CYCLE STRUCTURES IN GRAPHS

by

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ABSTRACT

The study of paths and cycles in graphs has long been fundamental in Graph Theory. One of the most popular questions in this area is determining when a graph contains a hamiltonian cycle, a cycle containing every vertex of the graph. We call such a graph a hamiltonian graph. We will investigate two different aspects of hamiltonian graphs. The first of these is graphs in which the hamiltonian cycle avoids certain subgraphs, and the second is the cycle lengths contained in a hamiltonian graph under given conditions.

Let G be a graph and H be a subgraph of G. If G contains a hamiltonian cycle C such that $E(C) \cap E(H)$ is empty, we say that C is an *H-avoiding hamiltonian cycle*. Let F be any graph. If G contains an H-avoiding hamiltonian cycle for every copy of H in G such that $H \cong F$, then we say that G is *F-avoiding hamiltonian*. We give minimum degree and degree-sum conditions which assure that a graph G is F-avoiding hamiltonian for various choices of F. In particular, we consider the cases where F is a union of k edge-disjoint hamiltonian cycles or a union of k edge-disjoint perfect matchings. If G is F-avoiding hamiltonian for any such F, then it is possible to extend families of these types in G. We also undertake a discussion of F-avoiding pancyclic graphs.

From there we turn our attention to pancyclic graphs. A graph of order n is said to be pancyclic if it contains cycles of all lengths from three to n. We consider hamiltonian graphs with two vertices of high degree sum. We determine the conditions on this degree sum that assure that the graph is pancyclic. We also consider what cycles must be present in the graph when the degree sum condition is reduced.

This abstract accurately represents the content of the candidate's thesis. I recommend its publication.

Signed ______________________________
Michael S. Jacobson

DEDICATION

For my Bell.

ACKNOWLEDGMENT

I would like to thank my advisor Michael S. Jacobson, without whose support and direction this thesis would not be possible. I would also like to thank my co-author, colleague and friend Mike Ferrara for his help, support and most of all, for not backing down. Finally, I would like to thank my children for their patience with me over the last four years.

CONTENTS

FIGURES

Figure

1. Introduction

1.1 Prologue

The study of paths and cycles in graphs has long been fundamental in Graph Theory. On of the most popular questions in this area is determining when a graph contains a spanning cycle, that is a cycle containing every vertex. Named for Sir William Rowan Hamilton, this problem traces its origin to the 1850's. Research into this problem has led to a number of deep and interesting results.

One of particular interest lies in studying what cycles are contained in hamiltonian graphs. We begin with a hamiltonian graph and, given some condition on that graph, investigate other cycles present in the graph.

In this chapter we provide some brief background material on graphs, emphasizing terminology and notation used in this paper. In Chapter 2 we introduce the idea of a hamiltonian cycle avoiding particular subgraphs in a graph and develop necessary conditions for a graph to contain a set of disjoint hamiltonian cycles that avoid a family of subgraphs. The result is extended to sets of hamiltonian paths and sets of cycles that avoid a family of subgraphs.

In Chapter 3 we investigate necessary conditions for a hamiltonian graph to contain cycles of all lengths, as well as determine what cycle lengths must be present under a given set of conditions. We begin with a degree sum condtion on two vertices of a hamiltonian cycle, using a result of Hakimi and Schmeichel ([13]) as a starting point. We then give analogous results for bipartite graphs, though these results lack sharpness.

1.2 Background

We begin with some fundamental terminology and notation related to graphs in general. We also include notation specific to the sections that follow. We suggest [8] or [29] as a general resource.

A *graph* $G = (V, E)$ is a finite set of nonempty objects called vertices together with a set of unordered pairs of distint vertices of G called edges. The vertex set of G is denoted by $V(G)$ and the edge set by $E(G)$. For a graph G with $u, v \in V(G)$, the edge $e = \{u, v\}$ is said to join u and v and we refer to u and v as being adjacent. We will denote an edge $e = \{u, v\}$ by uv. A graph defined in this way is called a *simple* graph, since there is at most one edge between any pair of vertices and a vertex is not adjacent to itself.

The *order* of a graph G is the cardinality of its vertex set, denoted $|V(G)|$, and the size of G is the cardinality of its edge set, denoted $|E(G)|$. For a vertex $v \in V(G)$, the degree of v, $d_G(v)$, is the number of edges incident with v and the neighbourhood of v, $N_G(v)$, is the set of vertices adjacent to v. When the context is clear, the subscript G is omitted. Let $\delta(G)$ and $\Delta(G)$ denote the minimum degree and maximum degree of G, respectively. A graph in which every vertex has degree r is referred to as an r-regular graph.

For a a graph $G = (V, E)$ and a set $S \subset V$ we define the *induced subgraph* $G[S]$ to be the graph with vertex set S and edge set $T = \{e | e \in E \,\text{and}\, e \in S\}$. That is, the edges of an induced subgraph include all edges of G that join vertices in S. A *subgraph* of a G is a graph $H = (S, T)$ where both $S \subseteq V$ and $T \subseteq E$ and the assignment of endpoints to edges in G' is the same as in G. Note that any subgraph of G is contained in an induced subgraph of G. If H is a subgraph of G we write $H \subseteq G$. In Figure 1.1 we show an example of a simple graph along

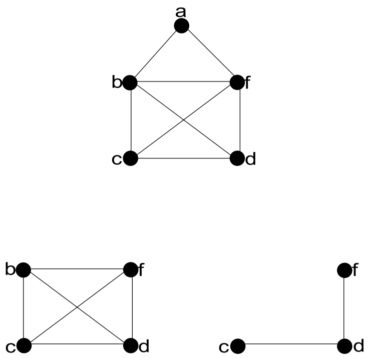

Figure 1.1: A graph and two subgraphs.

with two subgraphs. The subgraph on the left is induced, whereas the subgraph on the right is not, as it does not contain the edge cf. The *complement* $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

The disjoint union of graphs $G_1, \ldots, G_k$ is the graph G with $V(G) = V(G_1) \cup \ldots \cup V(G_k)$ and $E(G) = E(G_1) \cup \ldots \cup E(G_k)$. In the case where $G_i = H$ for all $1 \leq i \leq k$, we write $G = kH$. The *join* of graph G_1 and G_2 is the graph $G = G_1 + G_2$ with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{xy | x \in V(G_1), y \in V(G_2)\}$.

A *clique* in a graph G is a subgraph $H \subseteq G$ in which every pair of vertices of H are adjacent. If G itself is a clique, then we say that G is complete. The complete graph on n vertices is denoted by K_n. An independent set in a graph G is a set of pairwise nonadjacent vertices. A graph G is *bipartite* if $V(G)$ is the union of two disjoint independent sets called partite sets of G. A bipartite graph G with parts X and Y is complete if for every $x \in X$, $y \in Y$, xy is an edge of G. If the $|V(X)| = t$ and $|V(Y)| = n - t$, we denote G by $K_{t,n-t}$.

A *path* in a graph G is a sequence of distinct vertices $v_1v_2 \dots v_k$ such that v_iv_{i+1} is an edge of G for each i, $1 \leq i < k$. If, in addition, v_kv_1 is an edge of G, then the sequence is a *cycle*. The *order* of a path or cycle is the number of vertices in the sequence and the *length* is the number of edges contained in the path or cycle. Thus a path of length k has order $k+1$ while a cycle of length k has order k. The *distance* between two vertices $u, v \in V(G)$ is the length of the shortest path connecting u and v and is denoted by $d_G(u, v)$. A graph G is *connected* if every pair of vertices in G belongs to a common path; G is disconnected otherwise.

Of particular interest is a path or cycle containing all the vertices of G. Such a spanning path or cycle is called a *hamiltonian* path or cycle. If a graph G contains a hamiltonian path, we say it is traceable, and if it contains a hamiltonian cycle, we say that G is *hamiltonian*. Hamiltonian graphs have been widely studied, and a good reference for the recent status of such problems is [22]. Let $\sigma_2(G)$ denote the minimum degree sum over all pairs of nonadjacent vertices in G. Ore's Theorem [20], one of the classic results pertaining to hamiltonian graphs, states the following.

Theorem 1.1 (Ore 1960) *If G is a graph of order $n \geq 3$ with $\sigma_2(G) \geq n$ then G is hamiltonian.*

The requirement that $\sigma_2(G) \geq n$ is known as the Ore condition and we will refer to similar degree sum restrictions as Ore-type conditions.

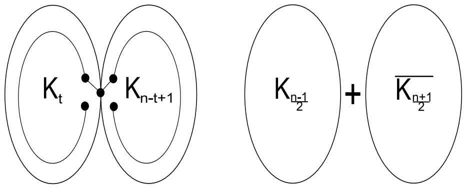

Figure 1.2: Nonhamiltonian graphs with $\sigma_2 = n - 1$

A graph is a *butterfly* if it is composed of two complete graphs intersecting in exactly one vertex. If G is isomorphic to a butterfly or if $K_{\frac{n-1}{2},\frac{n+1}{2}} \subseteq G \subseteq K_{\frac{n-1}{2}} + \overline{K_{\frac{n+1}{2}}}$, then G is nonhamiltonian and $\sigma_2(G) = n - 1$, demonstrating the sharpness of Ore's Theorem. In Figure 1.2 we show a butterfly of order n on the left and $K_{\frac{n-1}{2}} + \overline{K_{\frac{n+1}{2}}}$ on the right, demonstrating that neither is hamiltonian. In fact, it has been noted by several authors [1, 14, 18] that these are the only nonhamiltonian graphs with this property. Dirac's Theorem [9], another classic result, is an immediate corollary of Theorem 1.1.

Theorem 1.2 (Dirac 1952) *If G is a graph of order $n \geq 3$ with $\delta(G) \geq \frac{n}{2}$ then G is hamiltonian.*

We will refer to $\delta(G) \geq \frac{n}{2}$ as the Dirac condition.

Theorems 1.1 and 1.2 are the basis for much of the study related to cycles in graphs. They led to the use of minimum degree and degree sum conditions as determining factors for desired cycle structures. One such property introduced by Chartrand is the idea of k-ordered. A graph is *k-ordered (hamiltonian)* if for every ordered sequence of k vertices there is a (hamiltonian) cycle that encounters the vertices of the sequence in the given order. Ng and Schultz [19] were the first to investigate such graphs.

Theorem 1.3 *Let G be a graph of order n and let k be an interger with $3 \leq k \leq n$. If $deg(u) + deg(v) \geq n + 2k - 6$ for every pair u, v of nonadjacent vertices of G, then G is k-ordered hamiltonian.*

Corollary 1.4 *Let G be a graph of order n and let k be an interger with $3 \leq k \leq n$. If $deg(u) \geq \frac{n}{2} + k - 3$ for every vertex u of G, then G is k-ordered hamiltonian.*

Another property that is widely studied using minimum degree and degree sum conditions is pancyclicity. A graph G of order n is *pancyclic* if it contains a cycle of each length k for $3 \leq k \leq n$. Bondy [6] showed that any graph satisfying Ore's condition is either pancyclic or isomorphic to $K_{\frac{n}{2},\frac{n}{2}}$. Aldred, Holton and Min [3] relaxed Ore's condition and showed the following:

Theorem 1.5 *If G satisfies $\sigma_2(G) \geq n - 1$, then G is pancyclic unless G is isomorphic to one of the following graphs:*

1. *a graph of order n consisting of two complete graphs joined at a vertex,*
2. *a subgraph of the join of a complete graph of order $\frac{n-1}{2}$ and an empty graph of order $\frac{n+1}{2}$,*
3. *$K_{\frac{n}{2},\frac{n}{2}}$, or*
4. *C_5.*

Given conditions that assure a graph contains a hamiltonian cycle leads to the question: When does a graph contain more than one hamiltonian cycle? A classic result of Smith (presented by Tutte [28]) states that every edge of a 3-regular graph is contained in an even number of hamiltonian cycles. Thomason [25] extended Smith's result to all r-regular graphs where r is odd, and in fact to all graphs in which all vertices have odd degree. Thomassen [26] completed regular graphs for sufficiently large r.

Theorem 1.6 *If G is hamiltonian and r-regular with $r \geq 300$, then G has a second hamiltonian cycle.*

Thomassen also considered the question for bipartite graphs.

Theorem 1.7 ([27]) *Let $C = x_1y_1x_2y_2 \ldots x_ny_nx_1$ be a hamiltonian cycle in a bipartite graph G.*

1. *If all the vertices $y_1, y_2, \ldots, y_n$ have degree at least 3, then G has another hamiltonian cycle containing the edge x_1y_1.*
2. *If all the vertices $y_1, y_2, \ldots, y_n$ have degree $d > 3$ and if $P_1, P_2, \ldots, P_q$ ($0 \leq q \leq d-3$) are paths in C of length 2 of the form $y_{i-1}x_iy_i$, then G has at least $2^{q+1-d}(d-q)!$ hamiltonian cycles containing $P_1 \cup \ldots \cup P_q$.*

Another natural direction is to question when a graph has multiple edge disjoint hamiltonian cycles. Faudree, Rousseau and Schelp [11] gave an Ore-type condition for the existence of such hamiltonian cycles.

