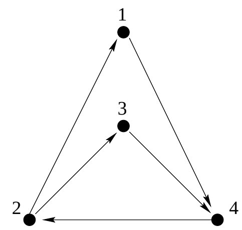

Figure 2.2. Digraph that is minimally strong but not unipathic.

Corollary 2.2.5 *If A is the adjacency matrix of a strongly unipathic digraph, then $B(A)$ is a $C4$-free, hence domino-free, bipartite graph.*

2.3 Examples With Dominos

Recall, $bp(G) = bc(G)$ if G is a domino-free bipartite graph or if G is a unipathic or minimally strong digraph. From Lemma 2.2.3 and Corollary 2.2.5, $B(A(G))$ is domino-free for these three classes of graphs. The following two examples show that we can have $bp(B) = bc(B)$ with B containing a large number of dominos relative to n. Furthermore, both examples have subclasses with

$$b(A(B)) = r(A(B)) = r_{z^+}(A(B)) = t(A(B)).$$

The first example is straight forward to construct from a bipartite graph and the resulting graph is such that $bp(B) = bc(B) = n$. Let B' be a bipartite graph with disjoint vertex sets $X = \{x_1, x_2, .., x_n\}$ and $Y = \{y_1, y_2, .., y_n\}$ where $n = 3m + 1$. Suppose B' has the form given in figure 2.3.

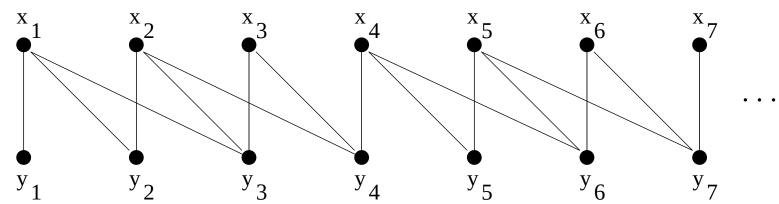

Figure 2.3. Bipartite graph with $bc(B') = bp(B') = n$ and more than $\lfloor \frac{n-1}{3} \rfloor$ dominos.

Now construct a bipartite graph B, with the same bipartition sets as B', by adding any edges to B' with the following two restrictions:

(1) If (x_i, y_j), $i \neq j$, is an edge in B, then (x_j, y_i) is not an edge in B.

(2) (x_{3j}, y_{3j-1}) and (x_{3j-2}, y_{3j+1}) are not an edges in B for $j \geq 1$.

B contains at least m dominos and $bc(B) = bp(B) = n$ since the set $\{(x_i, y_i) : i = 1, 2, ..., n\}$ is a perfect matching of B with no two edges in a four-cycle. Furthermore, the $B's$'s give a class of bipartite graphs with at least m dominos and

$$b(A(B')) = r(A(B')) = r_{z^+}(A(B')) = t(A(B')) = n.$$

The second example is not as straight forward as the first example, but its properties are more significant. For each k, $3 \leq k \leq n = 5m$, m any positive integer, we will construct a bipartite graph B_k on $2n$ vertices with at least m dominos,

$$b(A(B_k)) = r(A(B_k)) = r_{z^+}(A(B_k)) = t(A(B_k)) = k$$

for $k = 5j + i$, $i \in \{0, 1\}$, $j \in \{1, 2, 3, ...\}$, and

$$b(A(B_k)) = r(A(B_k)) = r_{z^+}(A(B_k)) = k$$

for all k. Furthermore, for each $n = 5m$, the graphs will have $11m$ edges for all k. We will construct these graphs from a matrix perspective. Let

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}, C_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and O be the 5×5 zero matrix. Let $n = 5m$, m a positive integer. For each k, $3 \leq k \leq n$, define the $5m \times 5m$ matrix $A_k = A_{k,m} = A(B_k)$ inductively.

$$A_{3,m} = \begin{bmatrix} C_1 & O & \cdots & O \\ C_1 & O & \cdots & O \\ \vdots & \vdots & & \vdots \\ C_1 & O & \cdots & O \end{bmatrix}, A_{4,m} = \begin{bmatrix} C_2 & O & \cdots & O \\ C_2 & O & \cdots & O \\ \vdots & \vdots & & \vdots \\ C_2 & O & \cdots & O \end{bmatrix},$$

$$A_{5,m} = \begin{bmatrix} C_3 & O & \cdots & O \\ C_3 & O & \cdots & O \\ \vdots & \vdots & & \vdots \\ C_3 & O & \cdots & O \end{bmatrix}, A_{6,m} = \begin{bmatrix} C_3 & O & O & \cdots & O \\ C_4 & & O & \cdots & O \\ \vdots & & \vdots & & \vdots \\ C_4 & & O & \cdots & O \end{bmatrix},$$

$$A_{7,m} = \begin{bmatrix} C_3 & O & O & \cdots & O \\ C_5 & & O & \cdots & O \\ \vdots & & \vdots & & \vdots \\ C_5 & & O & \cdots & O \end{bmatrix}, \quad \text{and}$$

$$A_{k,m} = \begin{bmatrix} C_3 & O & \cdots & O \\ O & & & \\ \vdots & & A_{k-5,m-1} & \\ O & & & \end{bmatrix} \quad \text{for} \quad k \geq 8.$$

Since $A_{k,m}$ is lower triangular, the k 1's on the diagonal form a set of k isolated 1's. Thus, $b(A_{k,m}) \geq k$. Using the inductive construction of $A_{k,m}$ it is not difficult, but tedious, to find a partition of $B_k = B(A_{k,m})$ consisting of k bicliques. Hence, $r_{z+}(A_{k,m}) \leq k$ for $3 \leq k \leq n$. This implies that $b(A_{k,m}) = r_{z+}(A_{k,m}) = k$. When the first k entries on the main diagonal are nonzero, they will form a set of independent 1's of largest cardinality in $A_{k,m}$. Thus,

$$t(A_{k,m}) = k \quad \text{for} \quad k = 5j + i, i \in \{0, 1\}, j \in \{1, 2, 3, ...\}.$$

Using the inductive construction of $A_{k,m}$, it is easy to verify that $r(A_{k,m}) = k$ for $k \geq 3$. In matrix form, a domino is a permutation of the submatrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Such submatrices occur at least m times in $A_{k,m}$. Thus, $B_k = B(A_k) = B(A_{m,k})$ has the desired properties. Note that for small k, the above construction gives a bipartite graph with many isolated vertices. These isolated vertices were included so the class could be constructed inductively. If $k \geq n - 3$, there are no isolated vertices.

3. Matrices

3.1 Nearly Reducible Matrices

The {0,1}-matrix A is *reducible* if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_1 & O \\ A_{21} & A_2 \end{bmatrix}$$

where A_1 and A_2 are square of order at least one and O is the zero matrix. If A is not reducible, then A is *irreducible*. An irreducible matrix A is called *nearly reducible* provided the replacement of any 1 with a 0 results in a reducible matrix. The matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

is an example of a nearly reducible matrix. The matrix

$$\begin{bmatrix} 0 & 1 & 0 & \mathbf{1} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

is irreducible, but not nearly reducible since the **1** in position (1,4) can be changed to a 0 without resulting in a reducible matrix. The following theorem is a well known result that was given in [BR92].

Theorem 3.1.1 *The matrix A is irreducible if and only if the digraph $D(A)$ is strongly connected.*

Corollary 3.1.2 *The matrix A is nearly reducible if and only if the digraph $D(A)$ is minimally strong.*

Corollary 3.1.3 follows from Corollary 3.1.2 and Theorem 2.2.2.

Corollary 3.1.3 *If A is a nearly reducible matrix, then*

$$b(A) = r(A) = r_{z^+}(A) = t(A).$$

The following example shows that, for any $n \geq 2$, we can construct a nearly reducible matrix where the simultaneous value of the ranks is k, for any k, $2 \leq k \leq n$. Let

$$A = \begin{bmatrix} B & C \\ O & D \end{bmatrix}$$

where B is an $(n-k+2) \times (n-k+1)$ matrix of the form

$$B = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

C is an $(n-k+2) \times (k-1)$ matrix of the form

$$C = \begin{bmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

D is an $(k-2) \times (k-1)$ matrix of the form

$$D = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

and O is the $(k-2) \times (n-k+1)$ zero matrix. The matrix A is nearly reducible since the digraph $D(A)$ is minimally strong. See figure 3.1. Clearly, $r(A) = k$. Hence,

$$b(A) = r_{z^+}(A) = r(A) = t(A) = k.$$

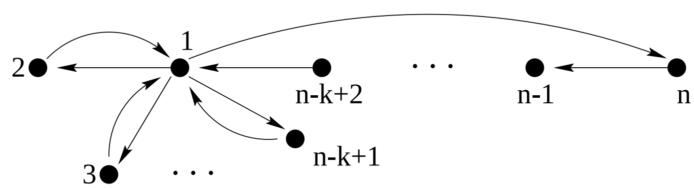

Figure 3.1. Minimally strong digraph with $b(A(D)) = r_{z^+}(A(D)) = r(A(D)) = t(A(D)) = k$.

Note that $D(A)$ is also a strongly unipathic digraph. Hence, this example shows that for every k, $2 \leq k \leq n$, we can find a strongly unipathic digraph on n vertices such that

$$b(A(D)) = r_{z^+}(A(D)) = r(A(D)) = t(A(D)) = k.$$

3.2 Nearly Decomposable Matrices

3.2.1 Preliminaries The {0,1}-matrix A is *partly decomposable* if there exists an integer k with $1 \leq k \leq n-1$ and permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} B & O \\ D & C \end{bmatrix}$$

where O is the $k \times (n-k)$ zero matrix. A is *fully indecomposable* if it is not partly decomposable. A fully indecomposable matrix is called *nearly decomposable* provided the replacement of any 1 with a 0 results in a partly decomposable matrix [BR92]. The matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

is an example of a nearly decomposable matrix. The matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & \mathbf{1} \end{bmatrix}$$

is fully indecomposable, but not nearly decomposable since the **1** in position (4,4) can be changed to a 0 without resulting in a partly decomposable matrix. Lemma 3.2.1 summarizes some properties of these matrices as discussed by Brualdi and Ryser in [BR92]. Lemma 3.2.2 gives a sufficient condition for a fully indecomposable matrix to be nearly decomposable.

Lemma 3.2.1 (Brualdi and Ryser [BR92]) *Let A be a $\{0,1\}$-matrix of order n. A is partly decomposable if and only if it has a minimum line cover other than the all rows cover or the all columns cover. Hence, A is fully indecomposable if and only if $t(A) = n$. Furthermore, every line of a fully indecomposable matrix has at least two ones.*

Lemma 3.2.2 *Let A be a fully indecomposable matrix. If $a_{ij} = 1$ implies row i or column j (or both) has exactly two 1's, then A is nearly decomposable*

Proof: Suppose A is a fully indecomposable matrix, $a_{ij} = 1$, and row i or column j has exactly two 1's. Without loss of generality, assume $a_{ij} = a_{ik} = 1$ and $a_{it} = 0$ for $1 \leq t \leq n$, $t \neq j$, and $t \neq k$.

$$\begin{array}{c} \\ i \end{array} \begin{array}{c} \begin{array}{ccccccccc} & & j & & & & k & & \end{array} \\ \left[\begin{array}{ccccccccc} & & \vdots & & & & \vdots & & \\ \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots \\ & & \vdots & & & & \vdots & & \end{array} \right] \end{array}$$

If $a_{ij} = 1$ is changed to $a_{ij} = \mathbf{0}$, the resulting matrix can be covered with column k and all rows except row i.

$$\begin{array}{c} \\ i \end{array} \begin{array}{c} \begin{array}{ccccccccc} & & j & & & & k & & \end{array} \\ \left[\begin{array}{ccccccccc} & & \vdots & & & & \vdots & & \\ \cdots & 0 & \mathbf{0} & 0 & \cdots & 0 & 1 & 0 & \cdots \\ & & \vdots & & & & \vdots & & \end{array} \right] \end{array}$$

Thus, A has a line cover that is not the all rows cover or the all columns cover. Therefore, by Lemma 3.2.1, A is nearly decomposable. ■

Theorem 3.2.3 shows how irreducible and fully indecomposable matrices are related. This will be used to verify matrices are fully indecomposable and to find bipartite graph representations of irreducible and nearly reducible matrices. The inductive structure given in Lemma 3.2.4 will be used to determine the boolean rank of nearly decomposable matrices.

Theorem 3.2.3 (Brualdi and Ryser [BR92]) *Let A be a $\{0,1\}$-matrix of order n. Let $A^\sharp$ be the matrix obtained from A by replacing each entry on the main diagonal with a 1. Then A is irreducible if and only if $A^\sharp$ is fully indecomposable.*

Lemma 3.2.4 (Brualdi and Ryser [BR92]) *Let A be a nearly decomposable $\{0,1\}$-matrix of order $n \geq 2$. Then there exists permutation matrices P and Q of order n and an integer m with $1 \leq m \leq n-1$ such that*

$$PAQ = \left[\begin{array}{cccccc|c} 1 & 0 & 0 & \cdots & 0 & 0 & \\ 1 & 1 & 0 & \cdots & 0 & 0 & \\ 0 & 1 & 1 & \cdots & 0 & 0 & F_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 & \\ 0 & 0 & 0 & \cdots & 1 & 1 & \\ \hline & & F_2 & & & & B \end{array}\right]$$

where B is a nearly decomposable matrix of order m. The matrix F_1 contains a unique 1 and it belongs to its first row and column j for some j with $1 \leq j \leq m$.

The matrix F_2 contains a unique 1 and it belongs to its last column and row i for some i with $1 \leq i \leq m$. If $m \geq 2$, then $m \neq 2$ and the element in position (i, j) of B is 0.

3.2.2 Ranks In this section we will discuss the term rank, boolean rank, nonnegative integer rank, and real rank of nearly decomposable matrices. The term rank, boolean rank, and nonnegative integer rank of a nearly decomposable matrix is equal to the order of the matrix; however, the real rank can be less. The term rank of a fully indecomposable matrix of order n is equal to n [BR92]. Thus, the term rank of a nearly decomposable matrix of order n is equal to n. This is stated in Lemma 3.2.5. This result will be used in determining the boolean rank and nonnegative integer rank of nearly decomposable matrices given in Theorem 3.2.6.

Lemma 3.2.5 (Brualdi and Ryser [BR92]) *If A is a nearly decomposable matrix, then $t(A) = n$.*

Theorem 3.2.6 *If A is a nearly decomposable matrix of order $n \geq 3$, then*

$$b(A) = r_{z^+}(A) = n.$$

Proof: To prove the theorem, we will use induction on n to show that if A is a nearly decomposable matrix of order $n \geq 3$, then A does not contain the submatrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Let A be a nearly decomposable {0,1}-matrix of order $n = 3$. By Lemma 3.2.4, there exists permutation matrices P and Q such that

$$PAQ = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 0 \\ \hline 0 & 1 & 1 \end{array}\right].$$

Thus, A does not contain the submatrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Now assume that any nearly decomposable matrix of order k, $3 \leq k < n$, does not contain the submatrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Let A be a nearly decomposable {0,1}-matrix of order $n \geq 3$. By Lemma 3.2.4, there exists permutation matrices P and Q of order n and an integer m with

$1 \leq m \leq n-1$ such that

$$PAQ = \left[\begin{array}{cccccc|c} 1 & 0 & 0 & \cdots & 0 & 0 & \\ 1 & 1 & 0 & \cdots & 0 & 0 & \\ 0 & 1 & 1 & \cdots & 0 & 0 & F_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 & \\ 0 & 0 & 0 & \cdots & 1 & 1 & \\ \hline & & F_2 & & & & B \end{array}\right]$$

where B is a nearly decomposable matrix of order m. The matrix F_1 contains a unique 1 and it belongs to its first row and column j for some j with $1 \leq j \leq m$. The matrix F_2 contains a unique 1 and it belongs to its last column and row i for some i with $1 \leq i \leq m$. If $m \geq 2$, then $m \neq 2$ and the element in position (i, j) of B is 0. By the induction hypothesis, B does not contain the submatrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

If A contained the submatrix

$$\begin{bmatrix} 1 & \mathbf{1} \\ \mathbf{1} & 1 \end{bmatrix},$$

then the unique 1 in F_1 and the unique 1 in F_2 must be the upper right **1** and lower left **1**, respectively, of this submatrix. This implies that $m = n - 1 \geq 2$ and hence the element in position (i, j) of B is 0. Thus, A cannot contain the submatrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

By Lemma 3.2.5, A has a set of n independent 1's. Since A does not contain any submatrices of the form

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

every set of independent 1's is isolated. Therefore, by Corollary 1.2.6,

$$b(A) = n. \quad \blacksquare$$

Corollary 3.2.7 *If A is a nearly decomposable matrix of order $n \geq 3$, then*

$$b(A) = r_{z^+}(A) = t(A) = n.$$

The only nearly decomposable matrices of order one or two are

$$A_1 = \begin{bmatrix} 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

respectively. Trivially, $t(A_2) = 2$ and

$$b(A_1) = r_{z^+}(A_1) = r(A_1) = t(A_1) = b(A_2) = r_{z^+}(A_2) = r(A_2) = 1.$$

The real rank of a nearly decomposable matrix of order n can be less than n. The following examples show that $r(A) = n, n-1$, and $n-2$ can be obtained for a nearly decomposable matrix A of order n. Let

$$A_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

The digraph $D(A_n - I)$ is strongly connected since it is a cycle of length n. Thus, by Theorem 3.1.1, $A_n - I$ is irreducible. Hence, by Theorem 3.2.3, A_n is fully indecomposable. Finally, by Lemma 3.2.2, A_n is nearly decomposable since every row and column of A_n has exactly two 1's. If n is odd, then $r(A_n) = n$. If n is even, then $r(A_n) = n - 1$ since the last row is a linear combination of the first $n - 1$ rows.

