CONDITIONAL COLORING

by

Mark R. Dillon

B.S., Purdue University, 1982, 1985

M.S., Purdue University, 1984

A thesis submitted to the

University of Colorado at Denver

in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Applied Mathematics

1998

This thesis for the Doctor of Philosophy

degree by

Mark R. Dillon

has been approved

by

Kathryn L. Fraughnaugh

David C. Fisher

J. Richard Lundgren

Tom Russell

Tom Altman

Date

Dillon, Mark R. (Ph.D., Applied Mathematics)

Conditional Coloring

Thesis directed by Associate Professor Kathryn L. Fraughnaugh

ABSTRACT

The *conditional chromatic number* $\chi(G, P)$ of a graph G with respect to a graphical property P is the minimum number of colors needed to color the vertices of G such that each color class induces a subgraph of G with property P. When P is the property that a graph contains no subgraph isomorphic to a graph F, we write $\chi(G, \neg F)$. The conditional chromatic number of a graph has been studied by various authors since 1968. We focus on two conditional chromatic numbers, specifically $\chi(G, \neg C_j)$ and $\chi(G, \neg P_j)$, where C_j is a cycle of length j for some fixed $j \geq 3$ and P_j is a path of length $j-1$ for some fixed $j \geq 2$. We find $\chi(G, \neg C_j)$ for graphs missing at most $j-1$ edges and $\chi(G, \neg P_j)$ for graphs missing at most $2j-5$ edges. To accomplish this, we characterize all Hamiltonian graphs of order n with at least $\binom{n}{2} - (n-1)$ edges and all graphs with no Hamiltonian paths with at least $\binom{n}{2} - (2n-5)$ edges. We determine both conditional chromatic numbers for all graphs with acyclic complements. We also determine a lower bound on $\chi(G, \neg P_j)$ in terms of the size of G. Finally, we show the problem of determining if $\chi(G, \neg P_3) \leq k$, for some $k \geq 0$, is NP-complete.

This abstract accurately represents the content of the candidate's thesis. I recommend its publication.

Signed ______________________

Kathryn L. Fraughnaugh

DEDICATION

To my wife, Wendy Tweten Dillon, whose love, understanding, patience, and encouragement made it possible to complete this dissertation.

ACKNOWLEDGEMENTS

I would like to express my deep gratitude to my advisor, Professor Kathryn Fraughnaugh for her tireless guidance, patience, inspiration and encouragement during the course of my research and throughout the preparation of this dissertation. My sincere thanks to Professor Richard Lundgren and Professor David Fisher who were always there to answer questions.

I would also like to thank my mother and father, Pat and Russell Dillon who always gave me the support I needed and my brothers and sister, Steve, Dave, Scott, Laura and Paul Dillon for their constant encouragement.

I also thank Mr. John Drubert, my sixth grade teacher, who was the first person to interest me in the field of mathematics and my close friends Doug Mckissack, Arthur Shulman and Kevin Williams for their constant encouragement.

Finally, I received, and gratefully acknowledge financial assistance from Hughes Aircraft Company without whose support, this thesis would never have been completed.

Contents

1 Introduction to coloring and its applications

Let G be a graph. A vertex coloring of G is an assignment of colors to its vertices so that no two adjacent vertices receive the same color. The chromatic number $\chi(G)$ is the minimum number of colors needed to color G. An equivalent definition of the chromatic number is the minimum integer k such that there is a partition of the vertices into k sets so that the subgraph induced by each set is an independent set.

Vertex coloring has a wide variety of applications. One such application is the following. In the United States Government, there are congressional committees with members of Congress serving on multiple committees. When assigning meeting times for these committees, one must not schedule simultaneous meetings for two committees that have a common member. A schedule solution is found by determining the minimum number of time slots required for the committees to meet. We can model this problem by constructing a graph G whose vertices represent committees. We draw an edge between two vertices if the committees represented by these vertices have a common member. Determining the minimum number of time slots required for the committees to meet is equivalent to determining $\chi(G)$. For further details, see Roberts [47] or Bodin and Friedman [8].

Another application is the channel assignment problem. Radio stations in a region are to be assigned transmitting frequencies so that radio stations which are geographically close, say 50 miles, receive different frequency assignments. The problem of assigning frequencies is a graph coloring problem. Let each vertex represent a radio station and draw an edge between two vertices if the radio stations represented by those vertices are within 50 miles. The number of frequencies required so that the radio stations do not interfere with each other is the vertex chromatic number of this graph. For more information regarding the channel assignment problem, see Cozzens and Roberts [20], Hale [28], Opsut and Roberts [43], or Pennotti [46].

The last application of vertex coloring we will discuss is the classic map coloring problem. This is a much studied problem and is discussed in most graph theory textbooks. For example, see Bondy and Murty [10], Harary [29], Chartrand and Lesniak [17], or Roberts [47]. Given a map with various countries, we would like to color the countries in such a way that we use the fewest number of colors, and if two countries share a common border, they receive a different color. This problem can be translated into a graph coloring problem by building a graph with each vertex representing a country and drawing an edge between two vertices if the countries represented by those vertices have a common border. To determine the minimum number of colors required to color the map, we find $\chi(G)$. This graph has the property that it is planar. The Four Color Theorem states that all planar graphs can be colored using at most four colors.

Now, suppose we relax the condition that there be no edges in each color class and allow certain configurations of edges. For example, in the committee scheduling application, we could allow each committee to have a time conflict with at most one other committee with a common member. In this case, each color class can contain isolated vertices and edges. We will call this problem the relaxed committee scheduling problem.

This is an example of a generalization for graph coloring called conditional graph coloring. We define the *conditional chromatic number* $\chi(G, P)$ of G with respect to a graph theoretical property P to be the minimum integer k such that there is a partition of the vertices into k sets so that the subgraph induced by each set has the property P.

Another application for conditional coloring is the circuit manufacturing problem. A designer draws an electrical circuit to be manufactured. Several circuit boards sandwiched one on top of the other may be required to build the entire circuit since, for this circuit to work properly, each circuit board must be built without intersecting edges. Now, the manufacturer would like to determine the minimum number of required circuit boards. The problem can be modeled as a conditional graph coloring problem. Let each vertex in our graph represent a junction in the circuit. Draw an edge between two vertices in the graph if the junctions represented by those vertices are connected in the circuit design. Determining the minimum number of circuit boards required is equivalent to determining $\chi(G, P)$, where P is the property that each color class be drawn with no intersecting edges, i.e., is planar. This particular conditional chromatic number is also known as the *vertex thickness* of a graph. For more information regarding vertex thickness, see Beineke and White [7] or Cimikowski [18]. For other electrical circuit problems, see Hutchinson [37] or Garey and Johnson [26].

We will study two types of conditional chromatic numbers. The first permits no paths of order j in a color class, and the second permits no cycles of order j in a color class. The relaxed committee scheduling problem is an example of conditional coloring with the property that no color class contains P_3. Studying these two numbers may give insight into and/or bounds on other conditional chromatic numbers or the original chromatic number. It is often the case that the understanding of a generalization leads to a better understanding of the original concept.

1.1 Overview of Thesis Results

This thesis will present some new results about the conditional chromatic number $\chi(G, \neg C_j)$ with respect to the property of having no cycles of a fixed length j and some new results about the conditional chromatic number $\chi(G, \neg P_j)$ with respect to the property of having no paths of a fixed length $j-1$.

The first chapter will provide some basic definitions from graph theory and review the existing literature regarding $\chi(G, \neg C_j)$ and $\chi(G, \neg P_j)$.

The second chapter will present and prove some basic results regarding $\chi(G, \neg C_j)$ and $\chi(G, \neg P_j)$. Some of these results will be needed in later chapters.

The third chapter will show that the problem of determining if a graph can be $\neg P_3$-colored using k colors is NP-complete.

The fourth chapter will answer the question: given any graph with acyclic complement, what is $\chi(G, \neg C_j)$ and $\chi(G, \neg P_j)$? A construction which illustrates that $\chi(G, \neg P_j) - \chi(G, \neg C_j) \geq a$ for any positive integer a is also provided.

The fifth chapter will determine $\chi(G, \neg C_j)$ for all graphs of large size. Dargen and Fraughnaugh [21] characterized $\chi(G, \neg C_j)$ when a graph is missing at most $j-2$ edges. In this chapter, we extend this result to graphs missing at most $j-1$ edges. To accomplish this, we characterize all Hamiltonian graphs of order n with at least $\binom{n}{2} - (n-1)$ edges.

The sixth chapter will determine $\chi(G, \neg P_j)$ for all graphs of large size and determine an upper bound on the size of G given a bound for $\chi(G, \neg P_j)$. To determine $\chi(G, \neg P_j)$ for all graphs of large size, we will characterize all graphs of order n with no Hamiltonian paths having at least $\binom{n}{2} - (2n-5)$ edges.

1.2 Definitions and Notation

A *graph* G consists of a finite nonempty set $V = V(G)$ of *vertices* and a collection $E = E(G)$ of distinct pairs of vertices, called *edges.* Throughout this paper, let G be a graph. The *size* or number of edges of G is denoted by $e = e(G)$, and the *order* or number of vertices of G is denoted by $n = n(G)$. A graph is *simple* if there is at most one edge between any distinct pair of vertices and there are no loops. For this paper, we will assume that all graphs are simple.

If u and v are vertices of G, we write the edge joining u and v as uv and call u and v *neighbors.* The *open neighborhood* $N(u)$ of a vertex u is the set of neighbors of u, and the *closed neighborhood* $N[u]$ of u is $N(u) \cup \{u\}$. The *degree* $d_G(v)$ of a vertex is defined to be the number of edges in G incident with that vertex. When it clear to which graph we are referring, we may write $d(v)$. The *minimum degree* $\delta(G)$ is the minimum degree among all vertices of G while the *maximum degree* $\Delta(G)$ is the maximum degree among all vertices of G. If $\delta(G) = \Delta(G) = r$, then all vertices have the same degree and G is *regular* of degree r or *r-regular.* A 3-regular graph is a *cubic* graph. If a vertex is not incident to any edge, then this vertex is an *isolated* vertex.