Theorem 1.8 *Let k be a positive integer.*

1. *If G is a graph of order $n \geq 60k^2$ such that $\sigma_2(G) \geq n + 2k - 2$, then G contains k edge disjoint hamiltonian cycles.*

2. If G is a graph of order $n \geq 6k$ and size at least $\begin{pmatrix} n-1 \\ 2 \end{pmatrix} + 2k$, the G contains k edge disjoint hamiltonian cycles.

Egawa [10] combined a minimum degree condition with a degree sum condition to prove the following result on edge disjoint hamiltonian cycles.

Theorem 1.9 *Let $n, k \geq 2$ be integers with $n \geq 44(k-1)$. If G is a graph of order n with $\sigma_2(G) \geq n$ and $\delta(G) \geq 4k-2$, then G contains k edge disjoint hamiltonian cycles.*

In considering conditions that guarantee a graph is pancyclic, the degree bounds can be reduced if hamiltonicity is assumed. In 1981 Amar, Flandrin, Fournier, and Germa [4] showed the following:

Theorem 1.10 *Let G be a hamiltonian, nonbipartite graph of order $n \geq 162$. If $\delta \geq \frac{2n+1}{5}$, then G is pancyclic.*

Hakimi and Schmeichel [13] showed that edge density could be reduced by considering a pair of consecutive vertices on a hamiltonian cycle, which was a variation of the bipartite results of Schmeichel and Mitchem [16].

Our focus will be the use of degree sum or Ore-type conditions on hamiltonian graphs. We will consider a cycle C in a clockwise direction and will use C^+ to represent that we are following the vertices in the cycle in a clockwise direction and C^- to represent that we are following the vertices in the cycle in a counterclockwise direction. Similarly, for a vertex x_i of the cycle, we represent the vertex succeeding x_i on C as x_i^+ and the vertex preceding x_i as x_i^-. For a set of vertices S of a graph, we use $N_C(S)$ to denote the set of neighbours of elements of S on C, $N^+(S)$ and $N^-(S)$ to represent the vertices on C that succeed and precede neighbours of S, resectively.

2. F-avoiding Hamiltonian Graphs

As noted in the previous chapter, the idea of imposing structure on a hamiltonian cycle has been considered, as is evidenced by k-ordering. In [15], Kronk extended a theorem of Pósa [21], giving a condition under which certain edges were contained in a hamiltonian cycle. Aldred, Holton, Porteous, and Plummer [2] considered matchings that contained certain edges while avoiding others. We are interested in a hamiltonian cycle in a graph G that avoids sets of edges.

2.1 F-Avoiding Hamiltonian

Let G be a graph and H be a subgraph of G. If G contains a hamiltonian cycle C such that $E(C) \cap E(H)$ is empty, then we say that C is an *H-avoiding hamiltonian cycle*. Let F be any graph. If G contains an H-avoiding hamiltonian cycle for every subgraph H of G such that $H \cong F$, then we say that G is *F-avoiding hamiltonian*. We note here that G is F-avoiding hamiltonian if and only if $G - E(H)$ is hamiltonian for every subgraph H of G such that $H \cong F$. We wish to determine conditions on G and F that assure that G is F-avoiding hamiltonian.

The *closure* of a graph G of order n, denoted $cl(G)$, is obtained by recursively joining nonadjacent vertices u and v whenever $d(u) + d(v) \geq n$ until no such pair of vertices exists. The following theorem from [7] will be used several times in this section.

Theorem 2.1 *A graph G is hamiltonian if and only if $cl(G)$ is hamiltonian.*

We first give two Ore-type conditions that assure G is F-avoiding hamiltonian.

Theorem 2.2 *Let G be a graph of order $n \geq 3$ and let F be a graph of order $t \leq \frac{n}{2}$ and maximum degree at most k. If $\sigma_2(G) \geq n+k$ then G is F-avoiding hamiltonian. This result is sharp for all choices of F.*

Proof: Let H be any subgraph of G that is isomorphic to F and let $G' = G - E(H)$. It suffices to show that G' is hamiltonian. We will, in fact, show that $cl(G')$ is hamiltonian implying the result by Theorem 2.1. Let v be any vertex in G' that is not a vertex of H and let w be any vertex in G' that is not adjacent to v. Then $d_{G'}(w) \geq d_G(w) - k$ and $d_{G'}(v) = d_G(v)$, so that

$$d_{G'}(v) + d_{G'}(w) \geq d_G(v) + d_G(w) - k \geq (n+k) - k = n.$$

This implies that v and w are adjacent in $cl(G')$ and, in fact, $d_{cl(G')}(v) = n-1$. Since $t \leq \frac{n}{2}$, this implies that at least $\frac{n}{2}$ vertices have degree $n-1$ in the closure, and hence all vertices of $cl(G')$ have degree $n-1$. Consequently, $cl(G') = K_n$, which is hamiltonian implying by Theorem 2.1, that G' is hamiltonian as well. The result follows.

Let H be any graph with $\Delta(H) \leq k$ and let x be a vertex of maximum degree in H. To see that the theorem is sharp, consider a the graph G on n vertices constructed from K_{n-1} and an additional vertex v of degree $\Delta(H)+1$, shown on the left of Figure 2.1. This graph has $\sigma_2 \leq n+k-1$ since $\Delta(H) \leq k$ and contains numerous copies of H with v playing the role of x. Removing the edges of any of these copies from G leaves a graph that is clearly not hamiltonian, as the degree of v would be one, as is shown on the right in Figure 2.1. ■

In Theorem 2.2, note that in order to assure the existence of a hamiltonian cycle that avoids any nonempty collection of edges in G, we must exceed the

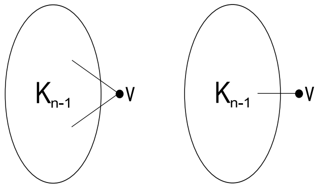

Figure 2.1: A graph G with $\sigma_2(G) \leq n + k - 1$ that is not H-avoiding hamiltonian.

Ore condition. Perhaps unexpectedly, this is not so when we consider the Dirac condition.

Theorem 2.3 *Let G be a graph of order $n \geq 3$ with $\delta(G) \geq \frac{n}{2}$. If E' is any subset of $E(G)$ such that $|E'| \leq \frac{n-6}{4}$ then there is a hamiltonian cycle in G containing no edge from E'. This result is sharp.*

Proof: It suffices to prove the theorem when $|E'| = \frac{n-6}{4}$. Let H be the subgraph of G induced by E'. Note that H has at most $\frac{n-6}{2}$ vertices. For convenience let G' be $G - E'$. We proceed by considering $cl(G')$. Each vertex in $G - V(H)$ still has degree at least $\frac{n}{2}$ in G', and as such the graph on $V(G) - V(H)$ is complete in $cl(G')$. Let $v \in V(H)$ be a vertex of degree $\Delta(H)$. Then

$$|V(H)| \leq \Delta(H) + 1 + 2(|E'| - \Delta(H)) \leq \frac{n-6}{2} - \Delta(H) + 1,$$

as $|V(H)|$ would be maximized in the case where those edges not adjacent to v form a matching in H. This implies that $|V(G) - V(H)| = n - |V(H)| \geq \frac{n+6}{2} + \Delta(H) - 1$. Thus, since $G - V(H)$ induces a clique in $cl(G')$ each vertex in $G - V(H)$ has degree at least $\frac{n+6}{2} + \Delta(H) - 2$. We now also note that each vertex in $V(H)$ has degree at least $\frac{n}{2} - \Delta(H)$ in G'. Let x and w be arbitrary vertices in G' chosen from $V(H)$ and $G - V(H)$ respectively. After closing $G - V(H)$, we have that

$$d(x) + d(w) \geq \left(\frac{n}{2} - \Delta(H)\right) + \left(\frac{n+6}{2} + \Delta(H) - 2\right) = n + 1 > n.$$

This implies that for any choice of x and w, xw is in $cl(G')$. Consequently, $cl(G')$ contains the join of $K_{|G-V(H)|}$ and $\overline{K_{|V(H)|}}$, which is hamiltonian since $|G-V(H)| > |V(H)|$. Thus, as $cl(G')$ is hamiltonian, G is H-avoiding hamiltonian, and the result follows.

To see that the theorem is sharp, let $k \geq 2$ be a positive integer, and let $n = 4k + 2$. We construct a graph H of order n by starting with the complete bipartite graph $K_{\frac{n}{2}-1,\frac{n}{2}+1}$ and adding the matching $e_1, \ldots, e_{k+1}$ to the partite set of size $\frac{n}{2}+1$. Removing any $k = \frac{n-2}{4}$ of the edges e_i yields a non-hamiltonian graph. Thus, if G is E'-avoiding hamiltonian, $|E'| \leq \frac{n-2}{4} - 1 = \frac{n-6}{4}$. ■

We now turn our attention to the problem of finding F-avoiding hamiltonian cycles in a graph G when the order of F is closer to the order of G. The next result can be obtained using techniques like those in the proof of Theorem 2.2, and is also a corollary of the theorem that follows, so here we provide only the sharpness example.

Theorem 2.4 *Let G be a graph of order $n \geq 3$ and let F be a graph with maximum degree k. If $\sigma_2(G) \geq n + 2k$ then G is F-avoiding hamiltonian. This result is sharp for all values of k.*

To see that the theorem is sharp for every value of k, let $n \geq 2k + 1$ be an odd integer and let B be any k-regular bipartite graph with partite sets of size $\frac{n-1}{2}$. In each partite set of B add an edge between every pair of vertices so that each partite set is a copy of $K_{\frac{n-1}{2}}$, forming a (no longer biparite) graph

B'. We then create the graph G by taking the join of B' and K_1 and we note that $\sigma_2(G) = n + 2k - 1$. If we remove the edges of the bipartite graph B, we are left with two cliques of order $\frac{n+1}{2}$ intersecting in a vertex, which is not hamiltonian. This implies that G is not B-avoiding hamiltonian, establishing the desired sharpness.

If we relax the degree condition in Theorem 2.4 slightly, it becomes possible that $G - E(H)$ is no longer hamiltonian. We can show however, that if $G - E(H)$ is not hamiltonian then it must fall into one of two exceptional classes.

Theorem 2.5 *Let $k \geq 0$ be an integer and let G be a graph on $n \geq 2k + 3$ vertices with $\sigma_2(G) \geq n + 2k - 1$. If F is a graph with maximum degree at most k, then G is F-avoiding hamiltonian, or there is some subgraph H of G such that $H \cong F$ and either $G - E(H)$ is a butterfly, or $K_{\frac{n-1}{2},\frac{n+1}{2}} \subseteq G \subseteq K_{\frac{n-1}{2}} + \overline{K_{\frac{n+1}{2}}}$.*

Before we begin we will give some useful notation and lemmas. Define G' to be the graph $G - E(H)$ and observe that $\sigma_2(G) \geq n + 2k - 1$ implies that the minimum degree of G is at least $2k + 1$ and hence that the minimum degree of G' is at least k+1. Moreover, for any vertices $x, y \in V(G)$ such that xy is not an edge in G, $d_{G'}(x) + d_{G'}(y) \geq n - 1$.

The following two lemmas will be used to prove Theorem 2.5.

Lemma 2.6 *Let G be a graph on $n \geq 2k + 3$ vertices with $\sigma_2(G) \geq n + 2k - 1$. Then for any subgraph $H \subset G$ with $\Delta(H) \leq k$, either G' is a butterfly or G' is 2-connected.*

Proof: Let H be a subgraph of G with $\Delta(H) \leq k$. We will show by way of contradiction that G' is a butterfly or is 2-connected.