The numerous examples of nearly decomposable matrices of order n we constructed all had real rank n or $n - 1$. This lead us to conjecture that the real rank of a nearly decomposable matrix is at least $n - 1$. However, the proof of this was illusive and we now know it is not true. Finding an example of a nearly decomposable matrix A of order n with $r(A) \leq n - 2$ was difficult, but was finally accomplished in the following way. From Lemma 3.2.4, if A is a nearly decomposable matrix of order $n \geq 2$ there exists permutation matrices

P and Q of order n and an integer m with $1 \leq m \leq n-1$ such that

$$PAQ = \left[\begin{array}{cccccc|c} 1 & 0 & 0 & \cdots & 0 & 0 & \\ 1 & 1 & 0 & \cdots & 0 & 0 & \\ 0 & 1 & 1 & \cdots & 0 & 0 & F_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 & \\ 0 & 0 & 0 & \cdots & 1 & 1 & \\ \hline & & & F_2 & & & B \end{array}\right]$$

where B is a nearly decomposable matrix of order m. The matrix F_1 contains a unique 1 and it belongs to its first row and column j for some j with $1 \leq j \leq m$. The matrix F_2 contains a unique 1 and it belongs to its last column and row i for some i with $1 \leq i \leq m$. The converse of Lemma 3.2.4 is not true; however, the inductive structure gives insight regarding constructions that may give a nearly decomposable matrix. We construct a nearly decomposable matrix A of order n with $r(A) = n-2$ by letting

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 1 & 1 & 0 & \\ 0 & 1 & 1 & F_1 \\ \hline & F_2 & & B \end{array}\right]$$

where B is a nearly decomposable matrix of order m with $r(B) = m-1$. The unique 1's in F_1 or F_2, must be placed so that $a_{ij} = 1$ implies row i or column j (or both) has exactly two 1's. Furthermore, B is such that a vector in the left nullspace and a vector in the right nullspace of B have zeros in the positions corresponding to the row or column of the unique 1 in F_1 or F_2, respectively.

The smallest matrix B that we could find that satisfied these conditions is

$$B = \left[\begin{array}{ccc|ccc|ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right]$$

where

$$\left[\begin{array}{ccc|ccc|ccccccccc} -1 & 1 & -1 & 1 & -1 & 1 & 1 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & -1 \end{array}\right] B = \mathbf{0}$$

and

$$B \left[\begin{array}{c} 1 \\ -1 \\ 1 \\ \hline -1 \\ 1 \\ -1 \\ \hline -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{array}\right] = \mathbf{0}.$$

Now, let

$$A = \left[\begin{array}{ccc||ccc|ccc|ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right].$$

With the assistance of computer software, we can show that $r(A) = 16 = n-2$. We can use similar reasoning as we did with A_n to show that A is nearly decomposable. It is not difficult, but tedious, to show the digraph $D(A-I)$ is strongly connected. Thus, by Theorem 3.1.1, $A - I$ is irreducible. Hence, by Theorem 3.2.3, A is fully indecomposable. Finally, by Lemma 3.2.2, A is nearly decomposable since $a_{ij} = 1$ implies row i or column j (or both) has exactly two 1's. This construction has been extended to construct nearly decomposable matrices of orders $n = 36$ and $n = 54$ with real rank $n - 3 = 33$ and $n - 4 = 50$, respectively. This leads us to make the following conjecture.

Conjecture 3.2.8 *For every nonnegative integer k, there exists a nearly decomposable matrix of order $n = 18k$ with $r(A) = n - k - 1 = 17k - 1$. Thus, there exists a class of nearly decomposable matrices such that*

$$\frac{r(A)}{n} \to \frac{17}{18} \qquad as \qquad n \to \infty.$$

Open problems: If A is a nearly decomposable matrix, find a lower bound for the ratio

$$\frac{r(A)}{n}.$$

Find a characterization of nearly decomposable matrices with respect to their real rank. Specifically, classify nonsingular nearly decomposable matrices.

3.3 Bipartite Graph Representations

Every {0,1}-matrix is the reduced adjacency matrix of a bipartite graph. In this section we will study the bipartite graph representation of irreducible, nearly reducible, fully indecomposable, and nearly decomposable matrices. In [LP77], a bipartite graph is called *elementary* if it is connected and each edge is contained in a perfect matching. A *minimal elementary* bipartite graph is one in which the removal of any edge results in a bipartite graph which is not elementary.

Theorem 3.3.1 (Brualdi and Ryser [BR92]) *A $\{0,1\}$-matrix A is fully indecomposable if and only if $B(A)$ is an elementary bipartite graph.*

Corollary 3.3.2 (Brualdi and Ryser [BR92]) *A $\{0,1\}$-matrix A is nearly decomposable if and only if $B(A)$ is a minimal elementary bipartite graph.*

From the proof of Theorem 3.2.6, bipartite graphs with nearly decomposable adjacency matrices of order $n \geq 3$ are C4-free. Thus, we can use Corollaries 3.3.2 and 3.2.7 to give a class of domino-free bipartite graphs with

$$b(A(B)) = r_{z^+}(A(B)) = t(A(B)) = n.$$

This is stated in Corollary 3.3.3.

Corollary 3.3.3 *If A is the reduced adjacency matrix of order $n \geq 3$ corresponding to a minimal elementary bipartite graph, then*

$$b(A) = r_{z^+}(A) = t(A) = n.$$

One can find a similar characterization of bipartite graphs corresponding to irreducible and nearly reducible matrices. To do this, we will need the following definitions. Let B be a bipartite graph with bipartition $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_n\}$. A subgraph $\bar{B}$ of B will be called *uniform* if $x_i \in v(\bar{B})$ if and only if $y_i \in v(\bar{B})$. See Figure 3.2. B will be called *sub-elementary* if each edge is contained in a perfect matching of a uniform subgraph $\bar{B}$ of B. A *minimal sub-elementary* bipartite graph is one in which the removal of any edge results in a bipartite graph that is not sub-elementary.

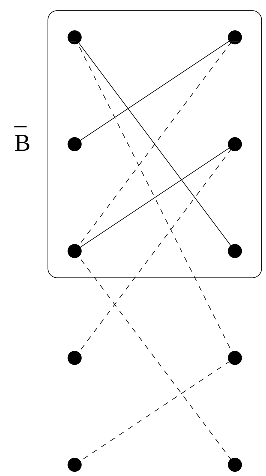

Figure 3.2. Perfect matching of a uniform subgraph $\bar{B}$.

Theorem 3.3.4 *Let A be a {0,1}-matrix. A is irreducible if and only if $B(A)$ is a sub-elementary bipartite graph.*

Proof: Suppose B is a sub-elementary bipartite graph with reduced adjacency matrix $A = A(B)$. Let $A^\sharp$ be the matrix obtained by replacing all the 0's on the main diagonal of A with 1's. Suppose $e \in e(B)$. Then e is an edge in a perfect matching M_H of some uniform subgraph $H \subseteq B$. M along with all edges (x_i, y_i) with $x_i \notin v(H)$ $(y_i \notin v(H))$ is a perfect matching of $B(A^\sharp)$. See Figure 3.3. $B(A^\sharp)$ is connected since $B(A)$ is sub-elementary. This implies that $B(A^\sharp)$ is an elementary bipartite graph. Hence, $A^\sharp$ is fully indecomposable. Thus, by Theorem 3.2.3, A is irreducible.

Conversely, let A be an irreducible matrix and $B = B(A)$ the corresponding bipartite graph with reduced adjacency matrix A. By Theorem 3.2.3, $A^\sharp$ is fully indecomposable and hence, by Theorem 3.3.1, $B(A^\sharp)$ is an elementary bipartite graph. Suppose $e \in e(B)$. Then $e \in e(B(A^\sharp))$ which implies e is in a perfect matching M of $B(A^\sharp)$. Removing all edges $(x_i, y_i) \notin B$ forms a perfect matching M_H of a uniform subgraph $H \subseteq B$. See Figure 3.3. Furthermore, $B(A^\sharp)$ is connected since it is elementary. Therefore, B is a sub-elementary bipartite graph. ■

For example, consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

As illustrated in Figure 3.4, every edge in $B = B(A)$ is in a perfect matching of one or more of the uniform subgraphs B_1, B_2, B_3, or B_4. Thus, by Theorem

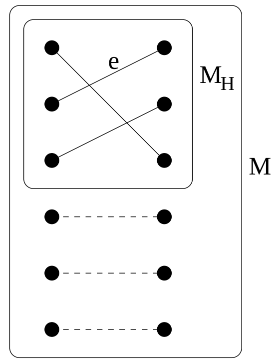

Figure 3.3. Perfect matchings M_H of $H \subseteq B(A)$ and M of $B(A^\sharp)$.

3.3.4, A is irreducible. One can easily verify that A is irreducible by observing that the digraph $D(A)$ is strongly connected. See Theorem 3.1.1. Now note that B is not an elementary bipartite graph since not every edge is in a perfect matching of B. In particular, the edge between the first vertex and the fourth vertex cannot be in a perfect matching that involves the fifth vertex in each set of the bipartition. Thus, A is not fully indecomposable. One can also observe that A is partly decomposable since row three and columns one, two, four, and five cover all the 1's in A and hence A has a minimum line cover that is not the all rows cover or the all columns cover.

Corollary 3.3.5 *Let A be a $\{0,1\}$-matrix. A is nearly reducible if and only if $B(A)$ is a minimal sub-elementary bipartite graph.*

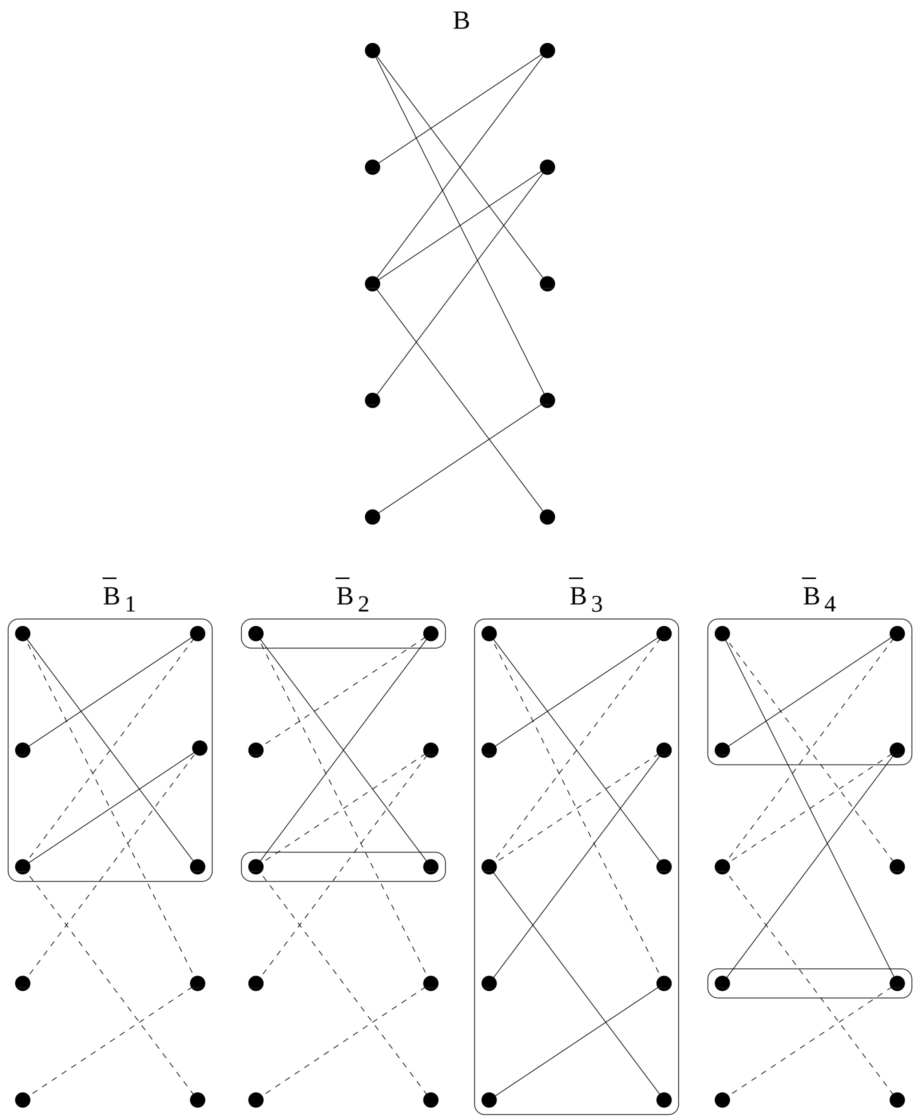

Figure 3.4. Every edge in the bipartite graph B is in a perfect matching of a uniform subgraph $\bar{B}_j$.

Corollary 3.3.6 follows from Corollaries 3.3.5 and 3.1.2. We can then use Corollaries 3.3.5, 3.3.6, and 3.1.3 and Lemma 2.2.3 to give a class of domino-free bipartite graphs with all four ranks equal. This is stated in Corollary 3.3.7.

Corollary 3.3.6 *Let A be a $\{0,1\}$-matrix. $B(A)$ is a minimal sub-elementary bipartite graph if and only if $D(A)$ is a minimally strong digraph.*

Corollary 3.3.7 *Let A be an adjacency matrix corresponding to a minimal sub-elementary bipartite graph, then*

$$b(A) = r(A) = r_{z^+}(A) = t(A).$$

Proposition 3.3.8 summarizes the relationships between irreducible and fully indecomposable matrices and their bipartite graph representations.

Proposition 3.3.8 *Let A be a square $\{0,1\}$-matrix. Let $A^\sharp$ be the matrix obtained from A by replacing each entry on the main diagonal with a 1. Then the following statements are equivalent.*

(1) $B(A)$ is a sub-elementary bipartite graph.

(2) A is an irreducible matrix.

(3) $A^\sharp$ *is a fully indecomposable matrix.*

(4) $B(A^\sharp)$ *is an elementary bipartite graph.*

Proof: (1) $\Leftrightarrow$ (2), (2) $\Leftrightarrow$ (3), and (3) $\Leftrightarrow$ (4) are Theorems 3.3.4, 3.2.3, and 3.3.1, respectively. Therefore, (1) $\Leftrightarrow$ (4). ■

4. Tournaments

4.1 Preliminaries

A digraph T is called a *tournament* if for each pair of vertices u and v, either (u, v) is an arc in T or (v, u) is an arc in T, but not both. We will call A an *n-tournament matrix* if A is the adjacency matrix corresponding to a tournament on n-vertices. $T(A)$ will denote the tournament with adjacency matrix A. A square $\{0,1\}$-matrix A is a tournament matrix if and only if $A + A^T + I = J$, where I is the identity matrix and J is the all 1's matrix. In this chapter we will discuss the ranks of various classes of tournament matrices, we will give several classes of tournament matrices with

$$b(A) = r(A) = r_{z^+}(A) = t(A),$$

and we will construct examples of tournament matrices to show that all of the above equalities can be strict inequalities. First, we will give some general results concerning the ranks of tournament matrices and a property of vectors in the null space of singular tournament matrices.