A *subgraph* H of G is a graph having all of its vertices and edges in G, and we write $H \subseteq G$. For any set S of vertices in G, the *induced subgraph* $\langle S \rangle$ is the maximal subgraph of G with vertex set S. If S is a nonempty set of edges in G, then $\langle S \rangle$ is the subgraph whose edge set is S and whose vertex set is the set of ends of edges in S. A graph is *complete* if every pair of vertices is joined by an edge. The complete graph on n vertices is K_n. The graph on n vertices with no edges is I_n.

The graph with distinct vertices $v_1, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$ is a *path* P_n. We say the vertices v_1 and v_n are *connected* by the path P_n. We sometimes call this path a (v_1, v_n)*-path*. The *length* of a path is the number of edges in it. The *distance* $d(u, v)$ from u to v in G is the length of a shortest path from u to v. The *diameter* of G is the maximum distance between any two vertices of G. If $u = v$ and there are at least three vertices on the path, then this path is a *cycle*. The graph which consists of a cycle on j vertices is C_j, and a graph which contains no cycles is *acyclic*. A graph G is *Hamiltonian* if it has a cycle containing all the vertices of G. A graph G has a *Hamiltonian path* if it has a path containing every vertex of G. A graph G is *Hamiltonian connected* if for every pair u and v of distinct vertices of G, there exists a Hamiltonian (u,v)-path.

A graph is *connected* if for any two vertices u and v, there is a path from u to v. A maximal connected subgraph of G is called a *component* of G. If G is not connected, then we say G is *disconnected*. The *connectivity* $\kappa(G)$ of G is the minimum number of vertices whose removal results in a disconnected or trivial graph. If the connectivity of G is κ, we say G is κ*-connected*. A connected acyclic graph is a *tree*, and a *forest* is a graph each of whose components is a tree.

In a tree T, we can make T a *rooted tree* by designating any vertex as the *root* vertex. Once we choose a root u, the *level* of a vertex v is $d(u, v)$. The root is the only vertex at level 0. Further, all adjacent vertices differ by exactly one level, and each vertex at level $i+1$ is adjacent to exactly one vertex at level i. The maximum level is the *height* $h_u(T)$ of the tree.

The *complement* $\bar{G}$ of G also has $V(G)$ as its vertex set, but two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

A *clique* of G is a complete subgraph of G. The *clique number* $\omega(G)$ of G is the maximum order among all cliques of G. An *independent set of vertices* of G is a set $I \subseteq V$ such that $xy \notin E$ for all $x, y \in I$. An *independent set of edges* of G has no two of its edges incident, and such a set is a *matching*.

Next, we will discuss some operations defined on graphs. Let G_1 and G_2 be two graphs with disjoint vertex sets. The *union* $G_1 + G_2$ of G_1 and G_2 has $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2)$. In general, mG is the pairwise vertex disjoint union of m copies of G. The *join* $G_1 \vee G_2$ of two graphs G_1 and G_2 has $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$.

The *cartesian product* $G_1 \times G_2$ of two graphs G_1 and G_2 has $V(G_1 \times G_2) = \{(u, v) : u \in V(G_1) \text{ and } v \in V(G_2)\}$, and (u_1, v_1) and (u_2, v_2) are adjacent whenever either $u_1u_2 \in E(G_1)$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1v_2 \in E(G_2)$. The *strong product* $G_1 \circ G_2$ of two graphs G_1 and G_2 has $V(G_1 \circ G_2) = \{(u, v) : u \in V(G_1) \text{ and } v \in V(G_2)\}$, and (u_1, v_1) and (u_2, v_2) are adjacent whenever either $u_1u_2 \in E(G_1)$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1v_2 \in E(G_2)$, or $u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)$. The *conjunction* $G_1 \wedge G_2$ of two graphs G_1 and G_2 has $V(G_1 \wedge G_2) = \{(u, v) : u \in V(G_1) \text{ and } v \in V(G_2)\}$, and (u_1, v_1) and (u_2, v_2) are adjacent whenever $u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)$.

A *vertex coloring* is an assignment of labels to the vertices of G so that any two adjacent vertices receive different labels. We think of each distinct label as a color and call each set of vertices assigned a fixed color a *color class*. The *vertex chromatic number* $\chi(G)$ of G is the minimum k for which G has a *k-coloring*, i.e., one using k colors. A graph is *k-colorable* if we can color it using k colors.

An *edge coloring* is an assignment of labels (or colors) to the edges of G so that any two incident edges receive a different label. The *edge chromatic number* of a graph is defined to be the minimum number of colors needed to edge color G. If we use the term "chromatic number" or "coloring" in this paper, we mean vertex chromatic number or vertex coloring.

A graph is said to be *embedded* in a surface S when it is drawn on S so that no two edges intersect. A graph is *planar* if it can be embedded in the plane; a *plane graph* has already been embedded in the plane. We will refer to the regions defined by a plane graph as its *faces*. The Four Color Theorem states that every planar graph is 4-colorable. For a proof of this result, see Appel and Haken [5].

A *bipartite graph* G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 and V_2. The *bipartition number* $b(G)$ *of a graph* G is given by $b(G) = \max\{e(B) : B \subseteq G$ and B is bipartite$\}$.

For other basic graph theory definitions and terminology, the reader is referred to Harary [29].

1.3 Conditional Coloring

An equivalent definition of vertex coloring is a partition of the vertex set so that the subgraph induced by each set of this partition is an independent set. Stated differently, each color class contains no path on two vertices.

This led several authors [15], [16], [35] to a more general concept of vertex colorings. Let P be any graph theoretic property. For example, P could be the property that a graph contains a clique of a certain order, that a graph does not contain a cycle of order 6, that a graph does not contain an induced cycle of order 6, or the maximum degree of a graph is 5.

We define a *P-coloring* of a graph to be an assignment of colors to its vertices so that the subgraph induced by each color class satisfies the property P. The *P-conditional chromatic number* $\chi(G, P)$ of G (or briefly *P-chromatic number*) is the minimum k for which G has a P-coloring with k colors. When P is the property that a graph consists entirely of isolated vertices, the P-chromatic number is the usual chromatic number.

When P is the property that a graph contains no subgraph (not necessarily induced) isomorphic to a graph F, we write $\chi(G, \neg F)$ for the P-chromatic number and refer to a P-coloring as a *$\neg F$-coloring* and the P-chromatic number as the $\neg F$-chromatic number. If $\chi(G, \neg F) \leq k$, then we say G is $\neg F$ k-colorable. When P is the property that a graph contains no induced subgraph isomorphic to a graph F, we write $\chi(G, \neg F!)$ for the P-chromatic number and refer to a P-coloring as a *$\neg F!$-coloring* and the P-chromatic number as the $\neg F!$-chromatic number.

The concept of partitioning the vertex set of a graph so that the subgraph induced by the vertices in each set in the partition has the property P seems to have been independently discovered by several authors around 1968 (see [15], [16], [35]) and again by several authors around 1985 (see [3], [4], [12], [14], [23], [30], [42]).

Mathematicians have studied various conditional chromatic numbers since 1968. In 1968, Chartrand, Kronk, and Wall [16] studied a conditional coloring number called the *arboricity* of a graph where color classes are acyclic. Hedetniemi [35] proved that the arboricity of a planar graph is at most three.

In 1968, Chartrand, Geller and Hedetniemi [15] proved several results about $\chi(G, \neg P_j)$. For example, if the diameter of G is d, then $\chi(G, \neg P_j) \leq d - j + 3$ for $j \geq 2$. Further, for every $j \geq 2$, the authors constructed a planar graph G such that $\chi(G, \neg P_j) = 4$. This proved that there is no stronger result than the Four Color Theorem for $\chi(G, \neg P_j)$ for $j \geq 2$. Also, in 1968, Sachs and Schäuble [48] proved that given $j \geq 2$ and $K \geq k \geq 1$, there exists a graph G with $\chi(G, \neg K_j) = k$ and a $\neg K_j$ K-coloring of G containing at least k color classes isomorphic to K_{j-1}.

In 1969, Kramer and Kramer [39] discussed $\chi(G, P)$, where P is the property that the subgraph induced by each color class has minimum degree j for some fixed $j \geq 0$. In 1970, Lick and White [41] published a paper with results for the same conditional chromatic number. In 1975, Cook [19] studied $\chi(G, \neg K_{1,j})$ for $j \geq 1$.

In 1977, Harary and Kainen [34] discussed $\chi(G, \neg K_3)$ for planar graphs. Also, Lesniak-Foster and Straight [40] published results for $\chi(G, P)$, where P is the property that each color class induces a complete graph or a graph with no edges. Sampathkumar, Prabha, Neeralagi, and Venkatachalam [50] published results for

$\chi(G, \neg P_j)$ for $j \geq 2$.

In 1978, Beineke and White [7] discussed the thickness of a graph. The concept of thickness is equivalent to $\chi(G, P)$, where P is the property that each color class is planar. The authors determined the thickness of complete and complete bipartite graphs.

In 1985, Harary [30] published a paper providing an overview of conditional coloring. This paper discussed the conditional vertex and edge chromatic numbers for several different properties and has become the standard for conditional coloring terminology.

In 1985, Andrews and Jacobson [3] studied $\chi(G, \Delta_t)$, where the maximum vertex degree in each color class is at most t. The authors related $\chi(G, \Delta_t)$ to $\chi(G)$ by proving that $\chi(G, \Delta_t) \geq \chi(G)/(t+1)$. Furthermore, they showed $\chi(G, \Delta_t) \geq n^2/(tn + n^2 - 2e)$, where n and e are the order and size of G. Harary and Fraughnaugh [31] also published results for $\chi(G, \Delta_t)$ in 1985.

Also in 1985, Mynhardt and Broere [42] discussed conditional coloring with respect to the property that each color class is a disjoint union of complete subgraphs. They also studied the problem of finding a graph subject to certain restrictions for which the conditional chromatic number is arbitrarily large. Mynhardt and Broere [12] also studied conditional coloring with respect to the property that each color class has no induced subgraph isomorphic to a graph F.

In 1986, Harary and Fraughnaugh [32] generalized the concept of bipartition number to that of a conditional bipartition number. Specifically, given a property P, a graph is *conditionally bipartite with respect to* P if $V(G)$ is the disjoint union of sets X and Y where the induced subgraphs $\langle X \rangle$ and $\langle Y \rangle$ both have property

P. The conditional bipartition number $b(G, P)$ is $\max\{e(B) : B \subseteq G$ and B is conditionally bipartite with respect to $P\}$. The authors studied $b(G, P)$ for several minimum and maximum degree properties.