Suppose that G' contains a vertex v such that $G' - v$ is disconnected, and let S_i and S_j be two components of $G' - v$ with $|V(S_i)| = s_i$ and $|V(S_j)| = s_j$. Since the minimum degree of G' is at least $k+1$, the minimum degree of $G' - v$ is at least k. Hence each component of $G' - v$ has at least $k+1$ vertices. Let x be a vertex in S_i. Then for all $j \neq i$, there exists $y \in S_j$ such that xy is not an edge in G, and therefore $d_{G'}(x) + d_{G'}(y) \geq n - 1$. (Note that this is true for every vertex of $G' - v$.) We also have that $d_{G'}(x) \leq s_i$ and $d_{G'}(y) \leq s_j$. Combining the inequalities we get $n-1 \leq d_{G'}(x)+d_{G'}(y) \leq s_i+s_j \leq n-1$, which implies that $s_i + s_j = n - 1$ (and also that $G' - v$ has exactly two components), $d_{G'}(x) = s_i$ $d_{G'}(y) = s_j$ and both x and y are adjacent v. This is true for all $x \in V(S_i)$, $y \in V(S_j)$, so G' is a butterfly. ■

Lemma 2.7 *Let G be a graph on $n \geq 2k+3$ vertices with $\sigma_2(G) \geq n + 2k - 1$ and let H be any subgraph of G with $\Delta(H) \leq k$. Let C be a longest cycle in G', with $|C| = t$. For any component S of $G' - C$ with $|N_C(S)| \geq 2$ we have the following:*

1. *For all $x \in V(S)$ and for all $x_i, x_j \in N_C(S)$ $\quad x_i^+x_j^+ \notin E(G')$ and $x_i^+x \notin E(G')$. (For all $x \in V(S)$ and for all $x_i, x_j \in N_C(S)$ $\quad x_i^-x_j^- \notin E(G')$ and $x_i^+x \notin E(G')$.) Furthermore, $|N_C(S)| \leq \frac{t}{2}$.*

2. *For all $x \in V(S)$ and for all $x_i \in N_C(S)$ such that $xx_i^+ \notin E(G)$, and for all $y \notin N_C^+(S) \cup V(S)$, $\;x_i^+y \in E(G')$. (For all $x \in V(S)$ and $x_i \in N_C(S)$ such that $xx_i^- \notin E(G)$, and for all $y \notin N_C^-(S) \cup V(S)$, $\;x_i^-y \in E(G')$.) Furthermore, x is adjacent to every vertex of $V(S) - x$.*

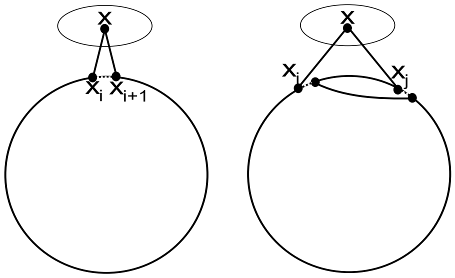

Figure 2.2: Extending a cycle C to a longer cycle.

Proof: Let G, $H\,and\,C$ satisfy the hypothesis. For convenience let $|N_C(S)| = \ell$ and note that we are considering $N_C(S)$ in the graph G'.

To show *1* let $x \in V(S)$ and $x_i, x_j \in N_C(S)$. Then xx_i^+ is not an edge of G' since $x_i x x_i^+ C^+ x_i$ would be a longer cycle in G', as is seen on the left in Figure 2.2. Suppose that $x_i^+ x_j^+$ is an edge in G'. Then $xx_iC^-x_j^+x_i^+C^+x_jx$ would be a longer cycle in G', as is seen on the right in Figure 2.2. The argument for x_i^- and x_j^- is similar.

To show *2* let $x \in V(S)$ and $x_i \in N_C(S)$ such that xx_i^+ is neither an edge in G' nor G. Then we know that $d_{G'}(x) + d_{G'}(x_i^+) \geq n - 1$. Since there are no edges between components of $G' - C$, $d_{G'}(x) \leq |V(S)| - 1 + \ell$. Recall from (*1*) that x_i^+ is not adjacent to any vertex of $N_C^+(S)$ nor $V(S)$, so $d_{G'}(x_i^+) \leq t - \ell + n - t - |V(S)| = n - |V(S)| - \ell$. Combining the inequalities yields $n - 1 \leq d_{G'}(x) + d_{G'}(x_i^+) \leq n - 1$. Therefore equality must hold, so x must be adjacent to every vertex in $V(S) - x$ and x_i^+ must be adjacent to every vertex in $G' - N_C^+(S) - V(S)$. The argument for x_i^- is similar. ■

We are now ready to prove Theorem 2.5.

Proof: Let G be a graph on $n \geq 2k+3$ vertices with $\sigma_2(G) \geq n+2k-1$ and let H be any subgraph of G with $\Delta(H) \leq k$. Let C be a longest cycle in G' and let $t = |V(C)|$. If C is a hamiltonian cycle, we are done. Suppose then that C is not a hamiltonian cycle.

We begin by showing that $G' - C$ is connected. Suppose otherwise and let $S_1, \ldots, S_h$ be the components of $G' - C$, with $|V(S_i)| = s_i$ for $1 \leq i \leq h$. Without loss of generality we will assume that $s_i \leq s_{i+1}$ for $1 \leq i \leq h-1$.

Let $x \in S_i$ and $y \in S_j$ for some distinct i and j. Then by part (*1*) of Lemma 2.7, $d_{G'}(x) \leq s_i - 1 + \frac{t}{2}$ and $d_{G'}(y) \leq s_j - 1 + \frac{t}{2}$ which implies that $d_{G'}(x) + d_{G'}(y) \leq s_i + s_j + t - 2 \leq n-2$ and hence $d_G(x) + d_G(y) \leq n + 2k - 2$. Consequently, as x and y are nonadjacent in G', xy must be an edge in H. Since x and y are arbitrary vertices of any two distinct components, this is true for every pair of vertices in different components of $G' - C$. At most k edges of H were incident with each vertex in each S_i, so $s_i \leq k$ for all $1 \leq i \leq h$.

Assume without loss of generality that $x \in S_1$ and consider the neighborhood of x on C. In G

$$d_C(x) \geq 2k + 1 - (s_i - 1) - \sum_{j \neq i} s_j \geq 2k + 2 - hs_h.$$

Since at most k edges of H were incident with each vertex in G, at most $(h-1)s_h$ edges between S_i and S_j are in H for all $j \neq i$, so in G'

$$d_C(x) \geq 2k + 2 - hs_h - (k - (h-1)s_h) = k + 2 - s_h.$$

At most $k - s_h$ of the non-neighbors of x on C in G' were neighbors of x on C in G. Therefore, there exist $x_i, x_j \in N_C(x)$ such that xx_i^+, xx_j^+ is neither an edge in G' nor G. By part (*2*) of Lemma 2.7 both x_i^+ and x_j^+ are adjacent to every vertex of G' not in $N_C^+(x) \cup V(S_1)$. Recall that $y \in S_j$, where $j \neq 1$. Then the cycle $xx_iC^-x_j^+yx_i^+C^+x_jx$ is longer than C, which contradicts that C is a longest cycle of G'. Therefore, $G' - C$ is connected.

Let S be the graph $G' - C$ and define the neighborhood of S in C to be $N_C(S)$. Suppose that $|N_C(S)| = \ell$. Note that if G' is not 2-connected, then by Lemma 2.6 G' is a butterfly. We will assume that G' is 2-connected.

Suppose that the order of S is at least $k+1$ and let x_i be in $N_C(S)$. Since S is connected, no vertex in S can be adjacent to x_i^+, but at most k edges incident with x_i^+ were in H. Consequently, as there are at least $k+1$ vertices in S, there exists $v \in V(S)$ such that x_i^+v is neither an edge in G' nor G, which implies $d_{G'}(x_i^+) + d_{G'}(v) \geq n-1$. Then by part *(2)* of Lemma 2.7 we know that for every $x_i \in N_C(S)$, $N_{G'}(x_i^+) = V(C) - N_C^+(S)$. This means that $d_{G'}(x_i^+) = t - \ell$ for all $x_i \in N_C(S)$. Since $d_{G'}(v) \leq |V(S)| - 1 + \ell = n - t - 1 + \ell$, we also have that $d_{G'}(x_i^+) + d_{G'}(v) \leq n - 1$, so equality must hold. Consequently v must be adjacent to every vertex in $N_C(S)$ and $V(S) - v$.

Suppose there is $x_j \in N_C(S)$ such that its predecessor x_j^- is not in $N_C^+(S)$; that is, there are at least two consecutive vertices on the cycle that are not in the neighbourhood of S. We have shown above that $x_j^+x_j^-$ and $x_{j-1}^+x_j \in E(G')$. Then the cycle $vx_{j-1}C^-x_j^+x_j^-C^-x_{j-1}^+x_jv$ has length $t+1$, which contradicts that C is a longest cycle of G'. So for each x_j, $(x_j^-)^-$ is in $N_C(S)$, implying that $|N_C(S)| = \frac{t}{2}$. Since G' is 2-connected, there exists $u \in V(S)$ with $u \neq v$ and $x_i \in N_C(S)$ such that $ux_i \in E(G')$. Then the cycle $vx_{i-1}C^-x_iuv$ has length $t+1$, which contradicts that C is a longest cycle of G'. Therefore, we will assume that S has order at most k. Suppose then that $|V(S)| = r$, where

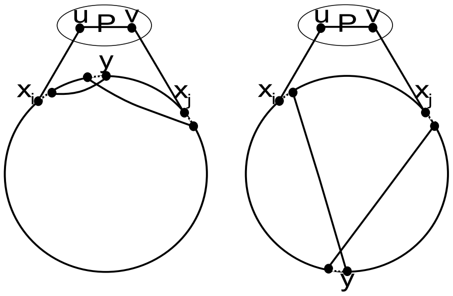

Figure 2.3: Extending a cycle C to a longer cycle.

$2 \leq r \leq k$. Since the minimum degree of G' is at least $k+1$, every vertex in S has at least $k+1-(r-1) = k-r+2 \geq 2$ neighbors on C. Let u, v be in $V(S)$ and x_i, x_j be in $N_C(S)$, such that $ux_i, vx_j \in E(G')$, and let P be any $u-v$ path in S. For every vertex $y \in V(C)$ such that $x_i^+y \in E(G')$ either $x_j^+y^- \notin E(G')$ or $x_j^+y^+ \notin E(G')$. Indeed, if y were to lie between x_i^+ and x_j^+ on C, the cycle $ux_iC^-x_j^+y^-C^-x_i^+yC^+x_jvPu$ would be longer than C, as can be seen in the left of Figure 2.3, and if y were to lie between x_j^+ and x_i^+ on C the cycle $ux_iC^-y^+x_j^+C^+yx_i^+C^+x_jvPu$ would be longer than C, as can be seen in the right of Figure 2.3. Thus $t \leq n-2$ implies that $d_{G'}(x_i^+) + d_{G'}(x_j^+) = d_C(x_i^+) + d_C(x_j^+) \leq n-2$ and therefore that $x_i^+x_j^+ \in E(G)$ for all $x_i, x_j \in N_C(S)$. Since $x_i^+x_j^+$ is not in $E(G')$ for any $x_i, x_j \in N_C(S)$, these edges must be in $E(H)$. But the minimum degree of G' is at least $k+1$, so $|N_C(S)| \geq k-r+2$ implies that there are at least $k-r+1$ such edges for each x_i^+. Then $r \leq k$ implies that there is at least one vertex $w \in V(S)$ for each x_i^+ such that wx_i^+ is neither an edge in G' nor G. (These w are not necessarily distinct.) By the same argument used above we see that $|N(S)| = \frac{t}{2}$, so we can find a cycle longer than C, which is a contradiction.

Hence we may assume that $|V(S)| = 1$. Let x be the vertex in S and suppose that $d_{G'}(x) < \frac{n-1}{2}$. Then there is a vertex $x_i^+ \in N_C^+(x)$ such that xx_i^+ is neither an edge in G' nor G and a vertex $x_j^- \in N_C^-(x)$ such that xx_j^- is neither an edge in G' nor G. Then by part (**b**) of Lemma 2.7, x_i^+ is adjacent to every vertex in $V(C) - N_C^+(x)$ and x_j^- is adjacent to every vertex in $V(C) - N_C^-(x)$. First suppose that $x_i^+ = x_{i+1}^-$; that is, x_i^+ is the only vertex between x_i and x_{i+1} on C. Since $d_{G'}(x) < \frac{n-1}{2}$, there exists a vertex x_m^-such that $x_m^- \notin N_C^+(x)$. Then the cycle $xx_iC^-x_mx_i^+x_m^-C^-x_{i+1}x$ is hamiltonian, which contradicts that C is a longest cycle in G'. So $x_i^+ \neq x_{i+1}^-$. By a similar argument we find that $x_j^- \neq x_{j-1}^+$. Hence by part (*2*) of Lemma 2.7 we know that $x_i^+x_j$ and $x_j^-x_j^+$ are edges in G'. Then the cycle $xx_iC^-x_j^+x_j^-C^-x_i^+x_jx$ is a hamiltonian cycle, which contradicts that C is a longest cycle in G'. Therefore we may assume that $d_{G'}(x) = \frac{n-1}{2}$. Observe that $N_C^+(x) \cup x$ is an independent set of order $\frac{n+1}{2}$. Since $n \geq 2k+3$, $\frac{n+1}{2} \geq k+2$, so for every vertex $y \in N_C^+(x) \cup x$ there is a vertex $z \in N_C^+(x)$ such that yz is neither an edge in G' nor G. Then every vertex in $N_C^+(x) \cup x$ is adjacent to exactly $N_C(x)$. It follows that $K_{\frac{n+1}{2},\frac{n-1}{2}} \subseteq G' \subseteq K_{\frac{n-1}{2}} + \overline{K_{\frac{n+1}{2}}}$. ■

The conclusion that there is a subgraph $H \cong F$ such that $G - E(H)$ either falls into the class of butterflies or is a supergraph of $K_{\frac{n-1}{2},\frac{n+1}{2}}$ is only feasible for certain choices of F. The following two corollaries reflect this.