4.2 General Results

In 1991, de Caen found a lower bound for the real rank of an n-tournament matrix. See Theorem 4.2.1. This was a significant result, not only because it was an open problem for many years, but because it allows only two possible values for the real rank, n or $n-1$. Hence, the nonnegative integer rank and the term rank are also n or $n-1$, since the real rank is a lower bound for these two ranks. See Corollary 4.2.2.

Theorem 4.2.1 (de Caen [dC91]) *If A is an n-tournament matrix, then*

$$r(A) \geq n - 1.$$

Corollary 4.2.2 *If A is an n-tournament matrix, then*

$$n - 1 \leq r(A) \leq r_{z^+}(A) \leq t(A) \leq n.$$

Corollary 4.2.3 *For every tournament matrix, at least two ranks are equal. Specifically, if A is an n-tournament matrix, then*

$$r_{z^+}(A) = r(A) = n - 1 \qquad \textit{or} \qquad r_{z^+}(A) = t(A) = n$$

Open problem: Classify singular tournaments as mentioned by de Caen [dC91]. See Katzenberger et. al [KS90] for some results on the form of a vector spanning the null space of a singular tournament.

From Corollary 4.2.2, we know that three of the ranks are either n or $n-1$. Classifying those tournament matrices with $r(A) = n-1$ and those tournament matrices with $r_{z^+}(A) = n-1$ are very difficult unsolved problems. However, we do know which tournament matrices have $t(A) = n-1$ and this classification is given in Theorem 4.2.4.

Theorem 4.2.4 *Let A be an n-tournament matrix. Then $t(A) = n-1$ if and only if there exists a strongly connected component of $T(A)$ containing exactly one vertex. Hence, $t(A) = n$ if and only if every strongly connected component of $T(A)$ has three or more vertices.*

Proof: Tournaments can be partitioned into strongly connected components $C_1, C_2, ..., C_k$ such that there is an arc from $u \in C_i$ to $v \in C_j$ if and only if $i < j$. Thus, there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_1 & & & 1's \\ & A_2 & & \\ & & \ddots & \\ 0's & & & A_k \end{bmatrix}$$

where A_j, $1 \leq j \leq k$, is the adjacency matrix of order n_j corresponding to the component C_j.

Suppose $t(A) = n-1$ and $n_j > 1$ for $1 \leq j \leq k$. Since C_j is a strongly connected tournament on three or more vertices, C_j contains a Hamiltonian cycle [Cam59, Fou59]. This implies that $t(A_j) = n_j$ for $1 \leq j \leq k$ and hence $t(PAP^T) = n$. This contradicts $t(A) = n - 1$. Thus, there exists a strongly connected component of $T(A)$ containing exactly one vertex.

Conversely, suppose $n_j = 1$ for some j. Then the submatrix A_j is a 0 on the diagonal of PAP^T. Suppose this 0 is in the position (m, m) of PAP^T. Then the first $m - 1$ rows and the last $n - m$ columns form a line cover of PAP^T. Thus,

$$t(A) = t(PAP^T) \leq m - 1 + n - m = n - 1.$$

Therefore, by Corollary 4.2.2, $t(A) = n - 1$. ■

Since $r(A) = n$ or $n - 1$ for an n-tournament matrix A, we know the null space of a singular tournament matrix is spanned by one vector. Maybee et. al [MP90] showed that if a vector spans the null space of a tournament matrix, then the sum of the squares of its components equals the square of the sum. This is called the *admissible* property and it will be used to prove that the real rank of a regular n-tournament is n.

Lemma 4.2.5 (Maybee and Pullman [MP90]) *If* $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ *spans the null space of an* n*-tournament matrix, then*

$$\left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{i=1}^{n} {x_i}^2.$$

4.3 Regular and Near-Regular Tournaments

A tournament is *regular* if all its vertices have equal scores. Thus, a *regular* n-tournament matrix is a tournament matrix with equal row sums. If A is a regular n-tournament, then n must be odd. A tournament is *near-regular* if the maximum difference between its scores is 1. Thus, a *near-regular* n-tournament matrix is a tournament matrix where half of the row sums equal $\frac{n}{2} - 1$ and half of the row sums equal $\frac{n}{2}$. If A is a near-regular n-tournament, then n must be even. There have been several proofs that regular tournament matrices are nonsingular, including de Caen [dC91] and Maybee and Pullman [MP90]. We give the following simple proof that uses Lemma 4.2.5.

Theorem 4.3.1 *A regular tournament matrix is nonsingular.*

Proof: Let A be a regular n-tournament matrix and let $\mathbf{v} = (v_1, v_2, ..., v_n)^T$ be a vector such that $A\mathbf{v} = \mathbf{0}$. Then

$$A \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \sum_{i \in A_1} v_i \\ \vdots \\ \sum_{i \in A_n} v_i \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $A_j = \{i : j \to i\}$. This implies that

$$\sum_{i \in A_1} v_i + ... + \sum_{i \in A_n} v_i = 0.$$

In the above sum, each v_i must appear exactly $\frac{n-1}{2}$ times which corresponds to the number of vertices which beat vertex i. Thus,

$$\frac{n-1}{2}(v_1 + ... + v_n) = 0$$

$$\Rightarrow \sum_{i=1}^{n} v_i = 0$$

$$\Rightarrow \left(\sum_{i=1}^{n} v_i \right)^2 = 0$$

$$\Rightarrow \sum_{i=1}^{n} v_i^2 = 0$$

$$\Rightarrow \mathbf{v} = \mathbf{0}.$$

Therefore, A is nonsingular. ■

Corollary 4.3.2 *If A is a regular n-tournament matrix, then*

$$r(A) = r_{z^+}(A) = t(A) = n.$$

The proof that near-regular tournament matrices are nonsingular is not as straight forward and it requires Theorem 4.3.3. Theorem 4.3.3 could also be used to prove nonsingularity for regular tournament matrices.

Theorem 4.3.3 (Katzenberger and Shader [KS90]) *Let* $\mathbf{s} = (s_1, s_2, ..., s_n)^T$ *be a score vector. If*

$$|\mathbf{s}|^2 < \frac{n^2(n-1)}{4},$$

then every n*-tournament with this score vector is nonsingular.*

Theorem 4.3.4 *For* $n \geq 4$*, a near-regular* n*-tournament matrix is nonsingular.*

Proof: The score vector of a near-regular n-tournament matrix is some permutation of

$$\mathbf{s} = (\frac{n}{2} - 1, ..., \frac{n}{2} - 1, \frac{n}{2}, ..., \frac{n}{2})^T$$

where each of the two different terms occurs exactly $\frac{n}{2}$ times. Thus,

$$|\mathbf{s}|^2 = \frac{n}{2}(\frac{n}{2} - 1)^2 + \frac{n}{2}(\frac{n}{2})^2 = \frac{n^3 + 2n - 2n^2}{4}$$

Now $n^3 + 2n - 2n^2 < n^3 - n^2$ for $n \geq 4$. Thus,

$$|\mathbf{s}|^2 < \frac{n^2(n-1)}{4}.$$

Therefore, by Theorem 4.3.3, near-regular tournament matrices are nonsingular. ■

Open problem: Is the boolean rank of a regular or near-regular n-tournament matrix, $n \geq 3$, equal to n as well?

4.4 Boolean Rank

By Corollary 4.2.2, the real, nonnegative integer, and term rank of an n-tournament matrix are each either n or $n-1$. However, the boolean rank can be much smaller. Bain et al. [BLM92] constructed the following tournament.

Let T be the tournament defined by

$$T = \begin{bmatrix} T_k & J-Q_{m-1}^T & J-Q_{m-2}^T & \cdots & J-Q_2^T & J-Q_1^T \\ Q_{m-1} & T_k & J-Q_{m-1}^T & \cdots & J-Q_3^T & J-Q_2^T \\ Q_{m-2} & Q_{m-1} & T_k & \cdots & J-Q_4^T & J-Q_3^T \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_2 & Q_3 & Q_4 & \cdots & T_k & J-Q_{m-1}^T \\ Q_1 & Q_2 & Q_3 & \cdots & Q_{m-1} & T_k \end{bmatrix}$$

where $k = m^2+m+1$, P is the permutation matrix of order k given by

$$P = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

and

$$
\begin{aligned}
A_1 &= P^{m^2} \\
A_2 &= P^{m^2-m-1} + P^{m^2-m} \\
&\vdots \\
A_i &= P^{m^2-(i-1)m-(i-1)} + P^{m^2-(i-1)m-(i-2)} + \cdots + P^{m^2-(i-1)m} \\
&\vdots \\
A_{m-1} &= P^{m+2} + P^{m+3} + \cdots + P^{2m} \\
A_m &= P + P^2 + \cdots + P^m \\
Q_1 &= A_1 \\
Q_2 &= A_1 + A_2 \\
&\vdots \\
Q_{m-1} &= A_1 + A_2 + \cdots + A_{m-1} \\
T_k &= A_1 + A_2 + \cdots + A_m.
\end{aligned}
$$

For example, if $n = mk = (2)(7) = 14$, then

$$T = \begin{bmatrix} 0&1&1&0&1&0&0&1&1&1&0&1&1&1 \\ 0&0&1&1&0&1&0&1&1&1&1&0&1&1 \\ 0&0&0&1&1&0&1&1&1&1&1&1&0&1 \\ 1&0&0&0&1&1&0&1&1&1&1&1&1&0 \\ 0&1&0&0&0&1&1&0&1&1&1&1&1&1 \\ 1&0&1&0&0&0&1&1&0&1&1&1&1&1 \\ 1&1&0&1&0&0&0&1&1&0&1&1&1&1 \\ 0&0&0&0&1&0&0&0&1&1&0&1&0&0 \\ 0&0&0&0&0&1&0&0&0&1&1&0&1&0 \\ 0&0&0&0&0&0&1&0&0&0&1&1&0&1 \\ 1&0&0&0&0&0&0&1&0&0&0&1&1&0 \\ 0&1&0&0&0&0&0&0&1&0&0&0&1&1 \\ 0&0&1&0&0&0&0&1&0&1&0&0&0&1 \\ 0&0&0&1&0&0&0&1&1&0&1&0&0&0 \end{bmatrix}$$

$$= \begin{bmatrix} 0&1&1&0&1&0&0 \\ 0&0&1&1&0&1&0 \\ 0&0&0&1&1&0&1 \\ 1&0&0&0&1&1&0 \\ 0&1&0&0&0&1&1 \\ 1&0&1&0&0&0&1 \\ 1&1&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \end{bmatrix} \begin{bmatrix} 1&0&0&0&0&0&0&1&0&0&0&1&1&0 \\ 0&1&0&0&0&0&0&0&1&0&0&0&1&1 \\ 0&0&1&0&0&0&0&1&0&1&0&0&0&1 \\ 0&0&0&1&0&0&0&1&1&0&1&0&0&0 \\ 0&0&0&0&1&0&0&0&1&1&0&1&0&0 \\ 0&0&0&0&0&1&0&0&0&1&1&0&1&0 \\ 0&0&0&0&0&0&1&0&0&0&1&1&0&1 \end{bmatrix}.$$

Since $A = BC$ using boolean arithmetic where B is 14×7 and C is 7×14, $b(T) \leq 7$.

Lemma 4.4.1 (Bain et al. [BLM92]) *For each integer $m \geq 2$ and n sufficiently large,*

$$min\{b(T) : T \text{ is an } n \times n \text{ tournament matrix}\} \leq \frac{n}{m}.$$

Open problem: Can this bound be improved, or is it best possible?

Conjecture 4.4.2 *Let T be the tournament defined above. Then $A(T)$ is nonsingular and hence*

$$r(A(T)) = r_{z^+}(A(T)) = t(A(T)) = n.$$

It has not been proven that $A(T)$ is nonsingular; however, by Theorem 4.4.3, we know that $t(T) = n$. Also, using Lemma 4.4.4, it has been shown that $A(T)$ is nonsingular for all possible values of k and m up through $n = mk = (10)(111) = 1110$.

We will need the following terminology in the proof of the next theorem. The *diagonal* beginning in column j of a matrix A of order n is the n elements

$$A(1,j), A(2,j+1), \cdots, A(n-j,n), A(n-j+1,1), A(n-j+2,2), \cdots, A(n,j-1).$$

Theorem 4.4.3 *If T is the tournament defined above, then $t(A(T)) = n$.*

Proof: Let T be the tournament defined above. We will show that the diagonal that begins with column $m^2 + 1$ of T will give a set of n independent 1's. The 1's on any diagonal will be independent, so we just need to show that every element on this diagonal is a 1.

$$T_k = A_1 + A_2 + \cdots + A_m$$

where $A_1 = P^{m^2}$. Thus, the diagonal of T_k that begins with column $m^2 + 1$ will be all 1's.

$$T_k = \begin{array}{c} \\ \\ m+1 \\ \\ \end{array} \overset{\quad m^2+1}{\begin{bmatrix} \cdots & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & \vdots \end{bmatrix}}_{(m^2+m+1)\times(m^2+m+1)}.$$

The diagonal of Q_{m-1} that begins with column $m+2$ will be all zeros since P^{m+1} does not occur in the sum that gives Q_{m-1}. Thus, the diagonal of $J - Q^T_{m-1}$ that begins with row $m+2$ will be all 1's. Thus, we have

$$\left[\ T_K \ \middle|\ J - Q^T_{m-1}\ \right] =$$

$$\begin{array}{c} \\ \\ \\ m+2 \\ \\ \\ \end{array} \overset{m^2+1\qquad\qquad\qquad\qquad}{\left[\begin{array}{cccc|cccc} \cdots & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \cdots \end{array}\right]}_{(m^2+m+1)\times 2(m^2+m+1).}$$

Lastly, since

$$Q_1 = P^{m^2} = \begin{array}{c} \\ m+2 \end{array}\!\!\!\begin{array}{c} m^2+1 \\ \left[\begin{array}{cccccc} & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \end{array}\right]_{(m^2+m+1)\times(m^2+m+1)} \end{array}$$

has a diagonal of all 1's beginning with row $m+2$, the diagonal of T beginning with column m^2+1 will give $mk = m(m^2+m+1) = n$ independent 1's. Therefore, $t(T) = n$. ■

Lemma 4.4.4 *Let $N(M)$ be the number of 1's in each row and column of a regular $\{0,1\}$-matrix M and let*

$$T = \begin{bmatrix} T_k & J-Q_{m-1}^T & J-Q_{m-2}^T & \cdots & J-Q_2^T & J-Q_1^T \\ Q_{m-1} & T_k & J-Q_{m-1}^T & \cdots & J-Q_3^T & J-Q_2^T \\ Q_{m-2} & Q_{m-1} & T_k & \cdots & J-Q_4^T & J-Q_3^T \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_2 & Q_3 & Q_4 & \cdots & T_k & J-Q_{m-1}^T \\ Q_1 & Q_2 & Q_3 & \cdots & Q_{m-1} & T_k \end{bmatrix}$$

be the tournament defined above. Then $A(T)$ is nonsingular if $\bar{N} =$

$$\begin{bmatrix} N(T_k) & k-N(Q_{m-1}^T) & k-N(Q_{m-2}^T) & \cdots & k-N(Q_2^T) & k-N(Q_1^T) \\ N(Q_{m-1}) & N(T_k) & k-N(Q_{m-1}^T) & \cdots & k-N(Q_3^T) & k-N(Q_2^T) \\ N(Q_{m-2}) & N(Q_{m-1}) & N(T_k) & \cdots & k-N(Q_4^T) & k-N(Q_3^T) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ N(Q_2) & N(Q_3) & N(Q_4) & \cdots & N(T_k) & k-N(Q_{m-1}^T) \\ N(Q_1) & N(Q_2) & N(Q_3) & \cdots & N(Q_{m-1}) & N(T_k) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{m(m+1)}{2} & k-\frac{(m-1)m}{2} & k-\frac{(m-2)(m-1)}{2} & \cdots & k-3 & k-1 \\ \frac{(m-1)m}{2} & \frac{m(m+1)}{2} & k-\frac{(m-1)m}{2} & \cdots & k-6 & k-3 \\ \frac{(m-2)(m-1)}{2} & \frac{(m-1)m}{2} & \frac{m(m+1)}{2} & \cdots & k-10 & k-6 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 3 & 6 & 10 & \cdots & \frac{m(m+1)}{2} & k-\frac{(m-1)m}{2} \\ 1 & 3 & 6 & \cdots & \frac{(m-1)m}{2} & \frac{m(m+1)}{2} \end{bmatrix}$$

is nonsingular.