Also in 1986, Domke, Laskar, Hedetniemi, and Peters [23] discussed $\chi(G, P)$, where P is the property that the subgraph induced by each color class is a complete r-partite graph for any r and $\chi(G, Q)$, where Q is the property that each color class is a disjoint union of complete subgraphs.

In 1987, Andrews and Jacobson [4] published a paper with results for $\chi(G, \Delta_t)$. Also in 1987, Brown and Corneil [14] studied conditional coloring with respect to the property that each color class has no induced subgraph isomorphic to a set of graphs.

In 1989, Akiyama, Era, Gervacio, and Watanabe [1] discussed the k-path chromatic number. The *k-path chromatic number* $\chi(G, P_k)$ of G is the smallest number c of distinct colors with which $V(G)$ can be colored such that each connected component of $\langle V_i \rangle$ is a path of order at most k, $1 \leq i \leq c$. The authors proved that $\chi(G, P_k) \leq \left\lceil \frac{r+1}{2} \right\rceil$ for r-regular graphs. Since the 2-path chromatic number is equivalent to our $\neg P_3$-chromatic number, we immediately get that $\chi(G, \neg P_3) \leq 2$ for cubic graphs. Also in 1989, Albertson, Jamison, Hedetniemi, and Locke [2] discussed conditional coloring with respect to the property that each color class is a disjoint union of complete subgraphs. In 1990, Baldi [6] published results for $\chi(G, \neg P_j)$ for $j \geq 2$.

In 1991, Harary and Hsu [33] provided theorems relating the conditional chromatic number of the Cartesian product, join, strong product, and conjunction of two graphs to the conditional chromatic number of the original pair of graphs.

In 1992, Borodin [11] discussed the k-cyclic chromatic number. A coloring of the vertices of a planar graph is *k-cyclic* if whenever two vertices lie in the boundary of the same face of size at most k, their colors are different.

There are also conditional coloring papers published in 1992 where a color class is not permitted to contain some *induced* subgraph. For example, Brown and Corneil [13], discussed uniquely $\neg H!$ k-colorable graphs where H is any graph. A graph G is *uniquely* k-colorable if G is k-colorable and there is only one k-coloring (up to a permutation of colors). In fact, Brown and Corneil conjectured that for all graphs of order at least two and for all nonnegative integers k, there exist uniquely $\neg H!$ k-colorable graphs. So far, they have shown this result whenever G is 2-connected or $\bar{G}$ is 2-connected.

In 1992, Johns and Saba [38] considered $\chi(G, \neg P_j)$. They proved that given integers $l \geq 1$ and $j \geq 2$, there exists a graph G such that $\chi(G, \neg P_j) = l$, namely $K_{(j-1)l}$. Also, there exists a graph such that $\chi(G, \neg P_j) - \chi(G, \neg P_{j+1}) = l$, namely $K_{j(j+1)l}$.

In 1993, Dargen and Fraughnaugh [21] determined $\chi(G, \neg C_j)$ for graphs missing up to $j-2$ edges and Sampathkumar [49] discussed $\chi(G, P)$, where P is the property that the subgraph induced by each color class has independence j for some fixed $j \geq 0$. Also in 1993, Hutchinson [37] applied thickness results to testing printed circuit boards for electrical shorts.

In 1995, Cimikowski [18] presented some heuristics for the graph thickness problem, i.e., decomposing a graph into the minimum number of planar subgraphs. The heuristics are based on some algorithms for finding a maximal planar subgraph of a nonplanar graph. He also proved that $T(G) \leq \lfloor \sqrt{2e/3} + 3/2 \rfloor$, where $T(G)$ is the thickness of G and e is the size of G.

For a survey of different properties studied, who studied them, and when they were studied, see Table 1.1.

Table 1.1. Conditional Coloring Properties studied

Color Class Property	Year	Reference
disconnected or trivial	1968 1970	Hedetniemi [35] Hedetniemi [36]
acyclic	1968 1968	Chartrand, Kronk, Wall [16] Hedetniemi [35]
has no K_3	1977	Harary, Kainen [34]
has no K_j for some fixed j	1968	Sachs, Schäuble [48]
has no P_j for some fixed j	1968 1977 1990 1992	Chartrand, Geller, Hedetniemi [15] Sampathkumar, Prabha, Neeralagi, Venkatachalam [50] Baldi [6] Johns, Saba [38]
complete or a graph without edges	1977	Lesniak-Foster, Straight [40]
has maximum degree j for some fixed j	1985 1985 1986 1987	Harary, Fraughnaugh [31] Andrews, Jacobson [3] Harary, Fraughnaugh [32] Andrews, Jacobson [4]
complete r-partite graph for any r	1986	Domke, Laskar, Hedetniemi, Peters [23]
has no induced subgraph isomorphic to graph F	1985 1985 1987	Broere, Mynhardt [12] Mynhardt, Broere [42] Brown, Corneil [14]
has no induced subgraph isomorphic to any graph F in a set of graphs **F**	1987	Brown, Corneil [14]
disjoint union of complete subgraphs	1985 1986 1989	Mynhardt, Broere [42] Domke, Laskar, Hedetniemi, Peters [23] Albertson, Jamison, Hedetniemi, Locke [2]
contains no $K_{1,j}$	1975	Cook [19]
minimum degree $\geq j$ for fixed j	1969 1970	Kramer, Kramer [39] Lick, White [41]

Table 1.1. Conditional Coloring Properties studied (cont.)

Color Class Property	Year	Reference
disjoint union of paths of order at most k	1989	Akiyama, Era, Gervacio, Watanabe [1]
j independent for fixed j	1993	Sampathkumar [49]
has no C_j for fixed j	1993	Dargen, Fraughnaugh [21]
planar	1978 1993 1995	Beineke, White [7] Hutchinson [37] Cimikowski [18]
k-cyclic	1992	Borodin [11]
contains no induced subgraph isomorphic to H	1992	Brown, Corneil [13]
survey of properties	1985	Harary [30]

In this thesis, we concentrate primarily on two conditional chromatic numbers, $\chi(G, \neg C_j)$ where C_j is a cycle of order $j \geq 3$ and $\chi(G, \neg P_j)$ where P_j is a path of order $j \geq 2$.

2 Basic Results for $\chi(\mathbf{G}, \neg\mathbf{P_j})$ and $\chi(\mathbf{G}, \neg\mathbf{C_j})$

This chapter provides the necessary background for the topics presented in Chapters 3, 4, 5, and 6. We start by proving some basic relationships for $\chi(G, \neg C_j)$ and $\chi(G, \neg P_j)$. The following straightforward result has been known since 1968 [15].

Theorem 2.1 *Let G be a graph of order n. If $j \geq 2$, then $\chi(G, \neg P_j) \leq \left\lceil \frac{n}{j-1} \right\rceil$.*

Proof. Let $n = a(j-1)+r$, where $0 \leq r < j-1$. Color G as follows: create a color classes of size $j-1$ and one color class of size r if $r > 0$. Hence $\chi(G, \neg P_j) \leq \left\lceil \frac{n}{j-1} \right\rceil$. □

Corollary 2.1 *If $j \geq 2$, then $\chi(K_n, \neg P_j) = \left\lceil \frac{n}{j-1} \right\rceil$.*

Proof. Since a complete subgraph of order at least j contains P_j, each color class of K_n can contain at most $j-1$ vertices. Therefore, $\chi(K_n, \neg P_j) \geq \left\lceil \frac{n}{j-1} \right\rceil$ and equality follows from Theorem 2.1. □

The next theorem provides the most basic relationship between $\chi(G, \neg C_j)$ and $\chi(G, \neg P_j)$.

Theorem 2.2 *If $j \geq 3$ and G is a graph, then $\chi(G, \neg C_j) \leq \chi(G, \neg P_j)$.*

Proof. Let $\mathcal{C}$ be any minimum $\neg P_j$-coloring of G. Since a color class that contains no P_j contains no C_j, the coloring $\mathcal{C}$ is also a $\neg C_j$-coloring of G. So, $\chi(G, \neg C_j) \leq \chi(G, \neg P_j)$. □

Now that we have an upper bound for $\chi(G, \neg P_j)$ and a relationship between $\chi(G, \neg C_j)$ and $\chi(G, \neg P_j)$, we can find an upper bound for $\chi(G, \neg C_j)$.

Corollary 2.2 *If $j \geq 3$ and G is a graph of order n, then $\chi(G, \neg C_j) \leq \left\lceil \frac{n}{j-1} \right\rceil$. Further, $\chi(K_n, \neg C_j) = \left\lceil \frac{n}{j-1} \right\rceil$.*

Proof. By Theorem 2.2 and Theorem 2.1, we get $\chi(G, \neg C_j) \leq \chi(G, \neg P_j) \leq \left\lceil \frac{n}{j-1} \right\rceil$. Let $G = K_n$. A subgraph induced by j or more vertices of K_n contains C_j, and therefore cannot be a color class of K_n. Hence, $\chi(K_n, \neg C_j) \geq \left\lceil \frac{n}{j-1} \right\rceil$. □

The next theorem relates two different conditional coloring numbers.

Theorem 2.3 *If $j \geq k \geq 2$, then $\chi(G, \neg P_j) \leq \chi(G, \neg P_k)$.*

Proof. Let $\mathcal{C}$ be any minimum $\neg P_k$-coloring of G. Since a color class that contains no P_k contains no P_j for $k \leq j$, the coloring $\mathcal{C}$ is also a $\neg P_j$-coloring of G. Therefore, $\chi(G, \neg P_j) \leq \chi(G, \neg P_k)$. □

One might ask if the above type of theorem is true for $\chi(G, \neg C_j)$. In fact, there is no relationship in general in either direction. For example, $\chi(C_4, \neg C_4) = 2$ and $\chi(C_4, \neg C_3) = 1$, which implies that $\chi(G, \neg C_4) \not\leq \chi(G, \neg C_3)$ for at least one graph. Further, $\chi(C_3, \neg C_4) = 1$ and $\chi(C_3, \neg C_3) = 2$, which implies that $\chi(G, \neg C_4) \not\geq \chi(G, \neg C_3)$ for at least one graph.

The next theorem provides a relationship between the conditional chromatic numbers of a graph and its subgraphs.