Corollary 2.8 *Let $k \geq 0$ be an integer and let G be a graph on $n \geq 2k+3$ vertices with $\sigma_2(G) \geq n+2k-1$. If F is a graph of order n with minimum degree at least 1 and maximum degree at most k, then either G is F-avoiding hamiltonian or there is some subgraph H of G such that $H \cong F$ and $K_{\frac{n-1}{2},\frac{n+1}{2}} \subseteq$*

$G - E(H) \subseteq K_{\frac{n-1}{2}} + \overline{K_{\frac{n+1}{2}}}$.

Corollary 2.9 *Let* $k \geq 0$ *be an integer and let* G *be a graph on* $n \geq 2k+3$ *vertices with* $\sigma_2(G) \geq n+2k-1$. *If* F *is a connected graph with maximum degree at most* k *and order at least* $\frac{n}{2}+1$, *then either* G *is* F*-avoiding hamiltonian or* F *is bipartite, has order at most* $n-1$, *and there is some subgraph* H *of* G *such that* $H \cong F$ *and* $G - E(H)$ *is a butterfly.*

The problem of determining when a graph contains k edge-disjoint hamiltonian cycles has long been of interest. In [11], it was shown that a graph G of sufficiently large order n with $\sigma_2(G) \geq n + 2k - 2$ contains k edge-disjoint hamiltonian cycles. The problem of finding disjoint hamiltonian cycles in bipartite graphs has also been examined [12]. Other results focus on finding k edge-disjoint hamiltonian cycles in graphs that satisfy the Ore condition. In [10], it is shown that if G is a graph of sufficiently large order n with $\sigma_2(G) \geq n$ and $\delta(G) \geq 4k-2$ then G contains k edge-disjoint hamiltonian cycles.

In light of these results, we present the following variation. Let H be a family of $k \geq 1$ edge-disjoint hamiltonian cycles in a graph G. If $G - E(H)$ is hamiltonian, then $G - E(H)$ contains a hamiltonian cycle C which, together with H, would comprise a family of $k+1$ edge-disjoint hamiltonian cycles in G. In fact, if G is F-avoiding hamiltonian for graph F isomorphic to k edge-disjoint hamiltonian cycles, then we are not only finding disjoint families of hamiltonian cycles, but in fact we are able to *extend* any family of k edge-disjoint hamiltonian cycles to a family of $k+1$ edge-disjoint hamiltonian cycles. Taking into account Corollaries 2.8 and 2.9, the following is an immediate consequence of Theorem

2.5.

Corollary 2.10 *Let $k > 0$ be an integer and let G be a graph on $n \geq 4k + 3$ vertices with $\sigma_2(G) \geq n + 4k - 1$ and let H be any collection of k edge-disjoint hamiltonian cycles in G. Then H can be extended to a family of $k + 1$ edge-disjoint hamiltonian cycles. This result is sharp.*

Corollary 2.10 complements the results mentioned above pertaining to the existence of k edge-disjoint hamiltonian cycles. To see that Corollary 2.10 is sharp, consider a graph G of even order $n \geq 4k + 4$ which is comprised of two disjoint cliques of order $\frac{n}{2}$, denoted G_1 and G_2, and a family H of k edge-disjoint hamiltonian cycles with the property that H is bipartite with partite sets $V(G_1)$ and $V(G_2)$. Then $\sigma_2(G) = n + 4k - 2$, but $G - E(H)$ is isomorphic to $2K_{\frac{n}{2}}$ which is not hamiltonian.

Since a hamiltonian cycle of even order can be viewed as the union of two disjoint perfect matchings, we also obtain the following result pertaining to extending families of perfect matchings.

Corollary 2.11 *Let $k > 0$ be an integer and let G be a graph of even order $n \geq 2k + 3$ with $\sigma_2(G) \geq n + 2k - 1$ and let H be any collection of k edge-disjoint perfect matchings in G. Then H can be extended to a family of $k + 2$ edge-disjoint perfect matchings in G. This result is sharp.*

To see that Corollary 2.11 is sharp, let t be an odd integer such that $2t \geq 2k - 1$. Consider a graph G of order $2t$ which is comprised of two disjoint cliques of order t, denoted G_1 and G_2, and a family H of k edge-disjoint perfect matchings with the property that H is bipartite with partite sets $V(G_1)$ and

$V(G_2)$. Then $\sigma_2(G) = n + 2k - 2$, but $G - E(H)$ is isomorphic to $2K_t$ which does not contain a perfect matching as t is odd.

As mentioned above, certain supergraphs of $K_{\frac{n-1}{2},\frac{n+1}{2}}$ and the class of butterflies serve to establish the sharpness of Ore's Theorem. That is, they are examples of nonhamiltonian graphs of with $\sigma_2 = n - 1$. If we let $k = 0$ in Theorem 2.5 we can see that these are in fact the only such graphs. As mentioned above, this fact was also noted in [1], [14] and [18].

Corollary 2.12 *Let G be a nonhamiltonian graph of order n with $\sigma_2(n) = n-1$. Then either G is a butterfly or $K_{\frac{n-1}{2},\frac{n+1}{2}} \subseteq G \subseteq K_{\frac{n-1}{2}} + \overline{K_{\frac{n+1}{2}}}$.*

A graph G is *traceable* if it contains a spanning path, that is, a path containing every vertex of G. Such a path is known as a *hamiltonian path.* We say that G is F-avoiding traceable if for any subgraph H of G such that $H \cong F$, $G - E(H)$ has a hamiltonian path. The following is an immediate corollary of Theorem 2.5, as hamiltonian graph, butterflies and graphs G satisfying $K_{\frac{n-1}{2},\frac{n+1}{2}} \subseteq G \subseteq K_{\frac{n-1}{2}} + \overline{K_{\frac{n+1}{2}}}$ are traceable.

Corollary 2.13 *Let $k \geq 0$ be an integer and let G be a graph on $n \geq 2k + 3$ vertices with $\sigma_2(G) \geq n + 2k - 1$. If F is a graph with maximum degree at most k, then G is F-avoiding traceable.*

2.2 F-avoiding Pancyclicity

A graph G is *pancyclic* if G contains a cycle of each length from 3 up to $|V(G)|$. The study of pancyclic graphs is a natural extension of the hamiltonian problem. Having developed necessary conditions for a graph G to be F-avoiding hamiltonian, we turn our attention to the analogous notion for pancyclic graphs. Let F and G be graphs. If $G - E(H)$ is pancyclic for every subgraph H of G such that $H \cong F$, then we say that G is *F-avoiding pancyclic*. In this section we will present several conditions on G and F which assure that G is F-avoiding pancyclic. In addition to Theorem 2.5, the following two theorems from [13] will be useful.

Theorem 2.14 *Let G be a graph of order n with $V(G) = \{v_0, \dots, v_{n-1}\}$ and hamiltonian cycle $v_0, \dots, v_{n-1}, v_0$. If $d(v_0) + d(v_{n-1}) \geq n$ then G is either pancyclic, bipartite or missing only an $(n-1)$-cycle.*

Theorem 2.15 *Let G be a graph of order n with $V(G) = \{v_0, \dots, v_{n-1}\}$ and hamiltonian cycle $v_0, \dots, v_{n-1}, v_0$. If $d(v_0) + d(v_{n-1}) \geq n+1$ then G is pancyclic.*

We begin with an ore-type condition for H-avoiding pancyclicity that leaves us with no exception graphs.

Theorem 2.16 *Let G be a graph of order n and let F be a graph with maximum degree k. If $\sigma_2(G) \geq n + 2k + 1$ then G is F-avoiding pancyclic. This result is sharp for all values of k.*

Proof: By Theorem 2.4 we know that $G' = G - E(F)$ is hamiltonian. Let x be a vertex of G with $d(x) = \delta(G)$. Then there is a vertex y of G with $d(y) \geq n + 2k + 1 - \delta(G)$. Let C be a hamiltonian cycle in G'. Then $d_{G'}(y) + d_{G'}(y^+) \geq n + k + 1 - \delta(G) + \delta(G) - k = n + 1$, so G' is pancyclic by Theorem 2.15.

To see that this result is best possible, let $n \geq 2k+2$ be an even integer and let H be any k-regular graph on $\frac{n}{2}$ vertices. We create the graph G by taking the join of two copies of H. Then $\sigma_2(G) = n + 2k$ and the removal of the edges of each copy of H leaves us with $K_{\frac{n}{2},\frac{n}{2}}$, which is not pancyclic since it contains no odd cycles. ■

The following is a well-known result of Bondy [6].

Theorem 2.17 *Let G be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n$ then either G is pancyclic or G is isomorphic to $K_{\frac{n}{2},\frac{n}{2}}$.*

If we relax the conditions on $\sigma_2(G)$ given in Theorem 2.16 slightly we obtain a similar result.

Theorem 2.18 *Let $k \geq 0$ be an integer and let G be a graph on $n \geq 6k+4$ vertices with $\sigma_2(G) \geq n + 2k$. If F is a graph with maximum degree at most k, then G is F-avoiding pancyclic or there is some subgraph H of G such that $H \cong F$ and $G - E(H)$ is $K_{\frac{n}{2},\frac{n}{2}}$. This result is sharp for all values of k.*

Proof: For simplicity we let $G' = G - E(H)$. By Theorem 2.4 we know that G' contains a hamiltonian cycle C. If $\sigma_2(G') \geq n$ the result follows by Theorem 2.17. Suppose that $\sigma_2(G') < n$ and that G' is not pancyclic. Let v be a vertex with degree $\delta(G') < \frac{n}{2}$. Then, as $\sigma_2(G) \geq n + 2k$ and $d_H(v) \leq k$ there are at least $\frac{n-1}{2} - k$ vertices of degree at least $n - \delta(G') \geq \frac{n+1}{2}$. Since $n \geq 6k+4$, $\frac{n-1}{2} - k > \frac{n}{3}$, and we can find two vertices x and y on C such that both x and y have degree at least $\frac{n+1}{2}$ and $1 \leq d_C(x,y) \leq 2$.

If $d_C(x,y) = 1$ then G' is pancyclic by Theorem 2.15. Therefore we may assume that $d_C(x,y) = 2$.

We assume without loss of generality that $x = y^{++}$ on C and for convenience let $C = v_0, v_1, \ldots, v_{n-1}, v_0$ with $x = v_0$ and $y = v_{n-2}$. By Theorem 2.14, since $d_{G'}(x) + d_{G'}(v_{n-1}) \geq (n - \delta(G')) + \delta(G') = n$, we need only show that G' contains an $(n-1)$-cycle. In $G'' = G' - v_{n-1}$, consider the hamiltonian path $v_0, \ldots, v_{n-2}$. We have $d_{G''}(v_0) + d_{G''}(v_{n-2}) \geq n + 1 - 2 = n - 1$, hence G" is hamiltonian and therefore G contains an $n - 1$-cycle. The result follows.

To see that the result is sharp, let $n \equiv 3 (\text{mod } 4)$ and let H be any k-regular graph on $\frac{n+1}{2}$ vertices. If we let G denote the join of H and $\overline{K_{\frac{n-1}{2}}}$ then $\sigma_2(G) = n + 2k - 1$ but $G - E(H)$ is isomorphic to $K_{\frac{n-1}{2}, \frac{n+1}{2}}$. ■

3. Cycles in Hamiltonian Graphs

In this chapter we will investigate the existence of cycles of various lengths in hamiltonian graphs. In the same paper in which Bondy proved Theorem 2.17 he stated the following "metaconjecture":

Conjecture 3.1 (Bondy's Metaconjecture) *Almost any nontrivial condition on a graph which implies that the graph is hamiltonian also implies that the graph is pancyclic. There may be a simple family of exceptional graphs.*

In light of Conjecture 3.1, it has been of interest to examine the relationship between hamiltonicity and pancyclicity in graphs. Schmeichel and Mitchem suggested that degree bounds in theorems guaranteeing pancyclicity could be reduced if hamiltonicity were assumed. Ore's Theorem and Theorem 2.17 both consider graphs where σ_2 is high enough to assure a considerable amount of global density. We wish to relax the Ore condition and examine the cycle structure of hamiltonian graphs that may be globally sparse yet have a small number of vertices with higher degree. Schmeichel and Mitchem proved Theorems 2.14 and 2.15, in which edge density is greatly reduced. These theorems serve to motivate our investigation, as they require only a single pair of vertices having high degree sum. However this pair of vertices must be consecutive on some hamiltonian cycle in G. Our goal is to extend Theorems 2.14 and 2.15 by examining hamiltonian graphs containing a pair of vertices that have a high degree sum but may lie far apart on every hamiltonian cycle.