Proof: Let $N(M)$ be the number of 1's in each row and column of a regular {0,1}-matrix M and let $\bar{N} =$

$$\begin{bmatrix} N(T_k) & k-N(Q_{m-1}^T) & k-N(Q_{m-2}^T) & \cdots & k-N(Q_2^T) & k-N(Q_1^T) \\ N(Q_{m-1}) & N(T_k) & k-N(Q_{m-1}^T) & \cdots & k-N(Q_3^T) & k-N(Q_2^T) \\ N(Q_{m-2}) & N(Q_{m-1}) & N(T_k) & \cdots & k-N(Q_4^T) & k-N(Q_3^T) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ N(Q_2) & N(Q_3) & N(Q_4) & \cdots & N(T_k) & k-N(Q_{m-1}^T) \\ N(Q_1) & N(Q_2) & N(Q_3) & \cdots & N(Q_{m-1}) & N(T_k) \end{bmatrix}$$

Let

$$T = \begin{bmatrix} T_k & J-Q_{m-1}^T & J-Q_{m-2}^T & \cdots & J-Q_2^T & J-Q_1^T \\ Q_{m-1} & T_k & J-Q_{m-1}^T & \cdots & J-Q_3^T & J-Q_2^T \\ Q_{m-2} & Q_{m-1} & T_k & \cdots & J-Q_4^T & J-Q_3^T \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_2 & Q_3 & Q_4 & \cdots & T_k & J-Q_{m-1}^T \\ Q_1 & Q_2 & Q_3 & \cdots & Q_{m-1} & T_k \end{bmatrix}$$

be the tournament defined above. $N(T_k) = N(Q_m) = \frac{m(m+1)}{2}$ and $N(Q_j) = \frac{j(j+1)}{2}$ since Q_j is the sum of $\frac{j(j+1)}{2}$ distinct powers of the permutation matrix P. Let $\mathbf{v}$ be a vector such that $T\mathbf{v} = \mathbf{0}$. Consider the system of km equations and km unknowns given by $T\mathbf{v} = \mathbf{0}$. Adding equations 1 through k, $k+1$ through $2k$,...,$(m-1)k+1$ through mk will give the following system of m

equations and mk unknowns.

$$\begin{aligned}
0 &= \frac{m(m+1)}{2}(v_1+\cdots+v_k)+\cdots+(k-1)(v_{(m-1)k+1}+\cdots+v_{mk}) \\
0 &= \frac{(m-1)m}{2}(v_1+\cdots+v_k)+\cdots(k-3)(v_{(m-1)k+1}+\cdots+v_{mk}) \\
&\vdots \\
0 &= 3(v_1+\cdots+v_k)+\cdots+\left(k-\frac{(m-1)m}{2}\right)(v_{(m-1)k+1}+\cdots+v_{mk}) \\
0 &= (v_1+\cdots+v_k)+\cdots+\left(\frac{m(m+1)}{2}\right)(v_{(m-1)k+1}+\cdots+v_{mk})
\end{aligned}$$

This is equivalent to

$$\bar{N}\begin{bmatrix} v_1+\cdots+v_k \\ v_{k+1}+\cdots+v_{2k} \\ \vdots \\ v_{(m-1)k+1}+\cdots+v_{mk} \end{bmatrix} = \mathbf{0}.$$

If $\bar{N}$ is nonsingular, then

$$\bar{N}\begin{bmatrix} v_1+\cdots+v_k \\ v_{k+1}+\cdots+v_{2k} \\ \vdots \\ v_{(m-1)k+1}+\cdots+v_{mk} \end{bmatrix} = \mathbf{0}$$

implies

$$\begin{aligned}
0 &= v_1+\cdots+v_k \\
0 &= v_{k+1}+\cdots+v_{2k} \\
&\vdots \\
0 &= v_{(m-1)k+1}+\cdots+v_{mk}
\end{aligned}$$

Thus, by Lemma 4.2.5,

$$\left(\sum_{i=1}^{n} v_i\right)^2 = \sum_{i=1}^{n} {v_i}^2 = 0.$$

$$\Rightarrow \mathbf{v} = 0. \quad \blacksquare$$

Conjecture 4.4.5 *Let $N(M)$ be the number of 1's in each row and column of a regular {0,1}-matrix M and let*

$$T = \begin{bmatrix} T_k & J - Q_{m-1}^T & J - Q_{m-2}^T & \cdots & J - Q_2^T & J - Q_1^T \\ Q_{m-1} & T_k & J - Q_{m-1}^T & \cdots & J - Q_3^T & J - Q_2^T \\ Q_{m-2} & Q_{m-1} & T_k & \cdots & J - Q_4^T & J - Q_3^T \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_2 & Q_3 & Q_4 & \cdots & T_k & J - Q_{m-1}^T \\ Q_1 & Q_2 & Q_3 & \cdots & Q_{m-1} & T_k \end{bmatrix}$$

be the tournament defined above. Then $\bar{N}$ =

$$\begin{bmatrix} N(T_k) & k - N(Q_{m-1}^T) & k - N(Q_{m-2}^T) & \cdots & k - N(Q_2^T) & k - N(Q_1^T) \\ N(Q_{m-1}) & N(T_k) & k - N(Q_{m-1}^T) & \cdots & k - N(Q_3^T) & k - N(Q_2^T) \\ N(Q_{m-2}) & N(Q_{m-1}) & N(T_k) & \cdots & k - N(Q_4^T) & k - N(Q_3^T) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ N(Q_2) & N(Q_3) & N(Q_4) & \cdots & N(T_k) & k - N(Q_{m-1}^T) \\ N(Q_1) & N(Q_2) & N(Q_3) & \cdots & N(Q_{m-1}) & N(T_k) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{m(m+1)}{2} & k - \frac{(m-1)m}{2} & k - \frac{(m-2)(m-1)}{2} & \cdots & k-3 & k-1 \\ \frac{(m-1)m}{2} & \frac{m(m+1)}{2} & k - \frac{(m-1)m}{2} & \cdots & k-6 & k-3 \\ \frac{(m-2)(m-1)}{2} & \frac{(m-1)m}{2} & \frac{m(m+1)}{2} & \cdots & k-10 & k-6 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 3 & 6 & 10 & \cdots & \frac{m(m+1)}{2} & k - \frac{(m-1)m}{2} \\ 1 & 3 & 6 & \cdots & \frac{(m-1)m}{2} & \frac{m(m+1)}{2} \end{bmatrix}$$

is nonsingular.

4.5 Classes With All Four Ranks Equal

4.5.1 Rotational Tournaments A regular tournament is called a *rotational tournament* if its vertices can be labeled $0, 1, 2, ...r$ in such a way that for some subset S of $\{1, 2, \ldots, r\}$, vertex i beats $i + j \bmod (r+1)$ for all $j \in S$. For example,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

is an adjacency matrix corresponding to a rotational tournament on $r + 1 = n = 9$ vertices determined by the set $S = \{1, 4, 6, 7\}$.

Lemma 4.5.1 *Let T be a rotational tournament on $n = r+1$ vertices, $0, 1, 2, \ldots, r$, given by a subset S of $\{1, 2, \ldots, r\}$. Then $|S| = \frac{r}{2}$ and*

$$j \in S \quad \Leftrightarrow \quad r + 1 - j \notin S.$$

Proof: Let T be a rotational tournament on $n = r + 1$ vertices, $0, 1, 2, \ldots, r$, given by a subset S of $\{1, 2, \ldots, r\}$. $|S| = \frac{r}{2}$ since, by definition, T is a regular tournament. Suppose $j \in S$. Since T is a tournament, $A(T) + A(T)^T + I = J$

where

$$A(T) = \begin{array}{c} \\ 0 \\ \\ \\ j \\ \\ \\ r+1-j \\ \\ r \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & & & j & & & r+1-j & & r \end{array} \\ \left[\begin{array}{ccccccccc} 0 & & & 1 & & & 0 & & \\ & 0 & & & 1 & & & \ddots & \\ & & \ddots & & & \ddots & & & 0 \\ 0 & & & 0 & & & 1 & & \\ & 0 & & & 0 & & & \ddots & \\ & & \ddots & & & \ddots & & & 1 \\ 1 & & & 0 & & & 0 & & \\ & \ddots & & & \ddots & & & \ddots & \\ & & 1 & & & 0 & & & 0 \end{array} \right] \end{array}$$

and J is the all 1's matrix. Hence, $r+1-j \notin S$. Similarly, if $r+1-j \notin S$, then $j \in S$. ∎

Theorem 4.5.2 *Let T be a rotational tournament on $n = r+1$ vertices, $0, 1, 2, \ldots, r$, given by a subset S of $\{1, 2, \ldots, r\}$. If there exists a $j \in S$ such that*

$$j + k \textit{ mod } (r+1) \in S \quad \Leftrightarrow \quad j - k \textit{ mod } (r+1) \notin S$$

for all $k \in \{1, 2, \ldots, r\}$, then $bc(T) = n$.

Proof: Let T be a rotational tournament on $n = r+1$ vertices, $0, 1, 2, \ldots, r$, given by a subset S of $\{1, 2, \ldots, r\}$. Suppose there exists a $j \in S$ such that

$$j + k \text{ mod } (r+1) \in S \quad \Leftrightarrow \quad j - k \text{ mod } (r+1) \notin S$$

for all $k \in \{1, 2, \ldots, r\}$. Let W be the set of 1's of the adjacency matrix $A(T)$ corresponding to the arcs

$$i \to i + j \text{ mod } (r+1) \text{ for } i = 0, 1, \ldots r.$$

By definition, W consists of n independent 1's; hence, we just need to show that all the 1's in W are isolated. Suppose two 1's of W are in a submatrix of $A(T)$ of the form

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

First, suppose the bold **1**'s in

$$\begin{bmatrix} \mathbf{1} & 1 \\ 1 & \mathbf{1} \end{bmatrix}$$

are in W. By the definition of a rotational tournament, we can assume the top row or the left column of this submatrix is in the top row or left column, respectively, of $A(T)$. Without loss of generality, assume the top row of the submatrix is the top row of $A(T)$. There are two cases to consider. If $j - k \geq 0$, then

$$A(T) = \begin{array}{c} \\ \end{array}\begin{array}{cc|cc|c} 0 & j-k & j & j+k & \\ \end{array}$$

$$A(T) = \left[\begin{array}{cc|cc|c}
0 & 1 & \mathbf{1} & 1 & \\
\ddots & \ddots & \ddots & \ddots & \\
 & 0 & 1 & \mathbf{1} & 1 \\
\hline
 & \ddots & \ddots & \ddots & \\
 & & 0 & 1 & 1 \\
 & & \ddots & \ddots & \\
 & & & 0 & 1 \\
 & & & & \ddots \\
 & & & & 0
\end{array}\right]$$

and $j - k \in S$. This contradicts the assumption that $j - k \bmod (r+1) \notin S$.

If $j - k < 0$, then

$$A(T) = \begin{array}{c} \\ 0 \\ \\ k-j \\ \\ \\ \\ \\ \\ \\ \end{array}\left[\begin{array}{c|ccc|c}
\multicolumn{1}{c}{0} & j & & j+k & \\
\hline
0 & \mathbf{1} & & 1 & \\
\ddots & \ddots & & & \ddots \\
1 & 0 & 1 & & 1 \\
\ddots & \ddots & \ddots & & \\
 & 1 & 0 & \mathbf{1} & \\
\hline
 & \ddots & \ddots & \ddots & \\
 & & 1 & 0 & 1 \\
 & & & \ddots & \ddots \\
 & & & 1 & 0
\end{array}\right],$$

$k - j \notin S$, and $r + 1 - (k - j) \in S$. But,

$$r + 1 - (k - j) \equiv r + 1 + j - k \equiv j - k \bmod (r + 1).$$

This contradicts $j - k \bmod (r + 1) \notin S$.

Second, suppose the bold **1**'s in the submatrix

$$\begin{bmatrix} 1 & \mathbf{1} \\ \mathbf{1} & 1 \end{bmatrix}$$

are in W. Again, without loss of generality, we can assume the top row of the submatrix is the top row of $A(T)$. If the two non-bold 1's are on the same diagonal modulo $r+1$, then

$$A(T) = \begin{array}{r} \\ 0 \\ \\ r+1-j \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & & j-k & & j & \end{array} \\ \left[\begin{array}{cc|ccc|c} 0 & & 1 & & \mathbf{1} & \\ & \ddots & & \ddots & & \ddots \\ 1 & & 0 & 1 & & \\ & \ddots & & \ddots & \ddots & \\ & & \mathbf{1} & 0 & 1 & \\ \hline & & & \ddots & \ddots & \ddots \end{array}\right] \end{array}$$

and $r+1=2k$, since the bold $\mathbf{1}$'s are on the same diagonal. This is a contradiction since regular tournaments have odd order. If the two non-bold 1's are not on the same diagonal modulo $r+1$, then there are three cases to consider.

First suppose $j+k<r+1$. Then the submatrix

$$\left[\begin{array}{cc} 1 & \mathbf{1} \\ \mathbf{1} & 1 \end{array}\right]$$

has the form

$$A(T) = \begin{array}{r} \\ 0 \\ \\ r+1-j-k \\ \\ r+1-j \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & & j-k & & j & & j+k & \end{array} \\ \left[\begin{array}{cc|ccc|cc} 0 & & 1 & & \mathbf{1} & 1 & \\ & \ddots & & \ddots & & \ddots & \ddots \\ 1 & & 0 & & 1 & 1 & \\ & \ddots & & \ddots & & \ddots & \ddots \\ 1 & & 1 & & 0 & 1 & \\ & \ddots & & \ddots & & \ddots & \ddots \\ & & \mathbf{1} & & 1 & 0 & \\ \hline & & & \ddots & & \ddots & \ddots \end{array}\right] \end{array}$$

in $A(T)$. This implies that,

$$r+1-j-k \equiv r+1-(j+k) \bmod (r+1) \notin S.$$

Thus, by Lemma 4.5.1, $j + k \in S$. This contradicts the assumption that $j - k \in S$.

Second, suppose $j + k \geq r + 1$ and the submatrix

$$\begin{bmatrix} 1 & \mathbf{1} \\ \mathbf{1} & 1 \end{bmatrix}$$

has the form

$$A(T) = \begin{array}{r} \\ 0 \\ \\ r+1-j \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{cc|ccccccc|c} 0 & & j-k & & j+k & & & & j & \\ 0 & & 1 & & 1 & & & & \mathbf{1} & \\ & \ddots & & \ddots & & \ddots & & & & \ddots \\ 1 & & 0 & & 1 & & 1 & & & \\ & \ddots & & \ddots & & \ddots & & \ddots & & \\ & & \mathbf{1} & & 0 & & 1 & & 1 & \\ \hline & & & \ddots & & \ddots & & \ddots & & \ddots \\ & & & & 1 & & 0 & & 1 & \\ & & & & & \ddots & & \ddots & & \ddots \\ & & & & & & 1 & & 0 & \\ & & & & & & & \ddots & & \ddots \end{array}$$

in $A(T)$. Then $j + k \in S$. This contradicts the assumption that $j - k \in S$.

Third, suppose $j + k \geq r + 1$ and the submatrix

$$\begin{bmatrix} 1 & \mathbf{1} \\ \mathbf{1} & 1 \end{bmatrix}$$

has the form

$$
A(T) = \begin{array}{r}
\\ 0 \\ \\ r+1-j \\ \\ \\ \\ \\ \\ \\ \\
\end{array}
\begin{array}{cc|ccccccc|c}
0 & & j+k & & j-k & & & & j & \\
\hline
0 & & 1 & & 1 & & & & \mathbf{1} & \\
 & \ddots & & \ddots & & \ddots & & & & \ddots \\
1 & & 0 & & 1 & & 1 & & & \\
 & \ddots & & \ddots & & \ddots & & \ddots & & \\
 & & \mathbf{1} & & 0 & & 1 & & 1 & \\
\hline
 & & & \ddots & & \ddots & & \ddots & & \ddots \\
 & & & & 1 & & 0 & & 1 & \\
 & & & & & \ddots & & \ddots & & \ddots \\
 & & & & & & 1 & & 0 & \\
 & & & & & & & \ddots & & \ddots
\end{array}
$$

in $A(T)$. Then $j+k \in S$. This contradicts the assumption that $j-k \in S$. Thus, two 1's of W cannot be in a submatrix

$$
\begin{bmatrix} 1 & \mathbf{1} \\ \mathbf{1} & 1 \end{bmatrix}
$$

of $A(T)$. Therefore, by Corollary 1.2.5, $bc(T) = bp(T) = n$. ■

Corollary 4.5.3 gives s sufficient condition, in graph theoretic terms, for rotational tournament T on n vertices to have the property that $bc(T) = n$. It follows directly from Theorem 4.5.2 and Corollary 4.3.2.