Theorem 2.4 *Let G be a graph with $H \subseteq G$. If $j \geq 2$, then $\chi(H, \neg P_j) \leq \chi(G, \neg P_j)$. If $j \geq 3$, then $\chi(H, \neg C_j) \leq \chi(G, \neg C_j)$.*

Proof. Let $\mathcal{C}$ be any minimum $\neg C_j(\neg P_j)$-coloring of G. Since $H \subseteq G$, the coloring $\mathcal{C}$ is also a $\neg C_j(\neg P_j)$-coloring of H. So, $\chi(H, \neg C_j) \leq \chi(G, \neg C_j)$ and $\chi(H, \neg P_j) \leq \chi(G, \neg P_j)$. □

Now that we have results for subgraphs for $\chi(G, \neg C_j)$ and $\chi(G, \neg P_j)$, we can derive an elementary lower bound for $\chi(G, \neg P_j)$ and $\chi(G, \neg C_j)$.

Theorem 2.5 *If G is a graph and $j \geq 3$, then $\chi(G, \neg P_j) \geq \chi(G, \neg C_j) \geq \left\lceil \frac{\omega(G)}{j-1} \right\rceil$.*

Proof. First of all, $K_{\omega(G)} \subseteq G$. By Corollary 2.2, $\chi(K_{\omega(G)}, \neg C_j) = \left\lceil \frac{\omega(G)}{j-1} \right\rceil$. By Theorem 2.4, we get $\chi(G, \neg C_j) \geq \chi(K_{\omega(G)}, \neg C_j) = \left\lceil \frac{\omega(G)}{j-1} \right\rceil$. By Theorem 2.2, we have $\chi(G, \neg P_j) \geq \chi(G, \neg C_j)$. □

This bound is attained when $G = K_n$, and therefore the bound is tight. In order to derive some upper bounds on the order of a color class of some graph G, we need some well known results regarding Hamiltonian graphs, Hamiltonian connected graphs, and graphs which have a Hamiltonian path. First, we state Ore's Theorem, Dirac's Theorem, and another theorem which appears as Corollary 4.6 in Bondy and Murty [10].

Theorem 2.6 *(Ore [44]) If G is a graph of order $n \geq 3$ such that for all distinct nonadjacent vertices u and v, $d(u) + d(v) \geq n$, then G is Hamiltonian.*

Theorem 2.7 *(Dirac [22]) If G is a graph of order $n \geq 3$ and each vertex has degree at least $\frac{n}{2}$, then G is Hamiltonian.*

Theorem 2.8 *(Ore [45], Bondy [9]) If G is a graph of order $n \geq 3$ and $e(G) \geq \frac{n^2-3n+6}{2}$, then G is Hamiltonian. Moreover, the only non-Hamiltonian graphs with n vertices and $\frac{n^2-3n+4}{2}$ edges are $K_1 \vee (K_1 + K_{n-2})$ and $K_2 \vee I_3$.*

The following theorem is also well known. Parts (*i*) and (*iii*) appear as an exercise in West [51] and part (*ii*) is simply a restatement of the first sentence in Theorem 2.8.

Theorem 2.9 *Let G be a graph of order n.*

If $n \geq 2$ and $e(G) \geq \binom{n}{2} - (n-4)$, then G is Hamiltonian connected. (*i*)

If $n \geq 3$ and $e(G) \geq \binom{n}{2} - (n-3)$, then G is Hamiltonian. (*ii*)

If $n \geq 2$ and $e(G) \geq \binom{n}{2} - (n-2)$, then G has a Hamiltonian path. (*iii*)

Proof. (*i*). (by induction on n). If $n \leq 3$, then (*i*) is vacuously true. If $n = 4$, then $G = K_4$ and a complete graph is Hamiltonian connected. If $n = 5$, then $G \in \{K_5, K_5 - e\}$ where e is an edge. These two graphs are Hamiltonian connected. If $n = 6$, then $G \in \{K_6, K_6 - e, K_6 - E(2K_2), K_6 - E(P_3)\} = \mathcal{G}$ where e is an edge. Each $G \in \mathcal{G}$ is Hamiltonian connected. So assume (*i*) holds for appropriate graphs on $n-1$ vertices.

Let G be a graph on $n \geq 7$ vertices and $e(G) \geq \binom{n}{2} - (n-4)$. We will show G is Hamiltonian connected. Let $\{u, v\} \subseteq V(G)$ with $u \neq v$.

If $d(u) \leq n-2$, then $e(G-u) \geq e(G) - (n-2) \geq \binom{n}{2} - (n-4) - (n-2) = \binom{n-1}{2} - (n-5)$ and, by the induction hypothesis, $G-u$ is Hamiltonian connected. Now $e(G) \geq \binom{n}{2} - (n-4)$ implies that $\delta(G) \geq 3$ for we would need to remove at least $n-3$ edges from a complete graph in order for $\delta(G) = 2$. Since $d(u) \geq 3$, we can choose $z \in N(u) - \{v\}$ and add uz to a Hamiltonian (z,v)-path in $G-u$ to form a Hamiltonian (u,v)-path in G.

If u is adjacent to every other vertex in G (i.e., $d(u) = n-1$), then $e(G-u) = e(G) - (n-1) \geq \binom{n}{2} - (n-4) - (n-1) = \binom{n-1}{2} - (n-4)$ and, by Theorem 2.8,

$G-u$ is Hamiltonian. Break an edge involving v (say vw) on the Hamiltonian cycle in $G-u$ and add the edge wu to obtain a Hamiltonian (u,v)-path in G. In either case, G has a Hamiltonian (u,v)-path. Therefore, G is Hamiltonian connected.

(ii). For a proof of Ore's Theorem, see Roberts [47].

(iii). If G is complete, then G is Hamiltonian. So, assume G is not complete. Since $e(G) \geq \binom{n}{2} - (n-2)$, we get $n \geq 3$. Let uv be any missing edge of G. Consider $G + uv$. Now, $e(G+uv) = e(G) + 1 \geq \binom{n}{2} - (n-3)$. By ($ii$), the graph $G + e$ has a Hamiltonian cycle. Therefore, G has a Hamiltonian path. □

We can see that all three statements in the previous theorem are exact since $K_2 \vee (K_1 + K_{n-3})$ has $\binom{n}{2} - (n-3)$ edges and is not Hamiltonian connected, $K_1 \vee (K_1 + K_{n-2})$ and $K_2 \vee I_3$ have $\binom{n}{2} - (n-2)$ edges and are not Hamiltonian (in fact, Theorem 2.8 points out that these are the only such graphs), and $K_1 + K_{n-1}$ has $\binom{n}{2} - (n-1)$ edges and contains no Hamiltonian path. Lastly, we state a sufficient condition on the size of a graph to obtain cycles of all orders.

Theorem 2.10 *If G is a graph with order $n \geq 3$ and $e(G) \geq \binom{n}{2} - (n-3)$, then $C_k \subseteq G$ for all $k = 3, 4 \ldots, n$.*

Proof. (by induction on n). If $n = 3$, then $G = K_3$ and the result follows immediately. Assume that $n \geq 4$ and that any graph of order $n-1$ with at least $\binom{n-1}{2} - (n-4)$ edges contains C_j for $j = 3, 4, \ldots, n-1$.

Let G be a graph of order n with $e(G) \geq \binom{n}{2} - (n-3)$. By Theorem 2.9, we get $C_n \subseteq G$. If G is complete, the result follows immediately. If G is not complete, then $\delta(G) \leq n-2$. Choose a vertex $v \in V(G)$ of minimum degree in G. Consider $G - v$. Then $e(G-v) = e(G) - \delta(G) \geq \binom{n}{2} - (n-3) - (n-2) =$

$\binom{n-1}{2} - (n-4)$. By the induction hypothesis, $G - v$ contains $C_3, C_4, \ldots, C_{n-1}$ as subgraphs. Therefore, G contains the same $C_3, \ldots, C_{n-1}$ as subgraphs. $\square$

Now that we have some basic results, we discuss the computational complexity of the following problem: Given a graph G and a positive integer k, can G be $\neg P_3$-colored using k colors? In the next chapter, we show that this problem is NP-complete.

3 NP-complete

A *polynomial time algorithm* is an algorithm whose worst-case running time is $O(n^k)$, where the input to the algorithm is of cardinality n and k is some constant. A *problem Π* is defined to be a binary relation on a set I of problem *instances* and a set S of problem *solutions*. For example, consider the problem SHORTEST-PATH of finding a shortest path between two given vertices in G. An instance for SHORTEST-PATH is a triple consisting of a graph and two vertices. A solution is a sequence of vertices in the graph G with perhaps the empty sequence denoting that no path exists. The problem SHORTEST-PATH itself is a relation that associates each instance of a graph and two vertices with a shortest path in the graph that connects the two vertices. For simplicity, the theory of NP-completeness restricts itself to *decision problems*, those having a yes/no solution. In this case, we can view an abstract decision problem as a function that maps the instance set I to the solution set $\{0, 1\}$. For example, a decision problem related to SHORTEST-PATH is as follows: *Given a graph G, two vertices $\{u, v\} \subseteq V(G)$, and a positive integer k, does there exist a path of length at most k?*

There are decision problems which can be solved in polynomial time and those which require superpolynomial time. A *polynomial time solvable problem* is one which can be solved using a deterministic polynomial time algorithm.

The complexity class *NP* is the class of problems where given a "yes" solution, we can verify this solution in polynomial time. A basic idea in the theory of NP-completeness is that of a *polynomial transformation.* Let Π_1 and Π_2 denote two decision problems. We say that there is a polynomial transformation from Π_1 to Π_2, written $\Pi_1 \propto \Pi_2$, if the following two conditions hold:

(a) There exists a function F transforming any instance I of Π_1 to an instance $F(I)$ of Π_2 such that the answer to I with respect to Π_1 is "yes" if and only if the answer to $F(I)$ is "yes" with respect to Π_2.

(b) There exists an polynomial time algorithm to compute $F(I)$.

A decision problem Π is *NP-complete* if $\Pi \in NP$ and for every problem $\Pi' \in NP$, $\Pi' \propto \Pi$.