3.1 Absolute Bounds

We begin by determining the degree sum of two vertices on a hamiltonian cycle, regardless of their placement, required to assure that the graph is pancyclic. We start with the following two useful lemmas.

Lemma 3.2 *Let G be a hamiltonian graph of order $n \geq 5$ with hamiltonian cycle C. If there exists a vertex x in $V(G)$ of degree $n-2$, then G is pancyclic.*

Proof: Let $C = xv_1 \dots v_{n-1}x$ and let v_i, for some $2 \leq i \leq n-2$, be the vertex nonadjacent to x. If xv_ℓ is an edge of G, then the cycle $C' = xv_\ell C^- x$ is a cycle of length $\ell + 1$ in G, so our assumption implies that G contains cycles of all lengths except possibly a cycle of length $(i+1)$. However, since $n \geq 5$, there exist vertices v_j and v_{j+i-1} in G that are both adjacent to x, and hence $xv_j C^+ v_{j+i-1} x$ is an $(i+1)$-cycle. Consequently, G is pancyclic. ■

Lemma 3.3 *Let G be a hamiltonian graph of order $n \geq 7$ with hamiltonian cycle C. If there exists a vertex x in $V(G)$ of degree $n-3$, then either G is pancyclic or G contains a cycle of each length except $(n-1)$.*

Proof: Let $C = xv_1 \dots v_{n-1}x$ and let v_i, v_j, $2 \leq i < j \leq n-2$, be the vertices of G that are not adjacent to x. As above, we may conclude that G contains cycles of all lengths except possibly cycles of length $(i+1)$ and $(j+1)$. If $i = 2$ and $j = n-2$ then G contains cycles of each length except possibly $n-1$. For all other i, j, since $n \geq 7$, there exist distinct vertices x_k and x_ℓ such that x_k, x_{k+i-1}, x_ℓ, $x_{\ell+j-1}$ are all adjacent to x. Then $xx_k C^+ x_{k+i-1} x$ is an $(i+1)$-cycle in G and $xx_\ell C^+ x_{\ell+j-1} x$ is a $(j+1)$-cycle in G, implying that G is pancyclic. ■

When considering a hamiltonian graph with two vertices of high degree sum, placing no restrictions on their proximity on a hamiltonian cycle, the sum that guarantees the graph is pancyclic turns out to be quite large.

Theorem 3.4 *Let* G *be a hamiltonian graph of order* $n \geq 9$*. If there exist* $x, y \in V(G)$ *such that* $d(x) + d(y) \geq 2n - 7$*, then* G *is pancyclic. This result is sharp.*

Proof: Assume that G is not pancyclic, and let $C = v_0 v_1 \ldots v_{n-1} v_0$, where $x = v_0$, be a hamiltonian cycle in G. Since $d(x) + d(y) \geq 2n - 7$, we may assume without loss of generality that $d(x)$ is at least $n - 3$. Lemmas 3.2 and 3.3 therefore imply that x has degree exactly $n - 3$ and furthermore, since by assumption G is not pancyclic, that G contains cycles of each length except $n - 1$. As such, we can see that v_2 and v_{n-2} must be the vertices of G that are not adjacent to x. We now consider the vertex y, which has degree at least $n - 4$.

It is useful to note that if y was either v_{n-1} or v_1, this would place x and y consecutive on a hamiltonian cycle, implying that G would be pancyclic by Theorem 2.15. Thus, we assume y is neither v_{n-1} nor v_1. We also note that y has at most three nonadjacencies, and two of these must be y^{--} and y^{++} since G does not contain a cycle of length $n - 1$.

Suppose then that $y = v_2, v_3$ or v_4. Then either yv_{n-1} or yv_{n-2} is an edge of G. If yv_{n-1} is an edge of G, then $yv_{n-1}C^{-}y^{++}xC^{+}y$ is an $(n-1)$-cycle in G, and if yv_{n-2} is an edge of G, then $yv_{n-2}C^{-}y^{+}xC^{+}y$ is an $(n-1)$-cycle in G. Similarly, if $y = v_{n-4}, v_{n-3}$ or v_{n-2} then either yv_1 or yv_2 is an edge of G. If yv_1

is an edge of G, then $yv_1C^+y^{--}xC^-y$ is an $(n-1)$-cycle in G and if yv_2 is an edge of G, then $yv_2C^+y^-xC^-y$ is an $(n-1)$-cycle in G.

If $y \neq v_2, v_3, v_4, v_{n-4}, v_{n-3}, v_{n-2}$, then at least one of yv_{n-1} and yv_1 is an edge of G, impling that one of the following is an $(n-1)$-cycle in G:

$$yC^+xy^{--}C^-v_1y,$$

or

$$yC^-xy^{++}C^+v_{n-1}y.$$

Having exhausted all cases, we may conclude that G is pancyclic.

To see that this result is sharp, consider the graphs G_1 and G_2 constructed as follows:

Let $V(G_1) = \{v_0, \ldots v_{n-1}\}$ and construct the hamiltonian cycle $C = v_0v_1 \ldots n_{n-1}v_0$ in G_1. Selecting $x = v_0$ and $y = v_{n-3}$ we let xv_i be an edge for each $i \neq 0, 2, n-5, n-2$ and yv_j be an edge for each $j \neq 2, n-5, n-3, n-1$. Then $d(x) + d(y) = 2n - 8$, but G_1 contains no $(n-1)$-cycle. See Figure 3.1.

Let $V(G_2) = \{v_0, \ldots v_{n-1}\}$ and construct the hamiltonian cycle $C = v_0v_1 \ldots n_{n-1}v_0$ in G_1. Selecting $x = v_0$ and $y = v_{n-4}$ we let xv_i be an edge for each $i \neq 0, 2, n-6, n-2$ and yv_j be an edge for each $j \neq 2, n-6, n-4, n-2$. Then $d(x) + d(y) = 2n - 8$, but G_2 contains no $(n-1)$-cycle. See Figure 3.2 ■

In both G_1 and G_2, the distance between x and y on the hamiltonian cycle C is at most four. If we require that this distance be at least five, the necessary degree sum for pancyclicity can be reduced slightly.

Theorem 3.5 *Let G be a graph of order $n \geq 9$ with hamiltonian cycle C. If there exist $x, y \in V(G)$ such that $dist_C(x, y) \geq 5$ and $d(x) + d(y) \geq 2n - 9$, then*

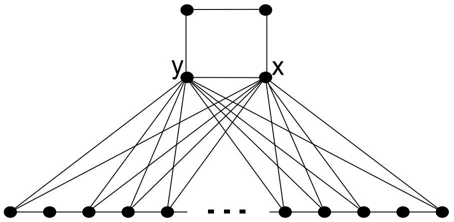

Figure 3.1: $d(x) + d(y) = 2n - 8$, but G contains no $(n-1)$-cycle

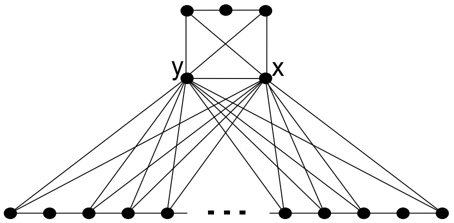

Figure 3.2: $d(x) + d(y) = 2n - 8$, but G contains no $(n-1)$-cycle

G is pancyclic. This result is best possible.

Proof: Assume that G is not pancyclic, and let $C = xv_1 \dots v_{n-1}x$. Since $d(x) + d(y) \geq 2n - 9$, we may assume without loss of generality that x has degree at least $n-4$. By Lemmas 3.2 and 3.3 and the assumption that G is not pancyclic, we may assume that x has degree at most $n-3$.

Suppose that the degree of x is exactly $n-3$ and therefore, by Lemma 3.3, G contains cycles of each length except $n-1$. Then we can see that v_2 and v_{n-2} are the vertices of G that are not adjacent to x. We now consider the vertex y, which has degree at least $n-6$.

Since $dist_C(x, y) \geq 5$ we know that $y \notin \{v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_0, v_1, v_2, v_3, v_4\}$. If at least one of yy^{--} or yy^{++} is an edge of G, then G contains an $(n-1)$-cycle and is pancyclic, so we assume that neither y^{--} nor y^{++} are adjacent to y. If $yv_1 \in E(G)$ or $yv_{n-1} \in E(G)$, then $xy^{-}C^{-}v_1yC^{+}x$ or $xC^{+}yv_{n-1}C^{-}y^{++}x$ is an $(n-1)$-cycle, so we may also assume that $yv_1, yv_{n-1} \notin E(G)$. Since y can have at most five nonadjacencies, one of yv_{n-2} and yv_2 is an edge of G, and G contains the $(n-1)$-cycle $xy^{+}C^{+}v_{n-2}yC^{-}x$ or $xC^{-}yv_2C^{+}y^{-}x$. Thus, if x has degree exactly $n-3$, G is pancyclic.

Suppose, then, that x has degree $n-4$ and let v_i, v_j and v_ℓ be the nonneighbors of x in G. We know that for $k \neq i, j, \ell$ G contains the $k+1$ cycle $xC^{+}v_kx$. If $i \neq n-j, n-\ell$, $j \neq n-\ell$, and $i, j, \ell \neq \frac{n}{2}$, then the cycle $C' = xv_{n-m}C^{+}x$ is a cycle of length $m+1$, where $m = i, j, \ell$, and G is pancyclic.

Without loss of generality, suppose then that $i = \frac{n}{2}$. If $j, \ell \neq 2, \frac{n+2}{2}$, the cycle $xv_{i+1}C^{-}v_2x$ has length $i+1$. If j or $\ell = 2$ or if j or $\ell = \frac{n+2}{2}$, then G contains either the $(i+1)$-cycle $xv_{i-1}C^{+}v_{n-2}x$ or $xv_{i-2}C^{+}v_{n-3}x$. Now we must find the cycles of lengths $j+1$ and $\ell+1$. If $j \neq n-\ell$, G contains the $(j+1)$- and $(\ell+1)$-cycles $C' = xv_{n-m}C^+x$ as above, where $m = j, \ell$. If $j = n-\ell \neq 2$, then $xv_{m+1}C^-v_2x$ or $xv_{n-m-1}C^+v_{n-2}x$ is a cycle of length $m+1$, where $m = j, \ell$. If $j = n-\ell = 2$, we will find an $(n-1)$-cycle by using the vertex y. As above, we know that $y \notin \{v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_0, v_1, v_2, v_3, v_4\}$, and that yy^{--}, yy^{++}, yv_1, and yv_{n-1} are not edges of G. Now, y can have at most four nonadjacencies, so G contains the $(n-1)$-cycle $xy^+C^+v_{n-2}yC^-x$.

Thus, we may assume without loss of generality that $i = n-j$ (and hence $i, j \neq n-\ell$). Observe that if $\ell \neq \frac{n}{2}$ the $(\ell+1)$-cycle $C' = xv_{n-\ell}C^+x$ is contained in G . If $\ell = \frac{n}{2}$ an $(\ell+1)$-cycle can be found in G as described above. We proceed by considering three cases.

Case 1: Assume $i \neq 2, \frac{n-1}{2}$. If $\ell \neq 2, i+1, j+1$, then $xv_{i+1}C^-v_2x$ is an $(i+1)$-cycle and $xv_{j+1}C^-v_2x$ is a $(j+1)$-cycle. If $\ell = 2, i+1$, or $j+1$, then $xv_{i+2}C^-v_3x$ is an $(i+1)$-cycle and $xv_{j+2}C^-v_3x$ is a $(j+1)$-cycle.

Case 2: Assume $i = \frac{n-1}{2}$ and $j = \frac{n+1}{2}$. If $\ell \neq 2, 3, j+1$, then $xv_{i+2}C^-v_3x$ is a cycle of length $i+1$ and $xv_{j+1}C^-v_2x$ is a cycle of length $j+1$. If $\ell = 2, 3$, or $j+1$, then $xv_{i-1}C^+v_{n-3}x$ is a cycle of length $i+1$ and $xv_{j-2}C^+v_{n-2}x$ is a cycle of length $j+1$.