Corollary 4.5.3 *Let A be an adjacency matrix corresponding to a rotational tournament T on $n = r+1$ vertices, $0, 1, 2, \ldots, r$, given by a subset S of*

$\{1, 2, \ldots, r\}$. If there exists a $j \in S$ such that

$$\begin{array}{ccccc} & & & & v_j \\ & & & \nearrow & \\ v_{j-k} & \rightarrow & v_0 & & \\ & & & \searrow & \\ & & & & v_{j+k} \end{array}$$

in T for all $k \in \{1, 2, \ldots, r\}$ modulo $r+1$, then

$$b(A) = r(A) = r_{z^+}(A) = t(A) = n.$$

Theorem 4.5.2 and Corollary 4.5.3 may appear to be quite restrictive. However, for every n one can easily construct a rotational tournament with the desired property. See Section 4.5.2. Furthermore, only one rotational tournament of order $n \leq 11$ does not have the specified property.

4.5.2 U_n Tournaments U_n is defined to be a tournament on n-vertices where $i \rightarrow j$ if and only if $j - i$ is even and positive or odd and negative. U_n is a regular tournament if n is odd and a near-regular tournament if n is even. Also, note that if n is odd, U_n is a rotational tournament. U_5 is given in Figure 4.1.

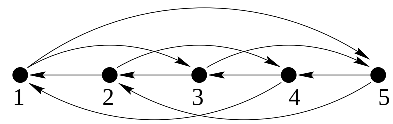

Figure 4.1. U_5

Now consider the adjacency matrices of U_n.

$$A(U_n) = \begin{bmatrix} 0 & 0 & \mathbf{1} & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 1 & \cdots & 1 & 0 & 0 \\ 0 & \mathbf{1} & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \qquad \text{for } n \text{ odd, and}$$

$$A(U_n) = \begin{bmatrix} 0 & 0 & \mathbf{1} & \cdots & 0 & 1 & 0 \\ 1 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & \cdots & 0 & 0 & \mathbf{1} \\ 0 & \mathbf{1} & 0 & \cdots & 1 & 0 & 0 \\ \mathbf{1} & 0 & 1 & \cdots & 0 & 1 & 0 \end{bmatrix} \qquad \text{for } n \text{ even.}$$

The sets $\{a_{13}, a_{24}, a_{35}, ..., a_{n-2,n}, a_{n-1,1}, a_{n2}\}$ and $\{a_{13}, a_{24}, a_{35}, ..., a_{n-2,n}, a_{n-1,2}, a_{n1}\}$ will give a set of n isolated **1**'s for n odd and n even, respectively. Thus, $b(A_n) = n$. Hence,

$$b(A(U_n)) = r(A(U_n)) = r_{z^+}(A(U_n)) = t(A(U_n)) = n.$$

Now we will give two classes of tournaments where all four ranks equal

$n-1$. A transitive tournament matrix has the form

$$A = \begin{bmatrix} 0 & \mathbf{1} & 1 & \cdots & & & 1 \\ & 0 & \mathbf{1} & & & & \\ & & 0 & & & & \\ \vdots & & & & & & \vdots \\ & & & & 0 & \mathbf{1} & 1 \\ & & & & & 0 & \mathbf{1} \\ 0 & & & \cdots & & & 0 \end{bmatrix}.$$

A has a zero column and the $\mathbf{1}$'s on the super diagonal give a set of $n-1$ isolated $\mathbf{1}$'s. Thus,

$$b(A) = r(A) = r_{z^+}(A) = t(A) = n - 1.$$

Note that transitive tournaments have a source and a sink which correspond to a zero row and a zero column in the adjacency matrix. Next, we will construct a class of tournaments without a source or sink and where all four ranks equal $n-1$. Define

$$A = \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \mathbf{1} \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & & & & & & 1 \\ \vdots & \vdots & \vdots & & U_m & & & & \vdots \\ 1 & 1 & 1 & & & & & & 1 \\ 1 & 1 & 1 & & & & & & 1 \\ 1 & 1 & 1 & & & & & & 1 \\ \mathbf{1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}.$$

where U_m is the regular or near-regular tournament matrix described previously. The $\mathbf{1}$'s in the top two rows and the $\mathbf{1}$ in the lower left position of A are isolated from the m isolated 1's in U_m, since they are each in a row or column with all other elements are 0. Thus, A has a set of $m + 3 = n - 1$ isolated 1's. Now suppose there was a set of n independent 1's in A. The above three mentioned **1**'s would have to be in this set which would leave no possible independent 1 for the third row. Thus, A is a tournament matrix with no source or sink and

$$b(A) = r(A) = r_{z^+}(A) = t(A) = n - 1.$$

Note that the above result will hold if the submatrix

$$A' = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & & & \\ 1 & & & \\ \vdots & & & U_m \\ 1 & & & \end{bmatrix}$$

of A is replaced by any matrix with a 0-row, no 0-column, and a set of $m = n-3$ isolated 1's.

4.6 Existence of Strict Inequalities

In all of the previous results and examples of tournament matrices,

$$n - 1 \leq r(A) = r_{z^+}(A) = t(A) \leq n.$$

The next example, given by Maybee and Pullman [MP90], is a class of singular irreducible tournament matrices. This example also shows it is possible to find a class of singular tournament matrices with $t(A) = n$. For $n \geq 6$, let

$$M = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

where

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

S is any $(n-3) \times (n-3)$ tournament matrix with

$$s_{n-3,1} = s_{12} = s_{23} = ... = s_{n-4,n-3} = 1,$$

the ith row of R is [0,0,0] if $s_{i1} = 0$ or [1,0,0] if $s_{i1} = 1$, and $Q = J_{3,n-3} - R^T$.

M is singular since

$$M \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ \mathbf{0} \end{bmatrix} = \mathbf{0}.$$

The 1's on the super diagonal and the 1 in the position $(0, 1)$ give a set of n independent 1's. Thus,

$$n - 1 = r(A) < t(A) = n.$$

The following is an example of a singular n-tournament matrix with $r_{z^+} = n$. Let

$$A = \left[\begin{array}{ccc|ccc|ccc} 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right].$$

A is a singular tournament matrix. In particular,

$$\begin{bmatrix} -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} A = A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \mathbf{0}.$$

Let

$$\mathbf{x}_l = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_r = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

be the vectors that span the left and right null spaces of A, respectively. Any biclique partition of $T(A)$ is equivalent to a sum of n or $n-1$ rank one matrices with nonnegative entries. That is,

$$A = A^{(1)} + A^{(2)} + \cdots + A^{(s)}$$

for $s = n$ or $s = n - 1$.

$$\begin{aligned}
\mathbf{x}_l^T A &= \mathbf{0}^T \quad \text{and} \quad A\mathbf{x}_r = \mathbf{0} \\
\Rightarrow \quad \mathbf{x}_l^T A^{(k)} &= \mathbf{0}^T \quad \text{and} \quad A^{(k)}\mathbf{x}_r = \mathbf{0} \quad \text{for} \quad 1 \leq k \leq s \\
\Rightarrow \quad \mathbf{x}_l^T A_{\cdot j}^{(k)} &= 0 \quad \text{for} \quad 1 \leq j \leq n \quad \text{and} \quad 1 \leq k \leq s \\
\text{and} \quad A_{i \cdot}^{(k)}\mathbf{x}_r &= 0 \quad \text{for} \quad 1 \leq i \leq n \quad \text{and} \quad 1 \leq k \leq s.
\end{aligned}$$

Suppose P is a biclique partition of $T(A)$ of size eight and suppose the bicliques in P have bipartition sets X_i and Y_i where all arcs go from the set X_i to the set Y_i for $1 \leq i \leq s$. Since

$$\mathbf{x}_l^T A_{\cdot j}^{(k)} = 0,$$

the biclique in the partition of $T(A)$ that contains the arc

$$v_3 \rightarrow v_1$$

must also contain the arc

$$v_4 \rightarrow v_1.$$

That is, since the first three elements of $\mathbf{x}_l$ are -1's and the last six are +1's, every bipartition set X_i of each biclique in P must contain an equal number of vertices from the subsets $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6, v_7, v_8, v_9\}$. Similarly, every bipartition set Y_i of each biclique in P must contain an equal number of vertices from the subsets $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $\{v_7, v_8, v_9\}$. This implies that each bipartition set in every biclique in P must have even order. Hence, the number of arcs in every biclique in P must be divisible by four. It follows that the pairs of arcs

$$\begin{array}{ccc} v_3 \to v_1 & \text{and} & v_4 \to v_1 \\ v_1 \to v_2 & \text{and} & v_5 \to v_2 \\ v_2 \to v_3 & \text{and} & v_6 \to v_3 \\ v_9 \to v_6 & \text{and} & v_9 \to v_7 \\ v_8 \to v_5 & \text{and} & v_8 \to v_9 \\ v_7 \to v_4 & \text{and} & v_7 \to v_8 \end{array}$$

must be in the same biclique in P. These arcs correspond to the bold **1**'s in

$$A = \left[\begin{array}{cccccc|ccc} 0 & \mathbf{1} & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \mathbf{1} & 1 & 0 & 1 & 1 & 1 & 1 \\ \mathbf{1} & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ \hline \mathbf{1} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & \mathbf{1} & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \end{array}\right].$$

$T(A)$ consists of $\frac{n(n-1)}{2} = 36$ arcs. Since $r_{z^+} = n - 1 = |P| = 8$, at least one biclique in P has eight or more arcs or at least two bicliques in P have six or more arcs. Thus, we can assume there is a biclique in P that has eight or more arcs since the number of arcs in each biclique is divisible by four. Simple observation shows that any biclique with eight or more vertices must be of one of the two forms given in Figure 4.2.

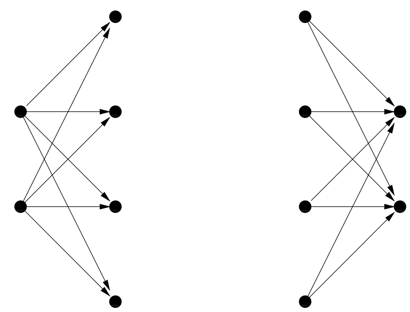

Figure 4.2. Only two possible forms of bicliques in the biclique partition P of $T(A)$

Without loss of generality, by symmetry, we can assume that one of the bicliques in P corresponds to a submatrix of the form

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

There are three possible submatrices of A of this form. These correspond to the bold **1**'s in the matrices given below.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ \mathbf{1} & 0 & 0 & 1 & \mathbf{1} & 0 & 1 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 & 1 & \mathbf{1} & \mathbf{1} & 1 & \mathbf{1} \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix},$$

$$\text{or} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 1 & \mathbf{1} & \mathbf{1} & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

In each case, one can easily verify that we get a contradiction. For example, suppose the biclique of size eight is represented by the bold **1**'s in

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ \mathbf{1} & 0 & 0 & 1 & \mathbf{1} & 0 & 1 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Every biclique must have at least four arcs, so there is only one possible choice for the biclique containing the arcs

$$v_8 \to v_5 \quad \text{and} \quad v_8 \to v_9.$$

This biclique is given by the additional **1**'s in

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \mathbf{1} & 1 & 1 & 1 & \mathbf{1} \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ \mathbf{1} & 0 & 0 & 1 & \mathbf{1} & 0 & 1 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

There is only one possible choice for the biclique containing the arcs

$$v_7 \to v_4 \quad \text{and} \quad v_7 \to v_8.$$

This biclique is given by the additional bold **1**'s in

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \mathbf{1} & 1 & 1 & 1 & \mathbf{1} \\ 0 & 0 & 1 & \mathbf{1} & 0 & 1 & 1 & \mathbf{1} & 1 \\ \mathbf{1} & 0 & 0 & 1 & \mathbf{1} & 0 & 1 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

There is only one possible choice for the biclique containing the arcs

$$v_1 \to v_2 \quad \text{and} \quad v_5 \to v_2.$$

This biclique is given by the additional bold **1**'s in

$$A = \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 1 & \mathbf{1} & 1 & \mathbf{1} \\ 0 & 0 & 1 & \mathbf{1} & 0 & 1 & 1 & \mathbf{1} & 1 \\ \mathbf{1} & 0 & 0 & 1 & \mathbf{1} & 0 & 1 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & 0 & 0 & 0 & 1 & \mathbf{1} & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Finally, there is only one possible choice for the biclique containing the arcs

$$v_2 \to v_3 \quad \text{and} \quad v_6 \to v_3.$$

This biclique is given by the additional bold **1**'s in

$$A = \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 1 & \mathbf{1} & 1 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 1 & \mathbf{1} & \mathbf{1} & 1 \\ \mathbf{1} & 0 & 0 & 1 & \mathbf{1} & 0 & 1 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & 0 & 0 & 0 & 1 & \mathbf{1} & 0 & 1 \\ 0 & 0 & \mathbf{1} & 1 & 0 & 0 & \mathbf{1} & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

This gives a contradiction since there is there is no possible biclique with four or more arcs that contains the arcs

$$v_9 \to v_6 \quad \text{and} \quad v_9 \to v_7.$$

All other cases are similar. Thus, there is no biclique partition of $T(A)$ consisting of eight bicliques. Thus,

$$n - 1 = r(A) < r_{z^+} = n.$$

Lastly, the following is an example of a n-tournament matrix with

$$n - 2 = b(A) < r(A) = r_{z^+}(A) < t(A) = n.$$

Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Then the factorization

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

shows that $r_{z^+}(A) = n - 1 = 8$. The factorization

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

using boolean arithmetic shows that $b(A) \leq n - 2 = 7$. The bold **1**'s in

$$A = \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & \mathbf{1} & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & \mathbf{1} & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & \mathbf{1} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

give a set of $n - 2 = 7$ isolated 1's. By Corollary 1.2.6, $b(A) = n - 2 = 7$.

Furthermore, the **1**'s in,

$$A = \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & \mathbf{1} & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & \mathbf{1} & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & \mathbf{1} \\ 1 & 1 & 1 & 1 & \mathbf{1} & 1 & 1 & 0 & 0 \end{bmatrix}$$

give a set of $n = 9$ independent 1's. Thus,

$$n - 2 = b(A) < r(A) = r_{z^+}(A) < t(A) = n.$$

Note: This is an example of an upset tournament matrix. This class of tournament matrices will be studied in detail in Chapter 5.

Open problem: Does there exist a singular n-tournament matrix (or class) with $b(A) = n$?

5. Upset Tournaments

5.1 Definitions and Standard Form

The *score-list* of a tournament is the multiset of the outdegrees of its vertices. An *upset tournament* is a tournament on $n \geq 4$ vertices with score-list

$$\{1, 1, 2, 3, ..., n-4, n-3, n-2, n-2\}.$$

An upset tournament is in standard form provided its vertices are labeled $v_1, v_2, ..., v_n$ such that the score of v_1 is 1, the score of v_n is $n-2$, the score of vertex v_i is $i-1$ for $2 \leq i \leq n-1$, and arcs (v_1, v_2) and (v_{n-1}, v_n) are arcs in the tournament. As stated in [PS], each upset tournament is isomorphic to exactly one upset tournament in standard form.

It is customary to represent an upset tournament by lining up the vertices $v_1, v_2, ..., v_n$ vertically from bottom to top and including in the picture only those arcs with upward orientation. An arc (v_i, v_j) of an upset tournament is an *upset arc* if the score of v_j is at least the score of v_i. In an upset tournament in standard form, (v_i, v_j) is an upset arc if and only if $i < j$. Equivalently, (v_i, v_j) is an upset arc if and only if it is present in the customary representation of an upset tournament. See Figure 5.1.

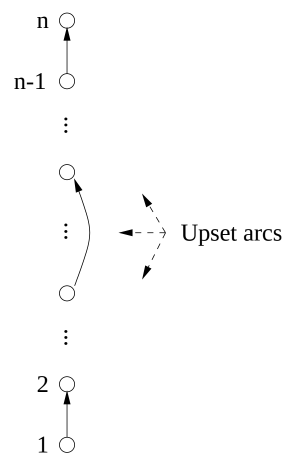

Figure 5.1. Upset tournament in standard form. Upset arcs are directed upward and non-upset arcs (not shown) are directed downward. The arcs $v_1 \to v_2$ and $v_{n-1} \to v_n$ are always present in the tournament.