The k-COLORING problem is stated as follows: given a graph $G = (V, E)$ and integer $3 \leq k \leq |V|$, is G k-colorable? We know from Garey and Johnson [27] that the k-COLORING problem is NP-complete for $k \geq 3$. The goal of this chapter is to determine how difficult it is to solve the problem of conditionally coloring a graph. Observe that the $\neg P_2$ k-COLORING problem is the usual k-COLORING problem. Does relaxing the condition for each color class being an independent set to each color class containing no P_j for $j \geq 3$ change the computational complexity of the colorability problem? We will show the answer is "no" when $j = 3$. The $\neg P_3$ k-COLORING problem is stated as follows: given a graph $G = (V, E)$ and integer $3 \leq k \leq \left\lceil \frac{|V|}{2} \right\rceil$ (Note: By Theorem 2.1, a graph can always be $\neg P_3$-colored with at most $\left\lceil \frac{|V|}{2} \right\rceil$ colors), does there exist a k-partition of the vertex set so that each subgraph induced by a set in the partition does not contain P_3 as a subgraph (not necessarily induced)?

Theorem 3.1 *The $\neg P_3$ k-COLORING problem Π is NP-complete for $k \geq 3$.*

Proof. Let $G = (V, E)$ be a graph of order n, and $\mathcal{C}$ a $\neg P_3$ k-coloring of G. The following is a polynomial time algorithm which can be used to check that $\mathcal{C}$ is a valid $\neg P_3$-coloring of G. Examine all triples of distinct vertices in each color class and determine whether the graph induced by each triple contains P_3. Determining if a triple contains P_3 is $O(1)$. Since each set in the partition can contain at most n vertices, there are $O(n^3)$ triples. This algorithm is of order n^3 and this shows that $\Pi \in NP$.

Next, we will show that a known NP-complete problem can be transformed to Π. Construct a graph $F(G)$ from $G = (V, E)$ as follows: for each $v \in V$, let $F(G)$ contain two vertices v_1 and v_2. We say that v_1 and v_2 are *associated* with the vertex v. In $F(G)$, join v_1 and v_2 to the vertices associated with each neighbor of v and join v_1 to v_2. Observe that this transformation is polynomial. See Figure 3.1 for the transformation of $G = C_4$.

We will show that $\chi(G) \leq k$ implies that $\chi(F(G), \neg P_3) \leq k$. Assume $\chi(G) \leq$

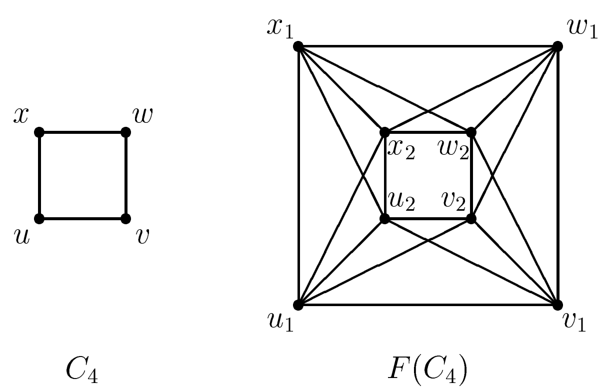

Figure 3.1. The construction of $F(C_4)$.

k. Let $H = F(G)$ and C be a k-coloring of G. Color H as follows: For each vertex $v \in V$, assign $C(v)$ to the two vertices associated with v in H. This coloring uses at most k colors to color H. To see that this coloring is a valid $\neg P_3$-coloring of H, suppose, by way of contradiction, that a color class of H contains $P_3 = uvw$. By construction, $ab \in E(H)$ if and only if either a and b are associated with the same vertex in G or there exists $cd \in E(G)$ such that a is associated with c and b is associated with d. Therefore, if two adjacent vertices in H are assigned the same color, then they must be associated with the same vertex in G since G has a valid k-coloring. Thus, $uv \in E(H)$ implies that there exists $c \in V(G)$ such that u and v are associated with c. Similarly, $vw \in E(H)$ and v being a vertex associated with c imply that v and w are associated with c. This is a contradiction since, by construction, there are exactly two vertices in H associated with a vertex in G. Therefore, $\chi(H, \neg P_3) \leq k$.

Next, we will show that $\chi(H, \neg P_3) \leq k$ implies that $\chi(G) \leq k$. Let $d : V(H) \to \{1, 2, \ldots, k\}$ be a $\neg P_3$ k-coloring of H. We color the vertices of G using Algorithm 3.1 presented in Figure 3.1. Let $c : V(G) \to \{1, 2, \ldots, k\}$ be the function that assigns colors to the vertices of G based on Algorithm 3.1.

Note 1: Observe that Algorithm 3.1 assigns a color to $z \in V(G)$ from one of the colors assigned to the two vertices in H associated with z.

Note 2: For every $v \in V(G)$ with associated vertices v_1 and v_2, there is at most one $w \in N_H(v_1, v_2)$ colored $c(v)$. Suppose by way of contradiction that there exist $x, y \in V(H)$ such that $\{x, y\} \subseteq N_H(v_1, v_2)$ and $d(x) = d(y) = c(v)$. By Note 1, we know that either v_1 or v_2 is colored $c(v)$, say v_1. But xv_1y forms P_3 in H, which is a contradiction.

Each time through the loop (Lines 7-22), a vertex in G is assigned a color in either Line 9 or Line 17. Since the number of vertices in a graph is finite, all the vertices of G are assigned a color. We will show that c is a valid k-coloring of G.

Input: Graphs G and H, and a $\neg P_3$-coloring $d : V(H) \rightarrow \{1, 2, \ldots, k\}$ of H.
Output: A k-coloring $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ of G.

1. For every $z \in V(G)$ with associated vertices z_1 and z_2 in H.
2.   If $d(z_1) = d(z_2)$, then let $c(z) = d(z_1)$.
3.   If $d(z_1) \notin d(N(z_1))$ and z is uncolored, then let $c(z) = d(z_1)$.
4.   If $d(z_2) \notin d(N(z_2))$ and z is uncolored, then let $c(z) = d(z_2)$.
5. If G has uncolored vertices, then
6.   Let previous_color $= -1$ and z be an uncolored vertex in G.
7.   Repeat steps 8-22 until all vertices of G are colored.
8.   If previous_color $\neq d(z_1)$, then
9.   let $c(z) = d(z_1)$,
10.   let previous_color $= d(z_1)$, and w be the unique vertex in $V(G)$ with which the vertex in $N_H(z_1)$ with color $d(z_1)$ is associated.
11.   If w is uncolored, then
12.   let $z = w$.
13.   else if there are uncolored vertices in G, then
14.   let previous_color $= -1$ and z be an uncolored vertex in G.
16.   else (previous_color $= d(z_1)$)
17.   let $c(z) = d(z_2)$,
18.   let previous_color $= d(z_2)$, and w be the unique vertex in $V(G)$ with which the vertex in $N_H(z_2)$ with color $d(z_2)$ is associated.
19   If w is uncolored, then
20.   let $z = w$.
21.   else if there are uncolored vertices in G, then
22.   let previous_color $= -1$ and z be an uncolored vertex in G.
23. Output c and stop.

Figure 3.1. Algorithm 3.1

Let $v \in V(G)$ with $c(v) = a$, $u \in N(v)$, v_1 and v_2 be the vertices in H associated with v, and u_1 and u_2 be the vertices in H associated with u. We will show that $c(u) \neq a$.

Assume v was assigned color a in Line 2 of Algorithm 3.1. Now, u cannot be assigned color a or else the subgraph induced by the color class in H containing v_1, v_2, and one of the vertices associated with u would contain P_3. Thus, all vertices assigned a color in Line 2 of Algorithm 3.1 receive a valid color.

Assume v was assigned color a in Line 3 of Algorithm 3.1, that is, $d(v_1) = a$ and the neighbors of v_1 (which include u_1 and u_2) are not colored a. Since u is assigned its color from one of the colors assigned to u_1 or u_2 (Note 1), we have $c(u) \neq a$. Thus, all vertices colored in Line 3 receive a valid color. A similar argument proves that every vertex colored in Line 4 receives a valid color.

We have just shown that if u or v were assigned a color in Lines 2, 3, or 4, then $c(u) \neq c(v)$. So assume u and v were assigned colors in Lines 9 or 17 of Algorithm 3.1. Since only one vertex is colored at at time, let's assume that u was already colored when v is colored.

Suppose that $c(u) = c(v) = a$. Assume u was assigned color a in Line 9 of Algorithm 3.1. In the loop (Lines 7-22) with $z = u$, in Line 9 we have $c(u) = d(u_1) = a$ and we let previous_color $= a$ (Line 10). Since we assumed that $c(v) = a$, either v_1 or v_2 is colored a (Note 1). Further, since there is at most one neighbor of u_1 colored a (Note 2), we must have $z = v$ (Line 12), i.e., v is the next vertex to be colored in the loop (Lines 7-22). When we go through the loop to color v, either it gets colored in Line 9 or 17. If v is colored in Line 9, then since previous_color $= a$, we know that the color of $c(v_1) \neq a$ (Line 8) and $c(v) \neq a$

(Line 9). So v must be colored in Line 17 and we know that $d(v_1) = a$ (Line 16) and $c(v) = d(v_2) = a$ (Line 17). Now, $d(v_2) \neq a$ or else v would have been colored in Line 2. Thus $c(v) \neq a$, which contradicts the assumption that $c(v) = a$. By applying the previous argument replacing x_1 with x_2 and starting with $z = x$ in Line 17, we again reach a contradiction.

Thus, for every adjacent $u, v \in V(G)$, we have $c(u) \neq c(v)$, which implies that Algorithm 3.1 produces a k-coloring of G. We have transformed a known NP-complete problem to our problem. Therefore, the $\neg P_3$ k-COLORING problem is NP-complete. $\square$

This construction for H does not help to prove that the $\neg P_j$ k-colorability problem is NP-complete for $j \geq 4$. We have tried other constructions for H for the $\neg P_j$ k-colorability problem with no success. Suggested areas for future research include determining whether the $\neg P_j$ k-colorability problem, the $\neg C_j$ k-colorability problem, and the $\neg K_j$ k-colorability problem are NP-complete for $j \geq 3$. Now, we turn our attention to finding $\chi(G, \neg P_j)$ and $\chi(G, \neg C_j)$ for graphs with acyclic complements.