Case 3: Assume $i = 2$ and $j = n-2$. As noted above, if at least one of yy^{--} or yy^{++} is an edge of G, then G contains an $(n-1)$-cycle, so we will assume that these edges are not in G.

If $\ell = y^-$, then if yv_1, or $yv_{n-2} \in E(G)$, the $(n-1)$-cycle $xy^{--}C^-v_1yC^+x$ or $xC^+yv_{n-2}C^-y^+x$ is contained in G. If $yv_1, yv_{n-2} \notin E(G)$, then $yv_{n-1} \in E(G)$, and G contains the $(n-1)$-cycle $xC^+yv_{n-1}C^-y^{++}x$. If $\ell = y^+$, the argument is

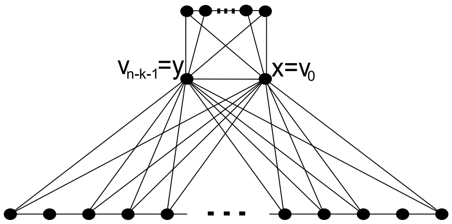

Figure 3.3: $d(x)+d(y)=2n-10$, but the pictured graph contains no $(n-1)$-cycle

similar, so we assume that $\ell \neq y^-, y^+$.

If $yv_1 \in E(G)$ or $yv_{n-1} \in E(G)$, then $xy^-C^-v_1yC^+x$ or $xC^+yv_{n-1}y^{++}x$ is an $(n-1)$-cycle, so we may also assume that $yv_1, yv_{n-1} \notin E(G)$. Since y can have at most four nonadjacencies, yv_2 must be an edge of G, and G contains the $(n-1)$-cycle $xC^-yv_2C^+y^-x$. Thus, if x has exactly three nonadjacencies, G is pancyclic. Having exhausted all cases, we may conclude that G is pancyclic.

To see that this result is sharp, consider the graph G^* defined as follows:

Let $V(G^*) = \{v_0, \ldots v_{n-1}\}$ and construct the hamiltonian cycle $C = v_0v_1 \ldots n_{n-1}v_0$ in G^*. Selecting $x = v_0$ and $y = v_{n-k-1}$ for some k satisfying $4 \leq k \leq \frac{n-2}{2}$, we let xv_i be an edge for each $i \neq 0, 2, n-k-3, n-k+1, n-2$ and yv_j be an edge for each $j \neq 2, n-k-3, n-k-1, n-k+1, n-2$. Then $d(x)+d(y) = 2n-10$, but G^* contains no $(n-1)$-cycle. See Figure 3.3. ■

3.2 Distance-dependent Bounds

Theorems 3.4 and 3.5 provide sharp results, and are important for completeness. The fact that the required degree sum is so large, while in some ways surprising, also makes them somewhat unsatisfying. We return to the notion introduced in [13] and examine the cycle structure of G when x and y are allowed to lie farther apart on a hamiltonian cycle. We begin by considering the case where the distance between x and y is two.

Theorem 3.6 *Let G be a graph of order n with hamiltonian cycle C. If there exist $x, y \in V(G)$ such that $dist_C(x, y) = 2$ and $d(x) + d(y) \geq n + 1$, then G is pancyclic.*

Proof: Let $C = v_0 v_1 \ldots v_{n-1} v_0$, where $x = v_0$ and $y = v_{n-2}$. We separate the proof into two cases based on whether xy is an edge of G.

Suppose that xy is an edge of G and consider the graph $G' = G - v_{n-1}$. This graph contains a hamiltonian cycle $C' = x, v_1, \ldots, v_{n-3}, y, x$ and $d_{G'}(x) + d_{G'}(y) \geq n-1$. Consequently, by Theorem 2.14 G' is either pancyclic, bipartite, or missing only an $(n-2)$-cycle. If G' is pancyclic, then G is also pancyclic and we are done.

Suppose then that G' is bipartite, and note that that x and y must lie in opposite partite sets of G'. Since G' is hamiltonian, each partite set of G' must have order $\frac{n-1}{2}$. Furthermore, if we let U and W denote these partite sets and assume, without loss of generality, that $x \in U$ and $y \in W$ then the fact that $d_{G'}(x) + d_{G'}(y) = n - 1$ implies that x is adjacent to each vertex in W and y is adjacent to each vertex in U. Since x and y are consecutive on the hamiltonian cycle C', we see that for each $2 \leq t \leq \frac{n-1}{2}$ there is a cycle of length $2t$ in G' (and hence G) that contains the edge xy. To obtain cycles of odd length in G we simply replace xy with the path $xv_{n-1}y$ to obtain a cycle of length $2t + 1$ for each t. These cycles, along with C and the triangle $xv_{n-1}yx$, imply that G is pancyclic.

Now suppose that both G' and G do not contain an $(n-2)$-cycle. We then consider the following pairs of possible edges in G': $(v_0v_1, v_{n-2}v_{n-4})$, $(v_0v_2, v_{n-2}v_{n-3})$ and $(v_0v_i, v_{n-2}v_{i-2})$, where $3 \leq i \leq n-3$. For convenience, we will refer to this collection of pairs as A_0. We claim that if both elements of any of the pairs in A_0 are in $E(G')$, then G' would contain a cycle of length $n-2$.

Indeed, were both elements of either $(v_0v_1, v_{n-2}v_{n-4})$ or $(v_0v_2, v_{n-2}v_{n-3})$ edges in G', it is not difficult to see that G' must contain a cycle of length $n-2$. If both elements of $(v_0v_i, v_{n-2}v_{i-2})$, where $3 \leq i \leq n-3$, were edges in G, then $v_0v_iC^+v_{n-2}v_{i-2}C^-v_0$ is an $(n-2)$-cycle in G'.

Consequently at most one element of each pair in A_0 is an edge of G', and hence of G. Since there are $n-3$ pairs and v_0v_{n-2}, v_0v_{n-1} and $v_{n-2}v_{n-1}$ are all edges of G, $d_G(x) + d_G(y) \leq n-3+4 = n+1$. Therefore equality must hold and exactly one element from each pair in A_0 must be an edge of G.

Similarly, examine the pairs $(v_0v_1, v_{n-2}v_{n-5})$, $(v_0v_2, v_{n-2}v_{n-4})$, $(v_0v_3, v_{n-2}v_{n-3})$ and $(v_0v_i, v_{n-2}v_{i-2})$, where $4 \leq i \leq n-3$. For convenience, we will refer to this collection of pairs as A_1. As is the case with A_0, at most one element from each pair in A_1 is an edge of G'. Using an argument similar to that given above, this implies that exactly one element from each pair in A_1 must be an edge of G.

By assumption, $v_{n-2}v_{n-3}$ is an edge of G and therefore, as $(v_{n-2}v_{n-3}, v_0v_3)$ is in A_1, v_0v_3 cannot be an edge of G. As $(v_0v_3, v_{n-2}v_1)$ is in A_0, $v_{n-2}v_1$ must be an edge of G. However, it then follows that $v_1v_2 \ldots v_{n-2}v_1$ is an $(n-2)$-cycle in G, contradicting our supposition. Thus if xy is an edge of G, G is pancyclic.

Suppose that xy is not an edge of G. Note first that since $d_G(x) + d_G(y) \geq n+1$, one of x or y must have degree at least $\frac{n+1}{2}$, and hence must be adjacent to two consecutive vertices on C. This implies that G contains a triangle.

Assume then that G does not contain a cycle of some fixed length $\ell \geq 4$. Then at most one element of each the pairs $(v_0v_i, v_{n-2}v_{\ell-4+i})$, where $1 \leq i \leq n - \ell + 1$, and each of the pairs $(v_0v_{n-\ell+j}, v_{n-2}v_j)$, where $1 \leq j \leq \ell - 3$, is an edge of G. Indeed, were it the case that for some i, both v_0v_i and $v_{n-2}v_{i+\ell-4}$ were edges in G, then $v_0v_iC^+v_{i+\ell-4}v_{n-2}C^+v_0$ would be a cycle of length ℓ in G. We handle the situation where both $v_0v_{n-\ell+j}$ and $v_{n-2}v_j$ are edges in G in a similar manner.

Since there are $n-2$ of these pairs and both v_0v_{n-1} and $v_{n-2}v_{n-1}$ are edges of G, $d(x) + d(y) \leq n - 2 + 2 = n$, which contradicts the hypothesis that $d(x) + d(y) \geq n + 1$. Hence, if xy is not an edge of G, G is pancyclic. ■

The following corollary is an immediate consequence of Theorem 2.15 and Theorem 3.6, since the conditions imply that there must be two vertices of degree sum at least $n+1$ at distance one or two apart on a hamiltonian cycle of G.

Corollary 3.7 *If G is hamiltonian with more than $\frac{n}{3}$ vertices of degree at least $\frac{n+1}{2}$, then G is pancyclic.*

As the vertices x and y are moved farther apart on the hamiltonian cycle, it becomes possible for cycle lengths to be missing from the graph. Those cycle lengths that may be missing from G are determined both by the degree sum of x and y along with the distance between x and y on the hamiltonian cycle.

Theorem 3.8 *Let G be a graph of order n with hamiltonian cycle C and let* *k be an integer satisfying* $1 \le k \le \frac{n-2}{2}$. *If there exist* $x, y \in V(G)$ *such that* $dist_C(x, y) = k+1$ *and* $d(x)+d(y) \ge n+k$, *then* G *contains cycles of all lengths* ℓ, *where* $3 \le \ell \le n-k$. *This result is sharp.*

Proof: Let $C = v_0 v_1 \dots v_{n-1} v_0$, where $x = v_0$ and $y = v_{n-k-1}$. If there is a vertex z in the path $P = v_{n-k} v_{n-k+1} \dots v_{n-1}$ that is adjacent to both x and y, then the cycle $C' = xC^+yzx$ is a hamiltonian cycle of the graph G' induced by $V(G) - V(P) + z$, which has order $n - k + 1$. In G', $dist_{C'}(x, y) = 2$ and $d_{G'}(x) + d_{G'}(y) \ge n + k - 2(k - 1) = (n - k + 1) + 1$, so by Theorem 3.6 G' is pancyclic. Consequently, G contains cycles of all lengths ℓ, where $3 \le \ell \le n - k + 1$.

Suppose that there is no vertex $z \in V(P)$ that is adjacent to both x and y. Since $d(x) + d(y) \ge n + k$, x and y have at least k common neighbors, all of which must be in $V(G) - \{V(P), x, y\}$. We separate the proof into two cases based on whether or not xy is an edge of G. For both cases we will consider the graph G'' induced by $V(G) - V(P)$.

Suppose that xy is an edge of G. The cycle $C'' = xC^+yx$ is a hamiltonian cycle of G'' with $d_{C''}(x, y) = 1$ and $d_{G''}(x) + d_{G''}(y) \ge n + k - k = n$. Since $|V(G'')| = n - k$ and $k \ge 1$, G'' is pancyclic by Theorem 2.15 and so G contains cycles of all lengths ℓ, where $3 \le \ell \le n - k$.

Now suppose that xy is not an edge of G. An observation similar to that made in the proof of Theorem 3.6 yields that G contains a cycle of length 3. As such, we will suppose that G contains no cycle of length ℓ, for some fixed ℓ between 4 and $n-k$. Recall that x and y are nonadjacent, share no neighbors in P and have a degree sum of at least $n + k$ in G. As a result, these two vertices have at least $k+2$ common neighbors in G'' and $d_{G''}(x) + d_{G''}(y) \geq n$.

We consider pairs of possible edges in G, as above. If $4 \leq \ell \leq k+2$, let A_0 denote the collection of pairs $(v_0v_i, v_{n-k-1}v_{\ell+i-4})$, where $1 \leq i \leq n-k-\ell+2$, and let A_1 denote the pairs $(v_0v_{(n-k-1)-(\ell-2)+j}, v_{n-k-1}v_{1+j})$, where $1 \leq j \leq \ell-4$. The reader should note that $v_{n-k-1}v_{\ell-3}$ appears in pairs from both A_1 and A_2, while $v_{n-k-1}v_1$ appears in neither. Additionally, it is useful to note that each v_0v_i, for $2 \leq i \leq n-k-2$ appears exactly once in $A_0 \cup A_1$ as does each $v_{n-k-1}v_j$, save the ones already mentioned, for $2 \leq j \leq n-k-2$.

We claim that if more than one element from any of these pairs is an edge in G, then G would contain a cycle of length ℓ. Indeed, consider $(v_0v_i, v_{n-k-1}v_{\ell+i-4}) \in A_0$, where $1 \leq i \leq n-k-\ell+2$, and assume that both elements of this pair were edges in G. Note that since x and y have at least $k+2$ common neighbors, there must be at least one such common neighbor v' that does not lie in the set $\{v_i, \ldots, v_{\ell+i-4}\}$. Then $v_0v'v_{n-k-1}v_{\ell+i-4}C^-v_iv_0$ is a cycle of length ℓ in G.