Lemma 5.1.1 (Poet and Shader [PS]) *Let T be an upset tournament in standard form. Then T has a unique path from vertex v_1 to vertex v_n, and this path consists of the upset arcs of T.*

The unique path given in Lemma 5.1.1 will be called the *upset path* of an upset tournament. The upset path includes the arcs (v_1, v_2) and (v_{n-1}, v_n) since the upset tournament is in standard form. See Figure 5.2. In an adjacency matrix of an upset tournament in standard form, the 1's above the main diagonal represent the upset path. For example, the **1's** in the upset tournament matrix

$$A = \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & \mathbf{1} & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & \mathbf{1} & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & \mathbf{1} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

represent the upset path of $T(A)$. There are $n-4$ vertices, $v_3, v_4, \cdots, v_{n-3}, v_{n-2}$, that may or may not be included in the upset path. Thus, there are 2^{n-4} non-isomorphic upset tournaments on n vertices.

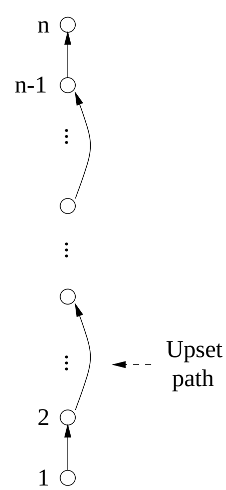

Figure 5.2. For upset tournaments in standard form, there exists a unique path from vertex v_1 to vertex v_n and this path includes the arcs (v_1, v_2) and (v_{n-1}, v_n).

5.2 Term Rank

As stated by Poet and Shader [PS98], upset tournaments are strongly connected. Thus, by Theorem 4.2.4, upset tournament matrices of order n have term rank n. This gives us Lemma 5.2.1. Lemma 5.2.2 gives another way of showing that upset tournaments have term rank n by constructing a set of n independent 1's.

Lemma 5.2.1 *Let A be an adjacency matrix corresponding to an upset tournament on $n \geq 4$ vertices. Then $t(A) = n$.*

Lemma 5.2.2 *Let A be an adjacency matrix corresponding to an upset tournament on $n \geq 4$ vertices in standard form. Let S be a set of 1's of A constructed by moving from row 1 to row n selecting for S the furthest right 1 in each row that is not in a column of a 1 already represented in S, if such a 1 exists. Then $|S| = n$ and hence $t(A) = n$.*

Proof: Let A be an adjacency matrix corresponding to an upset tournament on $n \geq 4$ vertices in standard form. Suppose S is a set of 1's of A constructed by moving from row 1 to row n and selecting for S the furthest right 1 in each row that is not in a column of a 1 already in S, if such a 1 exists. All 1's above the main diagonal of A are in S, since these correspond to arcs in the unique upset path. S contains the lone 1's from the the first two rows and last two columns of A, since A is in standard form. Furthermore, the 1 in S from row 1 is in column 2 and the 1 from column n is in row $n-1$. Now suppose that for some k, $3 \leq k \leq n-3$, all 1's in row k are in columns which contain 1's already in S. Then v_k is not a vertex in the upset path since otherwise it would be in S. Thus, row k has the form

$$\underbrace{1 \quad 1 \quad \cdots \quad 1}_{k-1 \quad 1's} \quad \underbrace{0 \quad 0 \quad \cdots \quad 0}_{k+1 \quad 1's}.$$

This implies that the 1's in S from rows 1 through $k-1$ must be from columns 1 through $k-1$. Since v_k is not a vertex in the upset path, there exists an upset arc (v_i, v_j) with $i < k < j$. Thus, the 1 in S from row i would be from column j. This contradicts all 1's in S from rows 1 through $k-1$ being from columns l through $k-1$. Therefore, $|S| = n$ and hence $t(A) = n$. ■

For example, the **1**'s in the upset tournament matrix

$$A = \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & \mathbf{1} & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & \mathbf{1} & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & \mathbf{1} \\ 1 & 1 & 1 & 1 & \mathbf{1} & 1 & 1 & 0 & 0 \end{bmatrix}$$

are the $n = 9$ independent 1's in the set S given by Lemma 5.2.2.

5.3 Real Rank and Nonnegative Integer Rank

Complete results concerning the real rank and nonnegative integer rank of upset tournaments are given in [KS90] and [Sha90]. Theorem 5.3.1 is an upset path formulation of a result given in [KS90].

Theorem 5.3.1 (Katzenberger and Shader [KS90]) *Let A be an adjacency matrix of order $n \geq 4$ corresponding to a upset tournament T in standard form. A is singular if and only if the upset path of T has one or more of the structures given in Figure 5.3.*

Theorem 5.3.2 (Shader [Sha90]) *Let A be an adjacency matrix of order $n \geq 4$ corresponding to a upset tournament. Then*

$$r(A) = r_{z^+}(A).$$

Corollary 5.3.3, which follows from Theorems 5.3.1 and 5.3.2, characterizes upset tournaments which have $bp(T) = n - 1$. Since $bp(T) = n - 1$ or $bp(T) = n$, this completely characterizes upset tournaments with respect to their biclique partitioning number.

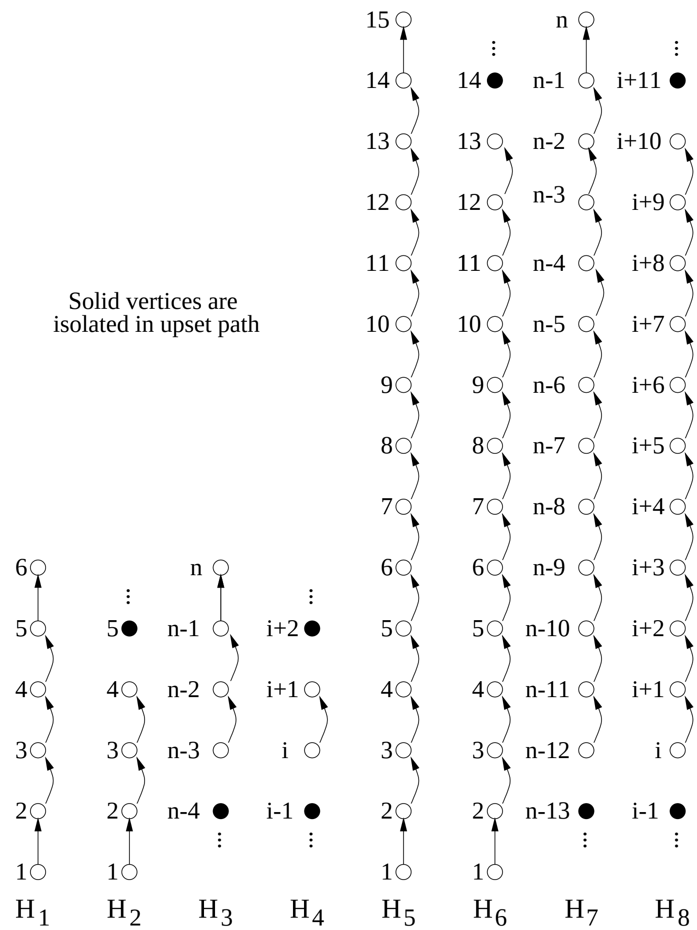

Figure 5.3. Structures of upset path that cause singularity.

Corollary 5.3.3 *Let T an upset tournament on n vertices in standard form. $bp(T) = n-1$ if and only if the upset path of T has one or more of the structures given in Figure 5.3.*

5.4 Boolean Rank

In this section, a characterization of upset tournament matrices with respect to their boolean rank and a best possible lower bound for the boolean rank is given. In addition, it is shown that the number of nonisomorphic upset tournaments with equal biclique cover and partition numbers can be given in terms of convolutions of the Fibonacci sequence. These results, together with Shader's work [Sha90], give a complete characterization of upset tournament matrices with respect to each rank and with respect to their biclique cover and partition numbers.

Theorem 5.4.1 *Let A be an adjacency matrix of order $n \geq 4$ corresponding to a upset tournament T in standard form. Then $bc(T) = b(A) = n - q$ if and only if the upset path of T contains q copies (total) of the subgraphs H_0 and H given in Figure 5.4.*

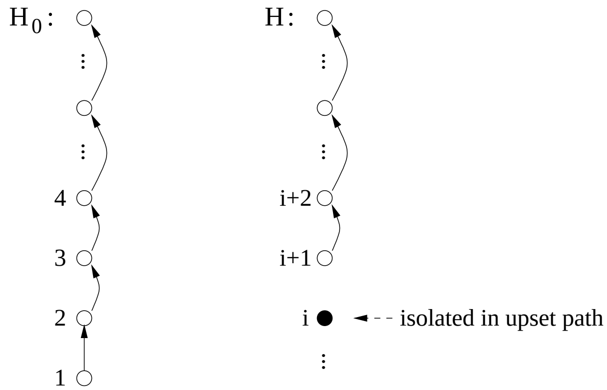

Figure 5.4. Subgraphs which reduce the biclique covering number.

Proof: Let A be an adjacency matrix of order $n \geq 4$ corresponding to an upset tournament T in standard form. First, we will show that if the upset path of T contains q copies of the subgraphs H_0 and H, then $bc(T) \leq n - q$. Second, we will show that if $bc(T) = n - q$, then the upset path of T contains at least q copies of the subgraphs H_0 and H. This will complete the proof since if the upset path of T contains q copies of the subgraphs H_0 and H and $bc(T) < n - q$, then we have a contradiction to the second statement. Furthermore, if $bc(T) = n - q$ and the upset path of T contains more than q copies of the subgraphs H_0 and H, then we have a contradiction to the first statement.

Suppose the upset path of T contains q copies of the subgraphs H_0 and H. We need to show that $bc(T) \leq n - q$. To do this, we will show that for each copy, $bc(T)$ is reduced by at least one. Suppose H_0 occurs in the upset path of T. Then

$$A(T) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ \\ j \\ \\ k \\ \\ \end{array} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & & j & & k & \\ \hline 0 & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ \mathbf{1} & 0 & 0 & \mathbf{1} & \cdots & 0 & \cdots & 0 & \cdots \\ \mathbf{1} & 1 & 0 & 0 & \cdots & \mathbf{1} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \\ \mathbf{1} & 1 & 1 & 0 & \cdots & 0 & \cdots & \mathbf{1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots & \\ \mathbf{1} & 1 & 1 & \mathbf{1} & \cdots & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \ddots \end{array}$$

and

$$\text{column } 1 = \text{column } 4 + \text{column } j + \text{column } k$$

using boolean arithmetic (1+1=1). Thus, the four columns $1, 4, j$, and k correspond to a subdigragh that can be covered with three bicliques and $bc(T)$ is reduced by one. Suppose H occurs in the upset path of T. Then

$$A(T) = \begin{array}{c} \\ \\ \\ \\ i \\ \\ \\ \\ j \\ \\ k \\ \\ \end{array} \begin{array}{ccccccccccc} & & & i & & & & j & & k & \\ \hline \ddots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \\ \cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \\ \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ \cdots & 0 & \cdots & \mathbf{1} & 0 & \mathbf{1} & \cdots & 0 & \cdots & 0 & \cdots \\ \cdots & 1 & \cdots & \mathbf{1} & 0 & 0 & \cdots & \mathbf{1} & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \\ \cdots & 1 & \cdots & \mathbf{1} & 1 & 0 & \cdots & 0 & \cdots & \mathbf{1} & \cdots \\ & \vdots & & \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots & \\ \cdots & 1 & \cdots & \mathbf{1} & 1 & \mathbf{1} & \cdots & 0 & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \ddots \end{array}$$

and

$$\text{column } i = \text{column } (i+2) + \text{column } j + \text{column } k$$

using boolean arithmetic. Thus, the four columns $i, i+2, j$, and k correspond to a subdigragh that can be covered with three bicliques and $bc(T)$ is reduced by one.

If all copies of H_0 and H in the upset path of T are pairwise arc disjoint, then q copies of H_0 and H will imply that $bc(T) \leq n - q$. Suppose that the upset path of T contains a copy of H_0 and a copy of H that are not arc disjoint. Since the overlap can occur in only one way,

$$
A(T) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ \\ i \\ \\ \\ \\ j \\ \\ k \\ \\ \end{array}
\begin{array}{cccccccccccccc}
1 & 2 & 3 & 4 & & i & & & & j & & k & \\
\end{array}
\left[\begin{array}{cccccccccccccc}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\mathbf{1} & 0 & 0 & \mathbf{1} & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\mathbf{1} & 1 & 0 & 0 & \cdots & 0 & \mathbf{1} & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \\
\mathbf{1} & 1 & 1 & \mathbf{1} & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\mathbf{1} & 1 & 1 & 0 & \cdots & \mathbf{1} & 0 & \mathbf{1} & \cdots & 0 & \cdots & 0 & \cdots \\
\mathbf{1} & 1 & 1 & \mathbf{1} & \cdots & \mathbf{1} & 0 & 0 & \cdots & \mathbf{1} & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \\
\mathbf{1} & 1 & 1 & \mathbf{1} & \cdots & \mathbf{1} & \mathbf{1} & 0 & \cdots & 0 & \cdots & \mathbf{1} & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots & \\
\mathbf{1} & 1 & 1 & \mathbf{1} & \cdots & \mathbf{1} & \mathbf{1} & \mathbf{1} & \cdots & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \ddots
\end{array}\right]
$$

and

$$
\begin{aligned}
\text{column } 1 &= \text{column } 4 + \text{ column } i + \text{ column } (i+1) \\
&+ \text{columns } (i+2) + \text{ column } j + \text{ column } k \\
&= \text{column } 4 + \text{ column } (i+1) \\
&+ \text{columns } (i+2) + \text{ column } j + \text{ column } k
\end{aligned}
$$

using boolean arithmetic. See Figure 5.5. Thus, the seven columns $1, 4, i, i+1, i+2, j$, and k correspond to a subdigragh that can be covered with five bicliques and $bc(T)$ is reduced by two. Similarly, suppose the upset path of T contains two copies of H that are not arc disjoint. Since the overlap can occur in only one way,

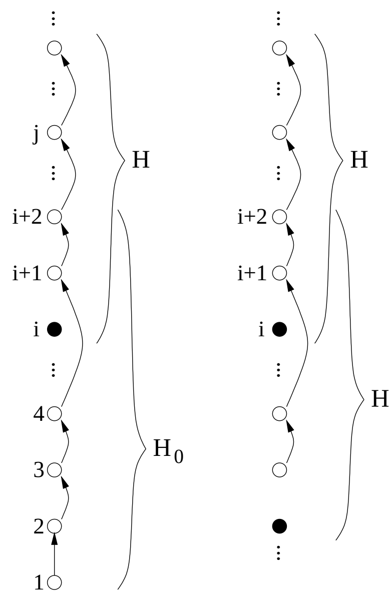

Figure 5.5. H_0 and an H or two H's can share at most one arc.

$$A(T) = \begin{array}{c} \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \\ k \\ \\ l \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} & i & & & & j & & & & k & & l & \end{array} \\ \left[\begin{array}{ccccccccccccc} \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \\ \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ \cdots & \mathbf{1} & 0 & \mathbf{1} & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ \cdots & \mathbf{1} & 0 & 0 & \cdots & 0 & \mathbf{1} & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \\ \cdots & \mathbf{1} & 1 & \mathbf{1} & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ \cdots & \mathbf{1} & 1 & 0 & \cdots & \mathbf{1} & 0 & \mathbf{1} & \cdots & 0 & \cdots & 0 & \cdots \\ \cdots & \mathbf{1} & 1 & \mathbf{1} & \cdots & \mathbf{1} & 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots \\ & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \\ \cdots & \mathbf{1} & 1 & \mathbf{1} & \cdots & \mathbf{1} & \mathbf{1} & 0 & \cdots & 0 & \cdots & 1 & \cdots \\ & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots & \\ \cdots & \mathbf{1} & 1 & \mathbf{1} & \cdots & \mathbf{1} & \mathbf{1} & \mathbf{1} & \cdots & 0 & \cdots & 0 & \cdots \\ & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \ddots \end{array}\right] \end{array}$$

and

$$\begin{aligned} \text{column } i &= \text{column } (i+2) + \text{ column } j + \text{ column } (j+1) \\ &+ \text{columns } (j+2) + \text{ column } k + \text{ column } l \\ &= \text{column } (i+2) + \text{ column } (j+1) \\ &+ \text{columns } (j+2) + \text{ column } k + \text{ column } l \end{aligned}$$

using boolean arithmetic. See Figure 5.5. Thus, the seven columns $i, i+2, j, j+1, j+2, k$, and l correspond to a subdigragh that can be covered with five bicliques and $bc(T)$ is reduced by two. Hence, for each copy of H_0 and H in the upset path of T, $bc(T)$ is reduced by one. Specifically, if the upset path of T contains q copies of the subgraphs H_0 and H, then $bc(T) \leq n - q$.