4 Determining the conditional chromatic number for graphs with acyclic complements

In this chapter we will show that $\chi(G, \neg C_j)$ and $\chi(G, \neg P_j)$ may have different values for many graphs. First, we will examine the values of $\chi(G, \neg C_j)$ for graphs whose complements are acyclic and then perform a similar analysis for $\chi(G, \neg P_j)$. Finally, the main theorem of this chapter shows that the difference between $\chi(G, \neg C_j)$ and $\chi(G, \neg P_j)$ can be made arbitrarily large in a particular family of graphs.

4.1 Determining $\chi(\mathbf{G}, \neg\mathbf{C_j})$ when $\bar{\mathbf{G}}$ is acyclic

To determine $\chi(G, \neg C_j)$ for $j \geq 3$, it is necessary to determine how large any color class can be in a minimum $\neg C_j$-coloring of G. If $j = 3$, then we will see the largest color class must be of size 4 or less and if $j \geq 4$, then the largest color class must be of size j or less. Therefore, we determine $\chi(G, \neg C_3)$ and $\chi(G, \neg C_j)$ for $j \geq 4$ in two separate theorems. We first address the size of the largest possible color class for $\chi(G, \neg C_3)$ with the following lemma.

Lemma 4.1 *If G is a graph of order $n \geq 5$ and $\bar{G}$ is acyclic, then G contains C_3 as a subgraph.*

Proof. Let G be a graph of order $n \geq 5$ with acyclic complement. Let H be a subgraph of G with five vertices. We may assume $\bar{H}$ is a tree (otherwise we remove edges from H until $\bar{H}$ is a tree). If $\bar{H}$ has a vertex v such that $d_{\bar{H}}(v) \geq 3$, then since $\bar{H}$ contains no cycles, $N_{\bar{H}}(v)$ is an independent set in $\bar{H}$ of size at least

3. If every vertex in $\bar{H}$ has degree at most 2, then $\bar{H} = P_5$, which clearly contains an independent set of size 3. In either case, the vertices from the independent set in $\bar{H}$ forms C_3 in H. Since $H \subseteq G$, the graph G contains C_3. □

The following theorem evaluates $\chi(G, \neg C_3)$ when $\bar{G}$ is acyclic.

Theorem 4.1 *Let G be a graph with acyclic complement. Let m be the number of edges in a maximum matching of $\bar{G}$. If $m \geq 0$, then $\chi(G, \neg C_3) = \left\lceil \frac{n-m}{2} \right\rceil$.*

Proof. Let A be a color class in a $\neg C_3$-coloring of G. By Lemma 4.1, $|A| \leq 4$. Let a be the number of color classes of size 4 and b the number of color classes of size 3. The only C_3-free graphs of order 4 whose complements are acyclic are P_4 and C_4. Each of these graphs is missing two independent edges. Therefore, the subgraph induced by each color class of size 4 must be missing two independent edges. The only C_3-free graphs of order 3 whose complements are acyclic are P_3 and $K_2 + K_1$. Each of these graphs is missing an edge. Therefore, the subgraph induced by each color class of size 3 must be missing an edge.

Since color classes are disjoint, we can form a matching M_1 in $\bar{G}$ with a pair of edges from each color class of size 4 and one from each color class of size 3. Thus $|M_1| = 2a + b$, and since m is the size of a maximum matching, we get $2a + b = |M_1| \leq m$. Thus, $\chi(G, \neg C_3) \geq \left\lceil \frac{n-4a-3b}{2} \right\rceil + a + b = \left\lceil \frac{n-2a-b}{2} \right\rceil \geq \left\lceil \frac{n-m}{2} \right\rceil$.

To construct a coloring, let M be a maximum matching in $\bar{G}$. We form as many color classes of size 4 as possible by using the endpoints of two edges in M in each, then based on the parity of m, zero (if m is even) or one (if m is odd) color class of size 3 using the endpoints of one edge in M and an additional vertex, and finally as many color classes of size 2 as possible. With this construction, the graph induced by a color class of size 4 is either C_4 or P_4. Further, the graph induced by a color class of size 3 is either P_3 or K_2+K_1. Each of these graphs is C_3-free and are valid color classes.

If m is even, there are $\frac{m}{2}$ color classes of size 4, and the rest are of size at most 2. With this coloring, we get $\chi(G, \neg C_3) \leq \left\lceil \frac{n-4(\frac{m}{2})}{2} \right\rceil + \frac{m}{2} = \left\lceil \frac{n-m}{2} \right\rceil$.

If m is odd and $n > 2m$, there are $\frac{m-1}{2}$ color classes of size 4 and one color class of size 3. The rest are of size at most 2. With this coloring, we get $\chi(G, \neg C_3) \leq \left\lceil \frac{n-4(\frac{m-1}{2})-3}{2} \right\rceil + \frac{m-1}{2} + 1 = \left\lceil \frac{n-m}{2} \right\rceil$.

If m is odd and $n = 2m$, there are $\frac{m-1}{2}$ color classes of size 4 and one color class of size 2. With this coloring, $\chi(G, \neg C_3) \leq \frac{m-1}{2} + 1 = \frac{m+1}{2} = \left\lceil \frac{m}{2} \right\rceil = \left\lceil \frac{n-m}{2} \right\rceil$.

□

Next we will complete the study of graphs whose complements are acyclic by determining $\chi(G, \neg C_j)$ for $j \geq 4$. Again, we need to determine how large any color class can be in a minimum $\neg C_j$-coloring of a graph. Recall that Theorem 2.10 guarantees cycles of all orders if a graph is missing at most $j-3$ edges. The following theorem shows that we still have long cycles for some graphs missing up to $n-1$ edges.

Lemma 4.2 *Let G be a graph of order $n \geq 5$ with acyclic complement. Then $C_{n-1} \subseteq G$. Furthermore, if $K_{1,n-2} \nsubseteq \bar{G}$, then G is Hamiltonian.*

Proof. We may assume $\bar{G} = T$ where T is a tree and $K_{1,n-2} \subseteq T$ only if $K_{1,n-2} \subseteq \bar{G}$. (Otherwise, remove edges from G until we get a graph whose complement is a tree. Since $n \geq 5$, this can be done without creating $K_{1,n-2}$ in the complement.) Choose a root vertex v_0 of T so that the height of the tree $h_{v_0}(T)$ is maximum.

Since $n \geq 5$, the diameter of the tree T is at least 2 and therefore, $h_{v_0}(T) \geq 2$. Further, $d_T(v_0) = 1$ and there is exactly one vertex v_1 at level 1. Let T_E be the subgraph of G induced by the vertices on even levels of T, and T_O be the subgraph of G induced by the vertices on odd levels of T.

Since T_E and T_O are complete graphs, every induced subgraph of T_E or T_O is Hamiltonian connected. Further, all vertices on level i are adjacent to all vertices on levels $i-2, i-3, \ldots, 1, 0$ in G. In each of the following cases, we will show $C_{n-1} \subseteq G$ and either $K_{1,n-2} \subseteq T$ or $C_n \subseteq G$.

Case 1. $\mathbf{h_{v_0}(T) \geq 4}$**.** Let v_i be a vertex on the tree at level i for $i = 2, 3, 4$.

Now, a Hamiltonian (v_0,v_4)-path from $T_E - v_2$ together with v_4v_1 together with a Hamiltonian (v_1,v_3)-path from T_O together with v_3v_0 forms $C_{n-1} \subseteq G$. Further, a Hamiltonian (v_0,v_4)-path from T_E together with v_4v_1 together with a Hamiltonian (v_1,v_3)-path from T_O together with v_3v_0 forms $C_n \subseteq G$.

Case 2. $\mathbf{h_{v_0}(T) = 3}$. Let $w_1, w_2, \ldots$ be the vertices on level 2 and $x_1, x_2, \ldots$ the vertices on level 3.

Case 2A. The number of vertices in level 2 is 1. Since $n \geq 5$, there are at least two vertices x_1 and x_2 on level 3. Then C_{n-1} is formed in G by using a Hamiltonian (x_1,x_2)-path in T_O together with $x_1v_0x_2$. In this case, we have $K_{1,n-2} \subseteq T$ induced by the vertices on levels 1, 2, and 3.

Case 2B. The number of vertices in level 2 is at least 2. Since $h_{v_0}(T) = 3$, there is at least one vertex x_1 on level 3. Assume without loss of generality that $w_1x_1 \in E(T)$.

If level 3 has only one vertex, then a Hamiltonian (v_0,w_2)-path from T_E together with $w_2x_1v_0$ forms $C_{n-1} \subseteq G$. Further, $K_{1,n-2} \subseteq T$ is induced by the vertices on levels 0, 1, and 2.

Assume level 3 has at least two vertices x_1, x_2 (and $w_1x_1 \in E(T)$). If $x_2w_1 \in E(T)$, then form $C_{n-1} \subseteq G$ by using a Hamiltonian (v_0,w_2)-path in T_E together with w_2x_2 together with a Hamiltonian (x_2,x_1)-path in $T_O - v_1$ together with x_1v_0. A Hamiltonian cycle is formed in G by using a Hamiltonian (v_0,w_2)-path in T_E together with w_2x_2 together with a Hamiltonian (x_2,x_1)-path in T_O together with x_1v_0.

Otherwise, without loss of generality assume $x_2w_2 \in E(T)$, and we can form $C_{n-1} \subseteq G$ by using a Hamiltonian (v_0,w_1)-path in T_E together with w_1x_2 together with a Hamiltonian (x_2,x_1)-path in $T_O - v_1$ together with x_1v_0. A Hamiltonian cycle is formed in G by using a Hamiltonian (v_0,w_1)-path in T_E together with w_1x_2 together with a Hamiltonian (x_2,x_1)-path in T_O together with x_1v_0.

Case 3. $\mathbf{h_{v_0}(T) = 2}$. Then $\bar{G} = K_{1,n-1}$ and $T_E = K_{n-1}$ which contains C_{n-1}. $\square$

We use the above result to get an upper bound on the size of a color class in a minimum $\neg C_j$-coloring of G. We complete our analysis of graphs whose complements are acyclic with the following theorem which determines $\chi(G, \neg C_j)$ when $j \geq 4$.