Next, consider $(v_0v_{(n-k-1)-(\ell-2)+j}, v_{n-k-1}v_{1+j})$, where $1 \leq j \leq \ell-4$, and assume that both elements of this pair were edges in G. Then $v_0C^+v_{j+1}x_{n-k-1}C^-v_{(n-k-1)-(\ell-2)+j}v_0$ is an ℓ-cycle in G.

Hence exactly one element from each of the $n-k-2$ pairs in $A_0 \cup A_1$ can be an edge of G. Taking into account that v_1v_{n-k-1} may be an edge of G, and recalling that either x or y, but not both, may be adjacent to each vertex in P, we have that $d(x) + d(y) \leq n-k-2+k+1 = n-1$. This contradicts the hypothesis that $d(x) + d(y) \geq n+k$.

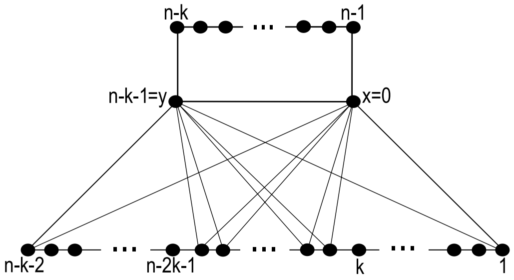

Figure 3.4: $d(x) + d(y) \geq n + k$, but G has no cycle of length longer than $n - k$

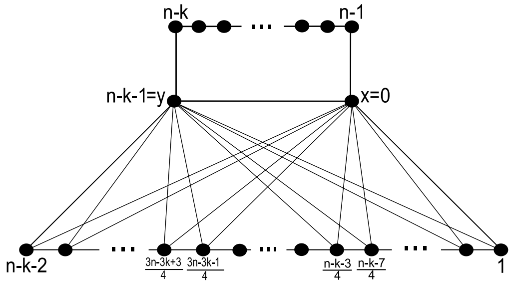

Figure 3.5: $d(x) + d(y) = n + k - 1$, but G contains no cycle of length $\frac{n-k+3}{2}$

If $k + 3 \leq \ell \leq n - k$, then arguments similar to those given previously yield that for each of the pairs $(v_0v_i, v_{n-k-1}v_{\ell-k+i-3})$, where $1 \leq i \leq n - \ell + 1$, and $(v_0v_{n-\ell+1-j}, v_{n-k-1}v_{k+2-j})$, where $n-\ell+2 \leq j \leq n-2k-1$, at most one element is an edge of G. There are $n - 2k - 1$ of these pairs and none of the $2k$ possible edges v_0v_{n-k-i} and $v_{n-k-1}v_i$, where $1 \leq i \leq k$, lie in any pair. This implies that $d(x) + d(y) \leq n - 2k - 1 + 2k + k = n + k - 1$, which contradicts the hypothesis that $d(x) + d(y) \geq n + k$.

Therefore G contains cycles of all lengths ℓ for $3 \leq \ell \leq n-k$. This result is sharp in the sense that, for $2 \leq k \leq \frac{n+4}{7}$, the graph in Figure 3.4 (for readability the v's are suppressed in Figures 3.4 and 3.5) has two vertices x and y satisfying $d(x) + d(y) \geq n + k$ and $d_C(x, y) = k + 1$, yet there are no cycles longer than

$n-k$. We also note that if we lower the degree sum of x and y, we may miss some cycles of length ℓ for $3 \leq \ell \leq n-k$. As an example see Figure 3.5 in which $d(x)+d(y)=n+k-1$, but there is no cycle of length $\frac{n-k+3}{2}$, provided that $2 \leq k \leq \frac{n}{5}$. ■

4. Cycles in Bipartite Hamiltonian Graphs

When studying hamiltonian graphs, and cycles in general in graphs, it is quite common to look for bipartite variations of a given result. However, a bipartite graph cannot be pancyclic, as it cannot contain cycles of odd lengths. We consider a bipartite graph of order $2n$ to be *bipancyclic* if it contains cycles of all even lengths from 4 to $2n$. In this chapter we will prove results for bipartite graphs that are analogous to those in the previous chapter. We begin by stating the classic result of Moon and Moser [17] that gives the Ore type condition for bipartite graphs.

Theorem 4.1 (Moon, Moser 1963) *If G is a balanced bipartite graph of order $2n$ with parts X and Y in which $d(x)+d(y) > n$ for every pair of nonadjacent vertices $x \in X$ and $y \in Y$, then G is hamiltonian.*

If we add the condition that G is hamiltonian in the assumptions, how much can we relax the degree sum conditions in order to assure that G is bipancyclic? Schmeichel and Mitchem showed that having two vertices of high degree sum "close" together on a hamiltonian cycle of G suffices.

Theorem 4.2 (Schmeichel, Mitchem 1982) *Let G be a bipartite graph containing hamiltonian cycle $C = v_1v_2 \dots v_{2n}$. If $d(v_1) + d(v_{2n}) > n + 1$, G is bipancyclic.*

Theorem 4.3 (Schmeichel, Mitchem 1982) *Let G be a bipartite graph containing hamiltonian cycle $C = v_1v_2 \dots v_{2n}$. If $d(v_2) + d(v_{2n}) > n + 2$, G is bipancyclic.*

The above two theorems will be used to as a starting point in the sections that follow.

4.1 Absolute Bounds for Bipartite Graphs

We begin this section by determining the degree sum of two vertices on a hamiltonian cycle, regardless of their placement, required to assure that a bipartite graph is bipancyclic, obtaining results analogous to those for graphs. We start with the following two useful lemmas.

Lemma 4.4 *Let G be a bipartite hamiltonian graph with parts X and Y such that $|X| = |Y| = n \geq 4$ and let C be a hamiltonian cycle of G. If there exists a vertex $v \in V(G)$ of degree $n-1$, then G is bipancyclic.*

Proof: Let $C = x_0y_0x_1y_1 \dots x_{n-1}y_{n-1}x_0$, where $v = x_0$ and let y_i, $1 \leq i \leq n-2$, be the vertex nonadjacent to v. If vy_ℓ is an edge of G, then the cycle $C' = vy_\ell C^- v$ is a cycle of length $2(\ell+1)$ in G, so our assumption implies that G contains cycles of all even lengths except possibly a cycle of length $2(i+1)$. We may assume without loss of generality that $i \neq 1$, otherwise we would consider C in the counterclockwise direction. Then $vy_1C^+y_{i+1}v$, is a $2(i+1)$-cycle. Hence, G is bipancyclic. ■

Lemma 4.5 *Let G be a bipartite hamiltonian graph with parts X and Y such that $|X| = |Y| = n \geq 6$ and let C be a hamiltonian cycle of G. If there exists a vertex $v \in V(G)$ of degree $n-2$, then G is either bipancyclic or missing only the $2(n-1)$-cycle.*

Proof: Let $C = x_0y_0x_1y_1 \dots x_{n-1}y_{n-1}x_0$, where $v = x_0$ and let y_i, y_j, $1 \leq i < j \leq n-2$, be the vertices nonadjacent to v. Then, as observed above,

G contains cycles of all even lengths except possibly $2(i+1)$ and $2(j+1)$. If $i=1$ and $j=n-2$, then G may not contain a $2(n-1)$-cycle. For all other i, j there are chords vy_k, vy_ℓ, vy_m and vy_p, $k < \ell$ and $m < p$, such that $vy_kC^+y_\ell v$ is a $2(i+1)$-cycle and $vy_mC^+y_pv$ is a $2(j+1)$-cycle. Hence G is bipancyclic or missing only the $2(n-1)$-cycle. ■

Similar to the case for graphs, when considering a hamiltonian bipartite graph with two vertices of high degree sum, placing no restrictions on their proximity on a hamiltonian cycle, the sum that guarantees the graph is bipancyclic is close to the order of the graph. The results for bipartite graphs come in pairs, as we must consider the case in which the vertices are in the same part separately from the case in which they are in different parts.

Theorem 4.6 *Let G be a bipartite hamiltonian graph with parts X and Y such that $|X| = |Y| = n \geq 8$ and let C be a hamiltonian cycle of G. If there exist $u, v \in V(X)$ such that $d(u) + d(v) \geq 2n-5$, then G is bipancyclic. This result is sharp.*

Proof: Assume that G is not bipancyclic, and let $C = x_0y_0x_1y_1 \ldots x_{n-1}y_{n-1}x_0$, where $v = x_0$, be a hamiltonian cycle in G. Since $d(u) + d(v) \geq 2n-5$, we may assume without loss of generality that $d(v)$ is at least $n-2$. Lemmas 4.4 and 4.5 therefore imply that v has degree exactly $n-2$ and that G is missing only the $2(n-1)$-cycle. As such, we can see that y_1 and y_{n-2} must be the vertices of G that are not adjacent to v. We now consider the vertex $u \in V(X)$, which has degree at least $n-3$.

It is useful to note that if u was either x_1 or x_{n-1}, this would place u and v distance exactly two apart on a hamiltonian cycle, implying that G would be bipancyclic by Theorem 4.3. Thus we may assume u is neither x_1 nor x_{n-1}. Since u has at most three nonadjacencies, and two of these must be u^{---} and u^{+++}, at least one of uy_2 and uy_{n-2} is an edge of G. We will assume without loss of generality that uy_{n-2} is an edge. If $u \neq x_{n-2}$ then $C' = x_0C^+uy_{n-2}C^-u^+x_0$ is a hamiltonian cycle. If $u = x_{n-2}$ then either uy_0 or uy_1 is an edge of G and G contains the $2(n-1)$-cycle

$$uy_0C^+u^{---}x_0C^-u$$

or

$$uy_1C^+u^-x_0C^-u.$$

Thus G is bipancyclic.

To see that this result is sharp, consider the graph G^* defined as follows: Let $V(G^*) = V(X) \cup V(Y)$, where $V(X) = \{x_0, x_1, \ldots, x_{n-1}\}$ and $V(Y) = \{y_0, y_1, \ldots, y_{n-1}\}$, and construct the hamiltonian cycle $C = x_0y_0x_1y_1 \ldots x_{n-1}y_{n-1}x_0$. Choose $v = x_0$ and $u = x_{n-2}$ with vy_i, uy_j an edge for all $i \neq 1, n-4, n-2$, $j \neq 1, n-4, n-1$. Then $d(u) + d(v) = 2n - 6$, but G^* does not contain a $2(n-1)$-cycle. See Figure 4.1. ∎

Theorem 4.7 *Let G be a bipartite hamiltonian graph with parts X and Y such that $|X| = |Y| = n \geq 8$ and let C be a hamiltonian cycle of G. If there exists $v \in V(X)$ and $u \in V(Y)$ such that $d(u) + d(v) \geq 2n - 4$, then G is bipancyclic. This result is sharp.*

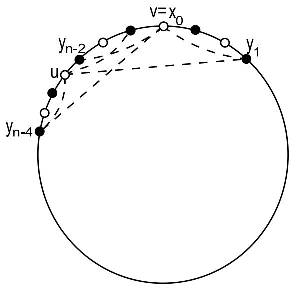

Figure 4.1: $d(u) + d(v) = 2n - 6$, but G^* contains no cycle of length $2(n-1)$

Proof: Assume that G is not bipancyclic, and let $C = x_0y_0x_1y_1 \ldots x_{n-1}y_{n-1}x_0$, where $v = x_0$, be a hamiltonian cycle in G. Since $d(u) + d(v) \geq 2n - 4$, we may assume without loss of generality that $d(v)$ is at least $n - 2$. Lemmas 4.4 and 4.5 therefore imply that v has degree exactly $n - 2$ and that G is missing only the $2(n-1)$-cycle. As such, we can see that y_1 and y_{n-2} must be the vertices of G that are not adjacent to v. We now consider the vertex $u \in V(Y)$, which has degree at least $n - 2$.

Note that if u was either y_0 or y_{n-1}, this would place x and y consecutive on a hamiltonian cycle, implying that G would be bipancyclic by Theorem 4.2. Thus we assume u is neither y_0 nor y_{n-1}. Since G contains no $2(n-1)$-cycle, the nonadjacencies of u must be u^{---} and u^{+++}. If $u \neq y_{n-3}$ then G contains the $2(n-1)$-cycle $x_0C^+ux_{n-1}C^-u^{++}x_0$. If $u = y_{n-3}$ then G contains the $2(n-1)$-cycle $ux_1C^+u^{--}x_0C^-u$. Hence G is bipancyclic.