Conversely, suppose $bc(T) = n - q$. We need to show that the upset path of T contains at least q copies of the subgraphs H_0 and H. If $q = 0$, the statement is trivially true. If $q > 1$, there exists a set S of m columns of A such that $D(S)$ can be covered with less than m bicliques. Existence of this set follows from the fact that we could choose S to be the set of all columns. Without loss of generality, we can assume that S is minimal in the sense that there is no subset $\bar{S} \subset S$ such that $D(\bar{S})$ can be covered with less than $|\bar{S}|$ bicliques. Every biclique in a biclique covering of $D(S)$ must involve at least two columns of S. To see this, suppose a biclique in a biclique covering of $D(S)$ involves just one column $c \in S$. Then $S - c \subset S$ such that $D(S - c)$ can be covered with $|S| - 2$ bicliques which contradicts the minimal property of S. Assume the columns in S are in the same order as they appear in A (in standard form) and that i is the index of the first column in S.

Suppose $i = 1$. Then

$$A = A(T) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & & & \\ 1 & & 0 & & \\ 1 & & & 0 & \\ \vdots & & & & \ddots \end{array} \right] \end{array}$$

First, we will show that the arc (v_3, v_4) must be included in the upset path and the arcs (v_3, v_1) and (v_3, v_4) must be covered by the same biclique. Since every biclique covering S must involve at least two columns of S, the arc

(v_3, v_1) must be covered with a biclique involving some column $c \neq 1, 3$ of S. Column two cannot be in S since the 1 in position $A(1, 2)$ is the only 1 in the first row. This implies that $c \neq 2$ and hence $c \geq 4$. Thus, the vertices v_1,v_2, v_3, and v_c must be included in the upset path of T. Similarly, since the 1 in position $A(2, 3)$ is the only 1 in row two, column three cannot be in S. The arc (v_4, v_1) must be covered with a biclique involving some other column j, $j \geq 5$ of S. Hence, the vertex v_4 must be in the upset path of T. Thus, the upset path of T must contain

$$1 \to 2 \to 3 \to 4 \to j$$

for some j, $5 \leq j \leq n$, and

$$A(T) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ \\ j \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & & j & \end{array} \\ \left[\begin{array}{cccccccc} 0 & \mathbf{1} & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \mathbf{1} & 0 & \cdots & 0 & \cdots \\ 1 & 0 & 0 & \mathbf{1} & \cdots & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots & \mathbf{1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 1 & 1 & 1 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \ddots \end{array} \right] \end{array}.$$

The arc (v_j, v_1) must be covered in $D(S)$ with a biclique involving some other column k of S with $5 \leq k < j$ or $j < k \leq n$. If $5 \leq k < j$, then column k is not in the upset path. This implies that $j < n$ since $4 \to j$ cannot be the end of the upset path. Thus, the upset path must have the form

$$1 \to 2 \to 3 \to 4 \to j \to l$$

for some $l > j$ and

$$A(T) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ \\ k \\ \\ j \\ \\ l \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & & k & & j & & l & & \end{array} \\ \left[\begin{array}{cccccccccccc} 0 & \mathbf{1} & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \mathbf{1} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ 1 & 0 & 0 & \mathbf{1} & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots & 0 & \cdots & \mathbf{1} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & & \vdots & \\ 1 & 1 & 1 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots & \\ 1 & 1 & 1 & 0 & \cdots & 1 & \cdots & 0 & \cdots & \mathbf{1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \ddots & \vdots & \\ 1 & 1 & 1 & 1 & \cdots & 1 & \cdots & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots & \ddots \end{array} \right] \end{array}.$$

Suppose $j < k \leq n$. Then column k must be in the upset path and the upset path contains the subgraph

$$1 \to 2 \to 3 \to 4 \to j \to k.$$

Thus, in both cases H_0 is in the upset path. Now since the upset path contains the subgraph H_0, columns $1, 2, j$, and k can be covered in $D(S)$ with three bicliques. By the minimal property of S, $|S| = 4$ and, in particular,

$$S = \{\text{columns } 1, 4, j, k\}.$$

Suppose $i > 1$. If v_i was in the upset path, then the biclique covering the arc to v_i in $D(S)$ could not involve any other columns in S. This implies that vertex v_i cannot be in the upset path of T since every biclique covering $D(S)$ must involve at least two columns of S. Thus, column i must be of the following form

$$
\begin{array}{c c}
 & \begin{array}{ccccc} & i & & & \end{array} \\
i & \left[\begin{array}{ccccc}
\ddots & \vdots & & & \\
\cdots & 0 & 0 & 0 & \cdots \\
 & 1 & 0 & & \\
 & 1 & & 0 & \\
 & \vdots & & & \ddots
\end{array}\right].
\end{array}
$$

Note that $i \neq 2$ since $v_1 \to v_2$ is always in the upset path when the upset tournament is in standard form. Since every biclique covering $D(S)$ must involve at least two columns of S, the arc (v_{i+1}, v_i) must be covered in $D(S)$ with a biclique involving some other column $c \geq i+2$ of S. Similarly, the biclique covering the arc (v_{i+2}, v_i) in $D(S)$ must involve a column $j \geq i+1$. Since $i+1$ is in the upset path, $j \neq i+1$; hence, $j > i+2$. Thus, $i+2$ is in the upset path. Furthermore, $c = i+2$ and

$$i+1 \to i+2.$$

Thus, the upset path must contain the subgraph

$$i+1 \to i+2 \to j$$

but it cannot contain i and

$$
A(T) = \begin{array}{c} \\ \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \end{array}
\left[\begin{array}{ccccccccc}
 & & & i & & & & j & \\
\ddots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \\
\cdots & 0 & \cdots & 0 & \mathbf{1} & 0 & \cdots & 0 & \cdots \\
 & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \\
\cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\
\cdots & 0 & \cdots & 1 & 0 & \mathbf{1} & \cdots & 0 & \cdots \\
\cdots & 1 & \cdots & 1 & 0 & 0 & \cdots & \mathbf{1} & \cdots \\
 & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \\
\cdots & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 & \cdots \\
 & \vdots & & \vdots & \vdots & \vdots & & \vdots & \ddots
\end{array} \right].
$$

Covering the arc (v_j, v_i) in $D(S)$ with a biclique involving some other column in S must involve a column $k \in S$ with $i+2 < k < j$ or $j < k \leq n$. If $i+2 < k < j$, then column k is not in the upset path. This implies that $j < n$ since $i+2 \to j$ cannot be the end of the upset path. Thus, the upset path must have the form

$$i+1 \to i+2 \to j \to l$$

for some $l > j$. Suppose $j < k \leq n$. Then column k must be in the upset path, the upset path contains the subgraph

$$i+1 \to i+2 \to j \to k,$$

and

$$
\begin{array}{c|ccccccccccc|}
 & & & & i & & & & j & & k & \\
 & \ddots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \\
 & \cdots & 0 & \cdots & 0 & \mathbf{1} & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
 & & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \\
i & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
 & \cdots & 0 & \cdots & 1 & 0 & \mathbf{1} & \cdots & 0 & \cdots & 0 & \cdots \\
 & \cdots & 1 & \cdots & 1 & 0 & 0 & \cdots & \mathbf{1} & \cdots & 0 & \cdots \\
 & & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \\
j & \cdots & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 & \cdots & \mathbf{1} & \cdots \\
 & & \vdots & & \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots & \\
k & \cdots & 1 & \cdots & 1 & 1 & 1 & \cdots & 0 & \cdots & 0 & \cdots \\
 & & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \ddots
\end{array}
$$

Thus, in both cases H is in the upset path. Now since the upset path contains the subgraph H, columns $i, i+2, j$, and k can be covered in $D(S)$ with three bicliques. By the minimal property of S, $|S| = 4$ and, in particular,

$$S = \{\text{columns } i, i+2, j, k\}.$$

Thus, if $bc(T) = n - q$, then the upset path of T contains at least q copies of the subgraphs H_0 and H. ■

Since the biclique covering number can be much less than the biclique partitioning number, which is bounded below by $n-1$, we would like a lower bound for the biclique covering number of upset tournaments. For an upset tournament T in standard form, the 1's in $A(T)$ corresponding to the upset arcs of T are all isolated 1's. Thus, by Corollary 1.2.5, the number of upset arcs of T give a lower bound for the biclique cover number of T. This simple result is stated in Lemma 5.4.2. Lemma 5.4.4 gives a best possible lower bound on the biclique covering number of upset tournaments. The proof of Lemma 5.4.4 follows directly from Lemma 5.4.3 and Theorem 5.4.1.

Lemma 5.4.2 *Let A be an adjacency matrix of order $n \geq 4$ corresponding to a upset tournament T in standard form. If the upset path of T consists of k arcs, then $bc(T) = b(A) \geq k$.*

Lemma 5.4.3 *If the upset path of an upset tournament in standard form on $n \geq 4$ vertices contains q copies of the subgraphs H_0 and H given in Figure 5.4, then*

$$q < \frac{n}{3}.$$

Furthermore, there exists a class of upset tournaments such that

$$\frac{q}{n} \rightarrow \frac{1}{3} \qquad as \quad n \rightarrow \infty.$$

Proof: To maximize q, we have to minimize the number of vertices needed to construct copies of H_0 and H. The subgraphs H_0 and H or two H's can share at most one arc. See Figure 5.5. Thus, q will be maximized if the upset path consists of the sequence $H_0, H, H, ...$ with exactly one arc shared between each copy. See Figure 5.6. Thus,

$$q \leq \frac{n}{3} - 1 < \frac{n}{3}.$$

For the example of an upset tournament on $n = 3k$, $k \geq 3$, vertices given in Figure 5.5,

$$\frac{q}{n} = \frac{k-1}{3k} \to \frac{1}{3} \quad \text{as} \quad n \to \infty. \quad \blacksquare$$

Lemma 5.4.4 *If the upset path of an upset tournament T in standard form on $n \geq 4$ vertices, with corresponding adjacency matrix A, contains q copies of the subgraphs H_0 and H given in Figure 5.4, then*

$$b(A) = bc(T) > \frac{n}{3}.$$

Furthermore, there exists a class of upset tournaments such that

$$\frac{b(A)}{n} = \frac{bc(T)}{n} \to \frac{2}{3} \qquad as \qquad n \to \infty.$$

Proposition 5.4.5, which follows from Lemma 5.2.1 and Theorem 5.3.2, summarizes the rank relationships for upset tournaments. As discussed in Section 1.2.7, there is no relationship between the boolean rank and the real rank

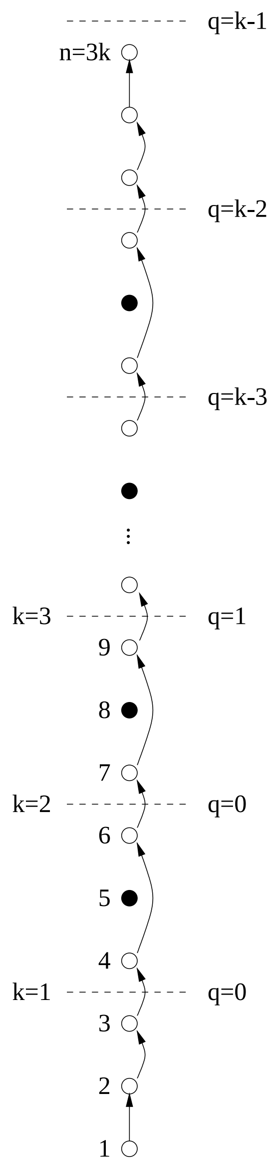

Figure 5.6. Upset path which maximizes copies of the subgraphs H_0 and H.

for any {0,1}-matrix. Upset tournaments give a large class of matrices with $b(A) \leq r(A)$.

Proposition 5.4.5 *If A is n-tournament matrix corresponding to an upset tournament, then*

$$\left.\begin{array}{c} \frac{2}{3}n < b(A) \\ n-1 \end{array}\right\} \leq r(A) = r_{z^+}(A) \leq t(A) = n.$$

5.5 Equal Biclique Cover and Partition Numbers

One of the problems of interest for any class of graphs is finding subclasses that have equal biclique covering and partitioning numbers. Since the biclique covering number and partitioning numbers are known for all upset tournaments, we can determine precisely which upset tournaments have equal biclique covering and partitioning numbers. Furthermore, we will show that the number of nonisomorphic upset tournaments with equal biclique covering and partitioning numbers can be given in terms of convolutions of the Fibonacci sequence.

First, we will characterize and count the upset tournaments on n vertices where the biclique cover and partition numbers both equal n. Second, we will characterize and count the upset tournaments on n vertices where the biclique cover and partition numbers both equal $n-1$. Third, we will put these results together to characterize and count the upset tournaments with equal biclique cover and partition numbers.

Theorem 5.5.1 *Let T be an upset tournament in standard form on $n \geq 6$ vertices. Then $bc(T) = n$ if and only if the upset path does not contain any arcs of the form (v_i, v_{i+1}) for $3 \leq i \leq n-3$.*

Proof: Let T be an upset tournament in standard form on n vertices. Suppose the upset path of T does not contain any arcs of the form (v_i, v_{i+1}) for $3 \leq i \leq n-3$. Thus, there are no copies of the subgraphs H_0 or H, given in Figure 5.4, in the upset path of T. Hence, by Theorem 5.4.1, $bc(T) = n$.

Conversely, suppose $bc(T) = n$ and suppose the upset path of T contains an arc of the form (v_i, v_{i+1}) for some i, $3 \leq i \leq n-3$. If $i = 3$, then the upset path contains a copy of H_0. If $i \geq 4$, then the upset path contains a copy of H. This contradicts Theorem 5.4.1. ■

Theorem 5.5.2 *Let T be an upset tournament in standard form on n vertices. Let b_n be the number of nonisomorphic upset tournaments on n vertices such that $bc(T) = n$. Then $b_4 = 1$, $b_5 = 2$, and $b_{n+2} = b_{n+1} + b_n$ for $n \geq 4$.*

Proof: $b_4 = 1$ and $b_5 = 2$ follow easily from the fact that all upset tournaments on $n \leq 5$ vertices have $bc(T) = n$. See Figure 5.7. The proof that $b_{n+2} = b_{n+1} + b_n$, for $n \geq 4$, is by induction n. Suppose $n = 4$. By Lemma 5.5.1, the upset path of an upset tournament on six vertices with $bc(T) = 6$ does not contain an arc of the form (v_3, v_4). As can be seen in Figure 5.7, there are only three nonisomorhic upset tournaments on six vertices with this property. The equality $b_4 + b_5 = b_6$ is illustrated by the removal of the circled vertices.

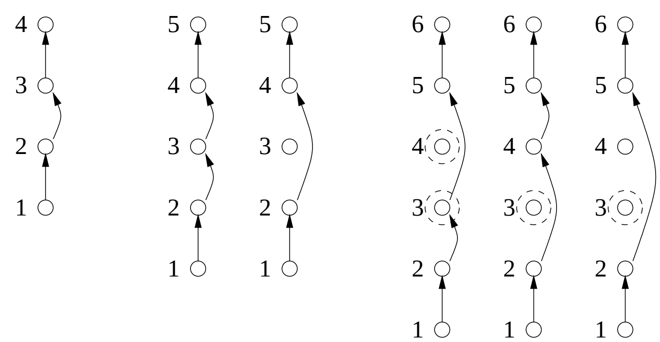

Figure 5.7. $b_4 + b_5 = b_6$.

Now suppose $b_{n+2} = b_{n+1} + b_n$ for any n, $4 \leq n < k$, for some k. First consider the number of upset tournaments on $k+2$ vertices with $bc(T) = k+2$ such that the arc

$$v_2 \to v_3$$

is in the tournament. Since $v_2 \to v_3$ we know that $v_3 \not\to v_4$. Thus, the vertex v_4 must be isolated in the upset path. Hence, the number of nonisomporhic upset tournaments on $k+2$ vertices with $bc(T) = k+2$ and

$$v_2 \rightarrow v_3$$

in the upset path is equivalent to the number of upset tournaments on k vertices for which $bc(T) = k$. These are equivalent since we are essentially removing the two vertices v_3 and v_4 from the tournament because their inclusion in or exclusion from the upset path is fixed. Second, consider the number of upset tournaments on $k+2$ vertices with $bc(T) = k+2$ such that

$$v_2 \not\rightarrow v_3.$$

This number is equivalent to the number of nonisomorphic tournaments on $k-1$ vertices for which $bc(T) = k-1$ since we are essentially removing the vertex v_3 from the tournament since its exclusion from the upset path is fixed. Thus, $b_{n+2} = b_{n+1} + b_n$. ■

Recall, the boolean rank is a lower bound for the nonnegative integer rank, and for upset tournaments, the nonnegative integer rank is always equal to the real rank. Thus, if the boolean rank of an upset tournament on n vertices is n, all four ranks must be n. Using these facts, we can restate Theorem 5.5.2 in terms of all four ranks. Specifically, Corollary 5.5.3 follows from Theorem 5.5.2 and Theorem 5.3.2.