Theorem 4.2 *Let G be a graph of order n with acyclic complement. Let $m \geq 0$ be the maximum number of pairwise vertex disjoint copies of $K_{1,j-2}$ in $\bar{G}$. If $j \geq 4$, then $\chi(G, \neg C_j) = \max\left(\left\lceil \frac{n-m}{j-1} \right\rceil, \left\lceil \frac{n}{j} \right\rceil\right)$.*

Proof. Let A be a color class in a $\neg C_j$-coloring of G. Notice that $\overline{\langle A \rangle}$ is acyclic since $\bar{G}$ is acyclic. If $|A| \geq j+1$, then by Lemma 4.2, we get $C_j \subseteq \langle A \rangle$, a contradiction. Hence $|A| \leq j$, and it follows that $\chi(G, \neg C_j) \geq \left\lceil \frac{n}{j} \right\rceil$.

Suppose $|A| = j$. We will show $K_{1,j-2} \subseteq \overline{\langle A \rangle}$. Suppose $j = 4$. Every graph of order 4 with acyclic complement contains either C_4 or $K_{1,2}$. Since A is a color class, we must have $K_{1,2} \subseteq \overline{\langle A \rangle}$. If $j \geq 5$, then by Lemma 4.2, we have $K_{1,j-2} \subseteq \overline{\langle A \rangle}$.

Let a be the number of color classes of size j. Since color classes are vertex disjoint, we get $a \leq m$. Therefore, $\chi(G, \neg C_j) \geq \left\lceil \frac{n-aj}{j-1} \right\rceil + a = \left\lceil \frac{n-a}{j-1} \right\rceil \geq \left\lceil \frac{n-m}{j-1} \right\rceil$ and $\chi(G, \neg C_j) \geq \max\left(\left\lceil \frac{n-m}{j-1} \right\rceil, \left\lceil \frac{n}{j} \right\rceil \right)$.

To show the inequality in the other direction, we produce a minimum $\neg C_j$-coloring. Consider $\{U_1, U_2, \ldots, U_m\}$, where each U_i is a $K_{1,j-2}$ in $\bar{G}$ and the U_i's are pairwise vertex disjoint. Let $S = V(G) - \bigcup_{i=1}^{m} V(U_i)$. Now, $n = (j-1)m + s$. We consider two cases: $m \leq s$ and $0 \leq s < m$.

If $m \leq s$, then $n \geq mj$, which implies that $\left\lceil \frac{n-m}{j-1} \right\rceil \geq \left\lceil \frac{n}{j} \right\rceil$. For $i = 1, \ldots, m$, let $V(U_i)$ together with a vertex from S form a color class of size j (the subgraph induced by each such color class is C_j-free since it contains a vertex of degree at most one). We partition the remaining $s - m$ vertices into as many color classes of size $j - 1$ as possible. Clearly each such subgraph is C_j-free. Thus, we have $\chi(G, \neg C_j) \leq m + \left\lceil \frac{s-m}{j-1} \right\rceil = \left\lceil \frac{(j-1)m+s-m}{j-1} \right\rceil = \left\lceil \frac{n-m}{j-1} \right\rceil$.

If $0 \leq s < m$, then $n < mj$, which implies that $\left\lceil \frac{n}{j} \right\rceil \geq \left\lceil \frac{n-m}{j-1} \right\rceil$. Let $r = \left\lceil \frac{m-s}{j} \right\rceil$ and $U = \{U_1, U_2, \ldots, U_{m-r}\}$. Further, let $T = S \cup (\bigcup_{i=m-r+1}^{m} V(U_i))$. For $i = 1, \ldots, m-r$, let $V(U_i)$ together with a vertex from T form a color class of size j. Note that $m - r = m - \left\lceil \frac{m-s}{j} \right\rceil = \left\lfloor \frac{mj-m+s}{j} \right\rfloor = \left\lfloor \frac{n}{j} \right\rfloor$, and there are sufficient elements in T to form $\left\lfloor \frac{n}{j} \right\rfloor$ color classes since $|T| = s + r(j-1) = s + \left\lceil \frac{m-s}{j} \right\rceil (j-1) \geq s + \frac{m-s}{j}(j-1) = \frac{n}{j}$. If there are vertices remaining in T, form one more color class B containing those vertices. To show $\langle B \rangle$ is C_j-free, we will show that $|B| < j$.

Now,

$$\begin{aligned}
|B| &= |T| - \left\lfloor \frac{n}{j} \right\rfloor \\
&= s + r(j-1) - (m-r) \\
&= rj - (m-s) \\
&< (\frac{m-s}{j} + 1)j - (m-s) \\
&= j
\end{aligned}$$

If $B = \emptyset$, then there were no remaining vertices after forming $m - r$ color classes of size j. Therefore, j divides n and $\chi(G, \neg C_j) \leq \left\lfloor \frac{n}{j} \right\rfloor = \left\lceil \frac{n}{j} \right\rceil$. If $B \neq \emptyset$, then j does not divide n and we get $\chi(G, \neg C_j) \leq \left\lfloor \frac{n}{j} \right\rfloor + 1 = \left\lceil \frac{n}{j} \right\rceil$. In either case, we have produced a $\neg C_j$-coloring with $\left\lceil \frac{n}{j} \right\rceil$ colors. □

Now that we have determined $\chi(G, \neg C_j)$ for graphs with acyclic complements, the next section gives $\chi(G, \neg P_j)$ for this same class of graphs.

4.2 Determining $\chi(\mathbf{G}, \neg \mathbf{P_j})$ when $\bar{\mathbf{G}}$ is acyclic

The following lemma determines the longest path in a graph whose complement is acyclic. This lemma will be used in the main theorem of this section which determines $\chi(G, \neg P_j)$ when $\bar{G}$ is acyclic.

Lemma 4.3 *If G is a graph of order $n \geq 2$ with acyclic complement, then $P_{n-1} \subseteq G$. Furthermore, if $\bar{G} \neq K_{1,n-1}$, then G has a Hamiltonian path.*

Proof. If $n = 2$, the result is trivial. If $n = 3$, then G is missing at most two edges and $P_2 \subseteq G$. Further, if $\bar{G} \neq K_{1,2}$, then $G = P_3$ or $G = K_3$ and both contain P_3. If $n = 4$, then $G \in \{K_4, K_4 - e, K_4 - E(2K_2), K_4 - E(P_3), P_4, K_4 - E(K_{1,3})\}$

where e is an edge. All graphs contain P_4 except $K_4 - E(K_{1,3})$, which contains P_3. So, assume $n \geq 5$. By Lemma 4.2, the graph G contains C_{n-1} as a subgraph and therefore $P_{n-1} \subseteq G$. If $\bar{G} \neq K_{1,n-1}$, then $\delta(G) \geq 1$ and the vertex in G not on C_{n-1} can be added to C_{n-1} to form $P_n \subseteq G$. □

The following theorem determines $\chi(G, \neg P_j)$ for graphs whose complements are acyclic.

Theorem 4.3 *Let G be a graph of order n with acyclic complement. Let $m \geq 0$ be the maximum number of pairwise vertex disjoint copies of $K_{1,j-1}$ in $\bar{G}$. If $j \geq 2$, then $\chi(G, \neg P_j) = \left\lceil \frac{n-m}{j-1} \right\rceil$.*

Proof. Let A be a color class in a $\neg P_j$-coloring of G. Notice that $\overline{\langle A \rangle}$ is acyclic since $\bar{G}$ is acyclic. If $|A| > j$, then by Lemma 4.3, we get $P_j \subseteq \langle A \rangle$, which contradicts the assumption that A is a color class. Therefore, $|A| \leq j$.

Suppose $|A| = j$. Now, $\overline{\langle A \rangle}$ is acyclic and is P_j-free. Therefore, by Lemma 4.3, we get $K_{1,j-1} = \overline{\langle A \rangle}$. Let a be the number of color classes of size j. Since color classes are vertex disjoint, we get $a \leq m$. Therefore, $\chi(G, \neg P_j) \geq \left\lceil \frac{n-aj}{j-1} \right\rceil + a = \left\lceil \frac{n-a}{j-1} \right\rceil \geq \left\lceil \frac{n-m}{j-1} \right\rceil$.

To show the inequality in the other direction, we produce a minimum $\neg P_j$-coloring. Consider $\{U_1, U_2, \ldots, U_m\}$, where each U_i is a $K_{1,j-1}$ in $\bar{G}$ and the U_i's are pairwise vertex disjoint. For each $i = 1, \ldots, m$, form a color class of size j using $V(U_i)$ (the subgraph induced by each such color class is P_j-free since it contains a vertex of degree zero). With the remaining vertices, form as many color classes of size $j-1$ as possible. With this coloring, we get $\chi(G, \neg P_j) \leq \left\lceil \frac{n-mj}{j-1} \right\rceil + m = \left\lceil \frac{n-m}{j-1} \right\rceil$. Therefore, $\chi(G, \neg P_j) = \left\lceil \frac{n-m}{j-1} \right\rceil$. □

The following corollary will be used several times in upcoming chapters and is stated here for reference.

Corollary 4.1 *If G is a graph with $\langle E(\bar{G})\rangle = K_{1,m}$ and $j \geq 2$, then*

$$\chi(G, \neg P_j) = \begin{cases} \left\lceil \frac{n}{j-1} \right\rceil & \text{if } 0 \leq m < j-1, \\ \left\lceil \frac{n-1}{j-1} \right\rceil & \text{if } j-1 \leq m \leq n-1. \end{cases}$$

Proof. If $m < j-1$, then $\bar{G}$ contains no copy of $K_{1,j-1}$ and by Theorem 4.3, $\chi(G, \neg P_j) = \left\lceil \frac{n}{j-1} \right\rceil$. If $m \geq j-1$, then $\bar{G}$ contains exactly one $K_{1,j-1}$ and by Theorem 4.3, $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$. □

Now, we can state the main theorem of this chapter.

4.3 The difference between the $\neg C_j$- and $\neg P_j$-chromatic numbers of a graph

Theorem 4.4 *If $a \geq 0$, $j \geq 4$ and $n \geq a(j-1)j$, then there exists a connected graph G of order n such that $\chi(G, \neg P_j) - \chi(G, \neg C_j) \geq a$.*

Proof. This is a proof by construction. Let $\bar{G}$ be the disjoint union of $I_{n-a(j-1)^2}$ and $a(j-1)$ pairwise vertex disjoint copies of $K_{1,j-2}$. See Figure 4.1 for the construction of $\bar{G}$ when $j = 5$, $a = 1$ and $n = 24$. In this case, $\bar{G}$ consists of I_8 and four copies of $K_{1,3}$. Further, $\chi(G, \neg C_5) = 5$ and $\chi(G, \neg P_5) = 6$.