To see that this result is sharp, consider the graph G^* defined as follows: Let $V(G^*) = V(X) \cup V(Y)$, where $V(X) = \{x_0, x_1, \ldots, x_{n-1}\}$ and $V(Y) = \{y_0, y_1, \ldots, y_{n-1}\}$, and construct the hamiltonian cycle $C = x_0y_0x_1y_1 \ldots x_{n-1}y_{n-1}x_0$. Choose $v = x_0$ and $u = y_{n-3}$ with vy_i, ux_j an edge for all $i \neq 1, n-2$ and

$j \neq 1, n-4, n-1$. Then $d(u) + d(v) = 2n - 5$, but G^* does not contain a $2(n-1)$-cycle. ■

4.2 Distance Dependent Bounds for Bipartite Graphs

We consider the effects of moving the two vertices of large degree sum farther apart on a hamiltonian cycle. We begin by considering the case that G has two vertices of large degree sum a distance three apart on a hamiltonian cycle of G.

Lemma 4.8 *Let G be a bipartite graph of order $2n$ with hamiltonian cycle $C = v_1 v_2 \dots v_{2n}$. If $d(v_1) + d(v_{2n-2}) > n + 2$, then G is bipancyclic.*

Proof: Let $x = v_1$ and $y = v_{2n-2}$. Suppose $xy \in (G)$. Define G' to be the graph induced by $V(G) - v_{2n-1}, v_{2n}$. Then G' contains the hamiltonian cycle $C' = xC^+yx$. Since G' has order $2(n-1)$ and $d_{G'}(x) + d_{G'}(y) > n = (n-1)+1$, G' is bipancyclic by Theorem 4.2. Hence G is bipancyclic.

Suppose $xy \notin E(G)$ and further suppose that G does not contain a cycle of length 2ℓ for some ℓ, $2 \leq \ell \leq n-1$. Then at most one of the elements of each pair $(xv_{2i}, yv_{2\ell+2i-5})$, where $1 \leq i \leq n - \ell + 1$, is an edge of G. If both elements of some pair were edges of G, then $xv_{2i}C^+v_{2\ell+2i-5}yC^+x$ would be a 2ℓ-cycle in G. Similarly, at most one of the elements of each pair $(xv_{2(n-\ell+j)}, yv_{2j+1})$, where $1 \leq j \leq \ell - 2$, is an edge of G. If both elements of some pair were edges of G, then G would contain the 2ℓ-cycle $xv_{2(n-\ell+j)}C^+yv_{2j+1}C^-x$. There are $n-2$ of these pairs. Together with the edges yv_{2n-1} and xv_{2n} all possible adjacencies of x and y are accounted for. This implies that $d(x) + d(y) \leq n - 2 + 2 = n$, which contradicts the assumption that $d(x) + d(y) > n + 2$. Consequently, G is bipancyclic. ■

As the vertices x and y are moved farther apart on the hamiltonian cycle, it becomes possible for cycle lengths to be omitted from the graph. As with graphs, those cycle lengths that may be missing from G are determined both by the degree sum of x and y along with the distance between x and y on the hamiltonian cycle.

Theorem 4.9 *Let G be a bipartite graph of order $2n$ with hamiltonian cycle $C = v_1v_2\dots v_{2n}$ and let k be an integer satisfying $2 \leq k \leq n-1$. If $d(v_1) + d(v_{2(n-k+1)}) > n+k$, then G contains cycles of all lengths 2ℓ, for $2 \leq \ell \leq n-k$.*

Proof: Let $x = v_1, y = v_{2(n-k+1)}$ and $P = v_{2(n-k)+3}C^+v_{2n}$. Observe that x and y are in different parts and that there are $2k-2$ vertices between x and y on C. Suppose that $xy \in E(G)$. Define G' to be the graph induced by $V(G) - V(P)$. Then G' contains the hamiltonian cycle $C' = xC^+yx$. Since G' has order $2n-(2k-2) = 2(n-k+1)$ and $d_{G'}(x)+d_{G'}(y) > n+k-(2k-2) = (n-k+1)+1$, G' is bipancyclic by Theorem 4.2. Therefore G contains cycles of all lengths 2ℓ, for $2 \leq \ell \leq n-k$.

So we may assume that $xy \notin E(G)$. Suppose there exist $u, v \in V(P)$ such that $uv \in E(C)$ and $yu, xv \in E(G)$. Define G' to be the graph induced by $V(G) - V(P) + u, v$. Then G' contains the hamiltonian cycle $C' = xC^+yuvx$. Since G' has order $2n - (2k-2) + 2 = 2(n-k+2)$ and $d_{G'}(x) + d_{G'}(y) > n+k-(2k-2)+2 = (n-k+2)+2$, G' is bipancyclic by Lemma 4.8. Thus G contains cycles of all lengths 2ℓ for $2 \leq \ell \leq n-k$.

We may now assume that x and y have no consecutive neighbours in $V(P)$, and so $d_P(x)+d_P(y) \leq k-1$. Suppose that G does not contain a cycle of length

2ℓ for some ℓ, $2 \le \ell \le n-k$. We will consider two cases.

Case 1: Let $2 \le \ell \le k+2$. For each of the pairs $(xv_{2(\ell+i-1)}, yv_{2i+1})$, where $1 \le i \le n-k-\ell+1$, at most one element is an edge of G. If both elements were edges of G, then G would contain the 2ℓ-cycle $xv_{2(\ell+i-1)}C^{+}yv_{2i+1}C^{-}x$. The edges xv_{2j} for $1 \le j \le \ell-1$ and yv_{2m+1} for $n-k-\ell+2 \le m \le n-k$ are not included in the pairs, and may therefore be edges of G. There are $n-k-\ell+1$ possible edges from the pairs and $\ell-1+(n-k-(n-k-\ell+2)+1 = 2\ell-2$ possible unpaired edges in $G-P, x, y$. Since xy is not an edge of G and $d_P(x)+d_P(y) \le k-1$, we have

$$d(x)+d(y) \le n-k-\ell+1+2\ell-2+k-1 = n+\ell-2 \le n+(k+2)-2 = n+k.$$

This contradicts the assumption that $d(x)+d(y) > n+k$, so G must contain cycles of all lengths 2ℓ, $2 \le \ell \le n-k$.

Case 2: Let $k+3 \le \ell \le n-k$. We consider the following pairs of possible edges: $(xv_{2i}, yv_{2(\ell+i-k)-1})$ for $1 \le i \le n-\ell$ and $(xv_{2(n-\ell+j)}, yv_{x(j+k)-3})$ for $1 \le j \le \ell-k$. In each pairing, at most one element is an edge of G or G would contain the 2ℓ-cycle $xv_{2i}C^{+}v_{2(\ell+i-k)-1}yC^{+}x$ or $xv_{x(j+k)-3}C^{+}yv_{2(n-\ell+j)}C^{-}x$, respectively. Since $k+3 \le \ell$ and $2 \le k$, $2(\ell-k)+1 \le 2\ell-3$. Hence the only possible edges not considered in the pairings are the edges yv_{2i+1} for $1 \le i \le k-2$. Recall that xy is not an edge of G and that x and y have no common neighbours in P. Therefore

$$d(x)+d(y) \le (n-k)+(k-1)+(k-2) = n+k-3,$$

which contradicts the assumption that $d(x)+d(y) > n+k$. This contradiction completes the proof. ■

Theorem 4.10 *Let G be a bipartite graph of order $2n$ with hamiltonian cycle $C = v_1v_2 \dots v_{2n}$ and let k be an integer satisfying $3 \leq k \leq n-1$. If $d(v_1) + d(v_{2(n-k)+3}) > n+k$, then G contains cycles of all lengths 2ℓ, for $2 \leq \ell \leq n-k$.*

Proof: Let $x = v_1, y = v_{2(n-k)+3}$ and $P = v_{2(n-k+2)}C^+v_{2n}$. Observe that x and y are in the same part and that there are $2k-3$ vertices between x and y on C. Suppose there is a vertex $z \in V(P)$ such that z is adjacent to both x and y. Define G' to be the graph induced by $V(G) - V(P) + z$. Then G' contains the hamiltonian cycle $C' = xC^+yzx$. Since G' has order $2n-(2k-4) = 2(n-k+2)$ and $d_{G'}(x) + d_{G'}(y) > n+k-(2k-4) = (n-k+2)+2$, G' is bipancyclic by Theorem 4.3. Hence G contains cycles of all lengths 2ℓ, for $2 \leq \ell \leq n-k$.

So we may assume that x and y have no common neighbours in $V(P)$. Suppose that G does not contain a cycle of length 2ℓ for some ℓ, $2 \leq \ell \leq n-k$. At this time we note that x and y have at least k common neighbours. We will consider two cases to complete the proof.

Case 1: If $2 \leq \ell \leq k-1$, then x and y have a common neighbour not in the path $v_{2i}C^+v_{2(\ell+i-2)}$, for $1 \leq i \leq n-\ell+1$. Since G does not contain a 2ℓ-cycle, at most one element of each pair $(xv_{2i}, yv_{2(\ell+i-2)})$, where $1 \leq i \leq n-k-\ell+3$, is an edge of G. If both elements of some pair were edges of G, then for some common neighbour z not in the path $v_{2i}C^+v_{2(\ell+i-2)}$, G would contain the 2ℓ-cycle $xv_{2i}C^+v_{2(\ell+i-2)}yzx$. Additionally, at most one element of each pair $(xv_{2(n-k-\ell+j+2)}, yv_{2j})$, where $1 \leq j \leq \ell-1$, is an edge of G. If both elements of some pair were edges of G, then $xv_{2(n-k-\ell+j+2)}C^+yv_{2j}C^-x$ would be a cycle of length 2ℓ. There are $n-k+2$ of these pairs, which account for all possible adjacencies of x and y in $V(G) - V(P) - x, y$. As x and y have no common neighbours in $V(P)$, this means that $d(x)+d(y) \leq (n-k+2)+\lceil \frac{2k-3}{2} \rceil = n+1$. However, $d(x)+d(y) > n+k \geq n+3$. Therefore we may assume that $k \leq \ell \leq n-k$.

Case 2: If $k \leq \ell \leq n-k$, then at most one of each pair $(xv_{2i}, yv_{2(\ell-k+i)})$, where $1 \leq i \leq n-\ell+1$, is an edge of G. If both elements of some pair were edges of G, then G would contain the 2ℓ-cycle $xv_{2i}C^{+}v_{2(\ell-k+i)}yC^{+}x$. Additionally, at most one element of each pair $(xv_{2(n-k-\ell+j+2)}, yv_{2j})$, where $1 \leq j \leq \ell-1$, is an edge of G. If both elements of some pair were edges of G, then as in case one $xv_{2(n-k-\ell+j+2)}C^{+}yv_{2j}C^{-}x$ would be a cycle of length 2ℓ. There are $n-1$ of these pairs, which account for all possible adjacencies of x and y in $V(G)-V(P)-x,y$. Recall that x and y have no common neighbours in $V(P)$. So we have that $d(x)+d(y) \leq (n-1)+\lceil \frac{2k-3}{2} \rceil = n+k-2$. This contradicts the assumption that $d(x)+d(y) > n+k$, and so G must contain cycles of all lengths 2ℓ for $2 \leq \ell \leq n-k$. ∎

5. Conclusion

We conclude with open problems and new directions. In Chapter 2 we presented many results on F-avoiding hamiltonian graphs. One of the directions remaining to be explored in that area is the bipartite variations of the results given. In our investigation of these questions we have encountered more open problems, such as determing the classes of graphs that are the exceptions to Moon and Moser's result Theorem 4.1.

In Chapter 4 we presented results for bipartite graphs that are analogous to the results of Chapter 3. However, the general results were all shown to be sharp, whereas the bipartite results were not. It is our belief that the results presented are not best possible and can be improved upon.

In addition to answering these questions, we have begun to investigate prism hamiltonian graphs. The *cartesian product* $G = G_1 \Box G_2$ has $V(G) = V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either

1. $u_1 = v_1$ and $u_2 v_2 \in E(G_2)$ or
2. $u_2 = v_2$ and $u_1 v_1 \in (G_1)$.

The *prism* of a graph G is the graph $G \Box K_2$. We say that a graph G is *prism hamiltonian* if $G \Box K_2$ is hamiltonian.

A completely different direction that we have taken is the study of graph saturation. Let $F = \{H_1, \ldots, H_t\}$ be a family of graphs. A graph G is F-*saturated* if G does not contain any member of F, but for any pair of nonadjacent vertices x and y in $V(G)$, $G+xy$ contains some member of F. We consider graph saturation from the perspective of games on graphs. We introduce the game of F-saturator, in which two players begin with the vertices of a graph G and none of its edges. The players alternately add edges of G in such a way as not to create any graph in F. For some specific families F and graphs G we have been able to determine when there is a winning strategy for one of the players.

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