Corollary 5.5.3 *Let b_n be the number of nonisomorphic upset tournament matrices of order n such that*

$$b(A) = r_{z^+}(A) = r(A) = t(A) = n.$$

Then $b_4 = 1$, $b_5 = 2$, and $b_{n+2} = b_{n+1} + b_n$ for $n \geq 4$.

Corollary 5.5.4 *Let b_n be the number of nonisomorphic upset tournament matrices of order n such that*

$$b(A) = r_{z^+}(A) = r(A) = t(A) = n.$$

Let a_n be the Fibonnaci sequence $a_0 = 1$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + a_n$ for $a \geq 2$. Then

$$b_n = a_{n-3} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-2}.$$

Proof: Let b_n be the number of nonisomorphic upset tournament matrices of order n such that

$$b(A) = r_{z^+}(A) = r(A) = n.$$

Let a_n be the Fibonnaci sequence $a_0 = 1$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + a_n$ for $a \geq 2$. From [Bog90], we have

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.$$

From Corollary 5.5.3, we have $b_4 = 1$, $b_5 = 2$, and $b_{n+2} = b_{n+1} + b_n$ for $n \geq 4$. Thus,

$$b_n = a_{n-3} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-2}. \quad \blacksquare$$

Theorem 5.5.5 *Let A be an adjacency matrix of order $n \geq 6$ corresponding to a upset tournament T in standard form. Then*

$$b(A) = r_{z^+}(A) = r(A) = n - 1.$$

if and only if the upset path of T has exactly one of the structures given in Figure 5.3 and the upset path of T does not contain any additional arcs of the form (v_i, v_{i+1}) for $3 \leq i \leq n-3$.

Proof: Let A be an adjacency matrix of order $n \geq 6$ corresponding to a upset tournament T in standard form. Suppose

$$b(A) = r_{z^+}(A) = r(A) = n - 1.$$

Since $b(A) = bc(T) = n - 1$, the upset path of T contains exactly one of the subgraphs H_0 or H given in Figure 5.4. Since A is singular, the upset path of

T has one or more of the eight structures given in Figure 5.3. Each of these structures contains exactly one copy of H_0 or H. Any additional arcs of the form (v_i, v_{i+1}) for $3 \leq i \leq n-3$ would give additional copies of the subgraphs H_0 and H. This would contradict $b(A) = n-1$. Thus, the upset path of T cannot contain any additional arcs of the form (v_i, v_{i+1}) for $3 \leq i \leq n-3$.

Conversely, suppose the upset path of T has exactly one of the structures given in Figure 5.3 and the upset path of T does not contain any additional arcs of the form (v_i, v_{i+1}) for $3 \leq i \leq n-3$. Then T contains exactly one of either the subgraph H_0 or the subgraph H given in Figure 5.4. Thus, by Theorem 5.4.1, $bc(T) = b(A) = n-1$. Now since the upset path of T has one the structures given in Figure 5.3, A is singular. Thus,

$$b(A) = r_{z^+}(A) = r(A) = n-1. \quad \blacksquare$$

Theorem 5.5.6 *Let c_n be the number of nonisomorphic upset tournament matrices of order n such that*

$$b(A) = r_{z^+}(A) = r(A) = n-1.$$

Let $\{a_n\}$ be the Fibonnaci sequence $a_0 = 1$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + a_n$ for $a \geq 2$. Then

$$c_n = \begin{cases} 0 & if \quad n \le 5 \\ 1 & if \quad n = 6 \\ 2 & if \quad n = 7 \\ 2a_{n-6} + \sum_{k=1}^{n-7} a_k a_{n-k-6} & if \quad 8 \le n \le 14 \\ 421 & if \quad n = 15 \\ 746 & if \quad n = 16 \\ 2a_{n-6} + 2a_{n-15} + \sum_{k=1}^{n-7} a_k a_{n-k-6} + \sum_{k=1}^{n-16} a_k a_{n-k-15} & if \quad n \ge 17 \end{cases}$$

Proof: The results $c_n = 0$ for $n \le 5$ and $c_6 = 1$ follow directly from Theorem 5.5.5, since there are no singular tournaments of order $n \le 5$ and only one of order $n = 6$. Suppose $n = 7$. From Theorem 5.5.5, we know that H_2 and H_3 are the two possible structures of an upset path of an upset tournament T satisfying

$$b(A(T)) = r_{z^+}(A(T)) = r(A(T)) = n - 1.$$

Furthermore, there is only one such tournament of each structure on $n = 7$ vertices. Thus, $c_7 = 2$.

Suppose $8 \le n \le 14$. From Theorem 5.5.5, H_2, H_3, and H_4 are the three possible structures of an upset path of an upset tournament T satisfying

$$b(A(T)) = r_{z^+}(A(T)) = r(A(T)) = n - 1.$$

There are $b_{n-3} = a_{n-6}$ upset tournaments having the structure H_2 and satisfying

$$b(A(T)) = r_{z^+}(A(T)) = r(A(T)) = n - 1,$$

since this is equivalent to the number of upset tournament matrices on $n-3$ vertices having full boolean rank. Similarly, $b_{n-3} = a_{n-6}$ will give the number of upset tournaments having the structure H_3 and satisfying

$$b(A(T)) = r_{z^+}(A(T)) = r(A(T)) = n-1.$$

For each i, $4 \le i \le n-4$, there are b_{n-i} possible configurations for the arcs in the upset path above the arc (v_i, v_i+1), since this is equivalent to the number of upset tournaments on $n-i$ vertices with full boolean rank. Similarly, there are b_i possible configurations for the arcs in the upset path below the arc (v_i, v_i+1), since this is equivalent to the number of upset tournaments on i vertices with full boolean rank. Thus, for each i, $4 \le i \le n-4$ there are $b_i b_{n-i} = a_{i-3}a_{n-i-3}$ upset tournaments that have the structure H_4 and that satisfy

$$b(A(T)) = r_{z^+}(A(T)) = r(A(T)) = n-1.$$

Thus,

$$c_n = \underbrace{2a_{n-6}}_{H_2\text{'s and } H_3\text{'s}} + \underbrace{\sum_{i=4}^{n-4} a_{i-3}a_{n-i-3}}_{H_4\text{'s}} \quad \text{if} \quad 8 \le n \le 14.$$

Shifting the index of the sum by three gives

$$c_n = \underbrace{2a_{n-6}}_{H_2\text{'s and } H_3\text{'s}} + \underbrace{\sum_{k=1}^{n-7} a_k a_{n-k-6}}_{H_4\text{'s}} \quad \text{if} \quad 8 \le n \le 14.$$

If $n = 15$, then c_n is given by the above expression plus one for the structure H_5. Specifically,

$$c_{15} = \underbrace{2a_9}_{H_2\text{'s and } H_3\text{'s}} + \underbrace{\sum_{k=1}^{8} a_k a_{9-k}}_{H_4\text{'s}} + \underbrace{1}_{H_5} = 421.$$

Now suppose $n \geq 17$. From Theorem 5.5.5, the structure of the upset path could be any of the six structures H_2, H_3, H_4, H_6, H_7, and H_8. The structures H_6,H_7, and H_8 are similar to the structures H_2, H_3 and H_4, respectively. Thus, we just need to adjust the indices by nine in the appropriate formulas since there are nine additional consecutive vertices in the upset paths having the structures H_6,H_7, and H_8. It follows that for $n \geq 17$,

$$c_n = \underbrace{2a_{n-6}}_{H_2\text{'s and } H_3\text{'s}} + \underbrace{2a_{n-15}}_{H_6\text{'s and } H_7\text{'s}} + \underbrace{\sum_{k=1}^{n-7} a_k a_{n-k-6}}_{H_4\text{'s}} + \underbrace{\sum_{k=1}^{n-16} a_k a_{n-k-15}}_{H_8\text{'s}}.$$

If $n = 16$, then the structure of the upset path could be any of the five structures H_2, H_3, H_4, H_6, and H_7. Thus, c_n is given by the above expression minus the term counting those tournaments whose upset path have the structure H_8. Hence,

$$c_{16} = \underbrace{2a_{10}}_{H_2\text{'s and } H_3\text{'s}} + \underbrace{2a_0}_{H_6\text{'s and } H_7\text{'s}} + \underbrace{\sum_{k=1}^{9} a_k a_{9-k}}_{H_4\text{'s}} = 746. \quad \blacksquare$$

Using Theorems 5.5.1 and 5.5.2, we can characterize upset tournaments with equal biclique cover and partition numbers. This is given in Corollary 5.5.7.

Corollary 5.5.7 *Let T be an upset tournament in standard form on $n \geq 6$ vertices. Then*

$$bc(T) = bp(T)$$

if and only if the upset path does not contain any arcs of the form (v_i, v_{i+1}) for $3 \leq i \leq n-3$ or the upset path of T has exactly one of the structures given in Figure 5.3 and the upset path of T does not contain any additional arcs of the form (v_i, v_{i+1}) for $3 \leq i \leq n-3$.

Combining Theorems 5.5.5 and 5.5.6 gives the number of upset tournaments with equal biclique cover and partition numbers. This is given in Corollary 5.5.8.

Corollary 5.5.8 *Let d_n be the number of nonisomorphic upset tournament matrices of order n such that*

$$bc(T) = bp(T).$$

Let $\{a_n\}$ *be the Fibonnaci sequence* $a_0 = 1,\ a_1 = 1,$ *and* $a_{n+2} = a_{n+1} + a_n$ *for* $n \geq 2$. *Then*

$$d_n = \begin{cases} 2 & \textit{if} \quad n \leq 5 \\ 4 & \textit{if} \quad n = 6 \\ 7 & \textit{if} \quad n = 7 \\ a_{n-3} + 2a_{n-6} + \sum_{k=1}^{n-7} a_k a_{n-k-6} & \textit{if} \quad 8 \leq n \leq 14 \\ 654 & \textit{if} \quad n = 15 \\ 1123 & \textit{if} \quad n = 16 \\ a_{n-3} + 2a_{n-6} + 2a_{n-15} + \sum_{k=1}^{n-7} a_k a_{n-k-6} + \sum_{k=1}^{n-16} a_k a_{n-k-15} & \textit{if} \quad n \geq 17 \end{cases}.$$

6. Summary

6.1 Bipartite

- If B is a bipartite domino-free graph, then $b(A(B)) = r_{z^+}(A(B))$.
- If B is a bipartite C4-free graph, then

$$b(A(B)) = r_{z^+}(A(B)) = t(A(B)).$$

- If B is a minimal elementary bipartite graph with reduced adjacency matrix $A(B)$ of order $n \geq 3$, then

$$b(A(B)) = r_{z^+}(A(B)) = t(A(B)) = n.$$

- If B is minimal sub-elementary bipartite graph, then

$$b(A(B)) = r(A(B)) = r_{z^+}(A(B)) = t(A(B)).$$

- There exists a class of domino-free bipartite graphs with

$$\frac{r(A)}{b(A)} = \frac{r(A)}{t(A)} = \frac{3}{4}$$

and a class with

$$\frac{b(A)}{t(A)} = \frac{r(A)}{t(A)} = \frac{3}{4}.$$

- For each k, $3 \leq k \leq n = 5m$, there exists a bipartite graph B containing at least m dominos with

$$b(A(B)) = r(A(B)) = r_{z^+}(A(B)) = k.$$

- For each k, $3 \leq k \leq n = 5m$, $k = 5j + i$, $i \in \{0, 1\}$, $j \in \{1, 2, 3, ...\}$, there exists a bipartite graph B containing at least m dominos with

$$b(A(B)) = r(A(B)) = r_{z^+}(A(B)) = t(A(B)) = k.$$

Open Problems:

- If $B(A)$ is a C4-free bipartite graph, find a lower bound for

$$\frac{r(A)}{b(A)}.$$

- If $B(A)$ is a domino-free bipartite graph, find lower bounds for

$$\frac{r(A)}{b(A)}, \frac{b(A)}{t(A)}, \quad \text{and} \quad \frac{r(A)}{t(A)}.$$

- Find new classes of bipartite graphs with $bc(B) = bp(B)$ or two or more of the matrix ranks equal for the corresponding adjacency matrix.

6.2 Digraphs

- If D is a strongly unipathic digraph, then

$$b(A(D)) = r(A(D)) = r_{z^+}(A(D)) = t(A(D)).$$

- If D is a minimally strong digraph, then

$$b(A(D)) = r(A(D)) = r_{z^+}(A(D)) = t(A(D)).$$

Open Problem:

- Find new classes of digraphs with $bc(B) = bp(B)$ or two or more of the matrix ranks equal for the corresponding adjacency matrix.

6.3 Nearly Reducible Matrices

- If A is a nearly reducible matrix, then

$$b(A) = r(A) = r_{z^+}(A) = t(A).$$

- For each k, $2 \leq k \leq n$, there exists a nearly reducible matrix A with

$$b(A) = r(A) = r_{z^+}(A) = t(A) = k.$$

- A is a nearly reducible matrix if and only if $B(A)$ is a minimal sub-elementary bipartite graph.

6.4 Nearly Decomposable Matrices

- If A is a nearly decomposable matrix of order $n \geq 3$, then

$$b(A) = r_{z^+}(A) = t(A) = n.$$

- There exists a class of nearly decomposable matrices of order n with $r(A) = n$ if n is odd and $r(A) = n - 1$ if n is even.

- There exists a nearly decomposable matrix of order n with $r(A) = n-2$.
- A is a nearly decomposable matrix if and only if $B(A)$ is a minimal elementary bipartite graph.

Open Problems:

- Find a lower bound for the ratio

$$\frac{r(A)}{n}$$

  if A is a nearly decomposable matrix.
- Find a characterization of nearly decomposable matrices with respect to their real rank.
- Specifically, classify singular nearly decomposable matrices.

6.5 Tournaments

- If A is an n-tournament matrix, then

$$n - 1 \leq r(A) \leq r_{z^+}(A) \leq t(A) \leq n.$$

- If A is an n-tournament matrix, then $t(A) = n$ if and only if every strong component of $T(A)$ has at least three vertices.
- If A is a regular or near-regular n-tournament matrix, $n \geq 3$, then

$$r(A) = r_{z^+}(A) = t(A) = n.$$

- For each integer $m \geq 2$ and n sufficiently large,

$$\min\{b(T) : T \text{ is an } n \times n \text{ tournament matrix}\} \leq \frac{n}{m}.$$

- There exists a class of n-tournament matrices with

$$b(A) = r(A) = r_{z^+}(A) = t(A) = n.$$

- There exists a class of n-tournament matrices with no source or sink and

$$b(A) = r(A) = r_{z^+}(A) = t(A) = n - 1.$$

- There exists a singular n-tournament matrix A with $t(A) = n$.
- There exists a singular n-tournament matrix A with $r_{z^+}(A) = n$.
- There exists a singular n-tournament matrix A with

$$n - 2 = b(A) < r(A) = r_{z^+}(A) < t(A) = n.$$

Open Problems:

- Classify singular tournament matrices.
- Improve the bounds for the boolean rank of tournament matrices or show the above is best possible.
- What is the boolean rank of regular or near-regular tournament matrices?
- Does there exist a singular n-tournament matrix with $b(A) = n$?

6.6 Upset Tournaments

- Upset tournament matrices have been classified with respect to each rank.
- If A is n-tournament matrix corresponding to a upset tournament, then

$$\frac{2}{3}n < b(A) \leq r(A) = r_{z^+}(A) \leq t(A) = n.$$

- There exists a class of upset n-tournament matrices such that

$$\frac{b(A)}{n} \to \frac{2}{3} \qquad \text{as} \qquad n \to \infty.$$

- The number of nonisomorphic upset tournaments with

$$bc(T) = bp(T) = n \quad \text{and} \quad bc(T) = bp(T) = n - 1$$

can be given in terms of convolutions of the Fibonacci sequence.

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