Observe that $\bar{G}$ contains $a(j-1)$ pairwise vertex disjoint copies of $K_{1,j-2}$ and no copy of $K_{1,j-1}$. By Theorem 4.3, $\chi(G, \neg P_j) = \left\lceil \frac{n}{j-1} \right\rceil$. Since $n \geq a(j-1)j$, Theorem 4.2 implies that $\chi(G, \neg C_j) = \left\lceil \frac{n-a(j-1)}{j-1} \right\rceil$. Therefore, $\chi(G, \neg P_j) - \chi(G, \neg C_j) = \left\lceil \frac{n}{j-1} \right\rceil - \left\lceil \frac{n-a(j-1)}{j-1} \right\rceil = \left\lceil \frac{n}{j-1} \right\rceil - \left\lceil \frac{n}{j-1} \right\rceil + a = a$. □

Since we have shown these two conditional chromatic numbers may be different, we will address each separately in the following two chapters.

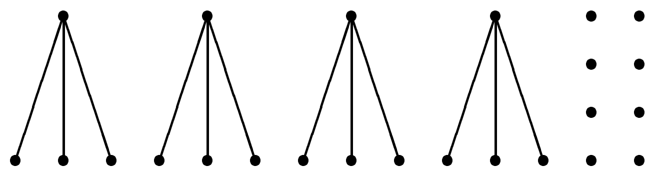

Figure 4.1. The construction of $\bar{G}$ for $j = 5$, $a = 1$ and $n = 24$

Some possible directions for further research include answering the following questions. What happens if we remove the restriction that the complement is acyclic and examine all graphs missing $n - 1$ edges or examine graphs whose complements are bipartite? Will the difference increase and, if so, by how much? Also, are there families of graphs where $\chi(G, \neg P_j) - \chi(G, \neg C_j)$ is $O(n^k)$ for $k \geq 1$?

The following questions also remain open. For the set of graphs of a fixed order n, how much can these two conditional chromatic numbers differ if we remove half of the edges from the complete graph? And, in general, given the size of G, what are upper and lower bounds for $\chi(G, \neg P_j) - \chi(G, \neg C_j)$? Which graphs attain these bounds?

5 Determining $\chi(\mathbf{G}, \neg \mathbf{C_j})$

Next we address the problem: given a graph of order n with e edges, what are the bounds on the conditional chromatic number? In 1995, Dargen and Fraughnaugh published the following theorem.

Theorem 5.1 *(Dargen and Fraughnaugh, [21]) If* $e(G) \geq \binom{n}{2} - (j-2)$*, then*

$$\chi(G, \neg C_j) = \begin{cases} \left\lceil \frac{n-1}{j-1} \right\rceil & \text{if } \bar{G} = K_{1,j-2} \\ & \text{or } \bar{G} = K_3 \text{ and } j = 5 \\ \left\lceil \frac{n}{j-1} \right\rceil & \text{otherwise.} \end{cases}$$

We can deduce from this theorem that we need to remove at least $j-2$ edges from a complete graph to obtain a graph whose $\neg C_j$-conditional chromatic number is $\left\lceil \frac{n-1}{j-1} \right\rceil$. A natural question one may ask is how many edges do we need to remove so that $\chi(G, \neg C_j) = \left\lceil \frac{n-2}{j-1} \right\rceil$? We will show that this number can be obtained by removing only two edges from a complete graph when $j = 3$.

In this chapter, we will determine which graphs are Hamiltonian given that the size of their complement is at most $n-1$. Using this information, the final theorem of this chapter will expand the knowledge of $\chi(G, \neg C_j)$ to the class of graphs missing exactly $j-1$ edges.

5.1 Preliminaries

To prove the final theorem, we will first address some special cases. We will soon see that the special cases for the final theorem are graphs missing a star of a particular order and graphs missing pairwise vertex disjoint complete subgraphs. First, we will address graphs missing a star of a particular order.

Theorem 5.2 *Let G be a graph of order n. If $j \geq 3$ and $e(G) \geq \binom{n}{2} - (2j-6)$ and $K_{1,j-2} \subseteq \bar{G}$, then $\chi(G, \neg C_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$.*

Proof. Let X be a set of vertices which induce $K_{1,j-2} \subseteq \bar{G}$. Since $K_{1,j-2} \subseteq \bar{G}$ and $e(\bar{G}) \leq 2j-6$, there is exactly one vertex v_0 of degree at least $j-2$ in $\bar{G}$. Therefore, $v_0 \in X$. Let A be a color class in a $\neg C_j$-coloring of G.

Suppose $v_0 \notin A$. If $|A| \geq j$, then $\langle A \rangle$ would be missing at most $j-4$ edges and by Theorem 2.10, we get $C_j \subseteq \langle A \rangle$. Therefore, for A to be a color class, we must have $|A| \leq j-1$. Suppose $v_0 \in A$, and consider $A - v_0$. If $|A - v_0| \geq j$, then by the same argument, we get $C_j \subseteq \langle A - v_0 \rangle$. Therefore, $|A - v_0| \leq j-1$, which implies $|A| \leq j$. Thus, there can be at most one color class of size j (one containing v_0), and $\chi(G, \neg C_j) \geq \left\lceil \frac{n-j}{j-1} \right\rceil + 1 = \left\lceil \frac{n-1}{j-1} \right\rceil$.

To show the inequality in the other direction, form one color class B of size j by using the vertices in X together with one vertex from $G - X$ and form as many other color classes as possible of size $j-1$ with the remaining vertices in $G - X$. Now $\langle B \rangle$ is C_j-free since v_0 is adjacent to at most one other vertex in $\langle B \rangle$. Therefore, $\chi(G, \neg C_j) \leq \left\lceil \frac{n-j}{j-1} \right\rceil + 1 = \left\lceil \frac{n-1}{j-1} \right\rceil$. □

Next, we would like to determine $\chi(G, \neg C_j)$ when G is missing a set of mutually disjoint complete graphs of a specific order. To accomplish this, we use the following two lemmas.

Lemma 5.1 *Let G be a graph of order $n \geq 3$ such that $G = \bigvee_{i=1}^{p} S_i$ where $S_i = I_{r_i}$ ($r_i \geq 1, p \geq 1$) for each i. Then either G is Hamiltonian or $max_i\{r_i\} \geq \left\lceil \frac{n+1}{2} \right\rceil$.*

Proof. Let $v \in V(G)$. Then $v \in S_i$ for some i and $d(v) = (n-1)-(r_i-1) = n-r_i$. Therefore, since each vertex in G is in some S_i, we get $\delta(G) = n - max_i\{r_i\}$. If

$max_i\{r_i\} < \left\lceil \frac{n+1}{2} \right\rceil$, then $\delta(G) > n - \left\lceil \frac{n+1}{2} \right\rceil = \left\lfloor \frac{n-1}{2} \right\rfloor$. Thus, $\delta(G) \geq n/2$ and by Dirac's Theorem (Theorem 2.7), G is Hamiltonian. □

Lemma 5.2 *Let G be a graph of order $n \geq 4$ such that $G = \bigvee_{i=1}^{p} S_i$ where $S_i = I_{r_i}$ $(1 \leq r_i \leq \left\lceil \frac{n}{2} \right\rceil, p \geq 1)$ for each i. Then either G contains an $(n-1)$-cycle or there exist r_1 and r_2 such that $r_1 = r_2 = \left\lceil \frac{n}{2} \right\rceil$.*

Proof. Assume $r_1 \leq \left\lceil \frac{n}{2} \right\rceil$ and $r_i < \left\lceil \frac{n}{2} \right\rceil$ when $i \neq 1$. Let $v \in S_1$ and $G' = G - v$. Then $G' = I_{r_1 - 1} \vee (\bigvee_{i=2}^{p} S_i)$ and $max_{i \geq 2}\{r_i, r_1 - 1\} < \left\lceil \frac{n}{2} \right\rceil$. By Lemma 5.1, G' is Hamiltonian, and therefore G contains an $(n-1)$-cycle. □

The following theorem determines $\chi(G, \neg C_j)$ when G is missing a set of mutually disjoint complete graphs of a specific order.

Theorem 5.3 *Let G be a graph of order n. If $j \geq 3$ and $\langle E(\bar{G}) \rangle = mK_{\lceil \frac{j+1}{2} \rceil}$, then*

$$\chi(G, \neg C_j) = \begin{cases} \left\lceil \frac{n-m}{j-1} \right\rceil & \text{if } j \text{ is odd} \\ \max\left(\left\lceil \frac{n-m}{j-1} \right\rceil, \left\lceil \frac{n}{j} \right\rceil \right) & \text{if } j \text{ is even} \end{cases}$$

Proof. Observe that $\langle E(\bar{G}) \rangle = mK_{\lceil \frac{j+1}{2} \rceil}$ if and only if $G = \bigvee_{i=1}^{m+n-\lceil \frac{j+1}{2} \rceil m} S_i$ where $S_i = I_{\lceil \frac{j+1}{2} \rceil}$ for $i = 1, 2, \ldots, m$ and $S_i = I_1$ for $i = m+1, m+2, \ldots, n - \left\lceil \frac{j+1}{2} \right\rceil m$. Assume j is odd. Then $\left\lceil \frac{j+1}{2} \right\rceil = \frac{j+1}{2}$.

Let A be a color class in a $\neg C_j$-coloring of G. Suppose that $|A| > j+1$. Let $B \subseteq A$ such that $|B| = j+2$. Then $\langle B \rangle = \bigvee_{i=1}^{k} I_{q_i}$ where $1 \leq q_1 \leq (j+1)/2$, $1 \leq q_2 \leq (j+1)/2$, and $1 \leq q_i \leq (j-1)/2$ for $3 \leq i \leq k$ since $|B| = j+2$ implies that $\langle B \rangle$ can contain at most two copies of $I_{\frac{j+1}{2}}$. By removing one vertex from each of I_{q_1} and I_{q_2}, form $C \subseteq B$ such that $|C| = j$. Now, $\langle C \rangle = I_{q_1 - 1} \vee I_{q_2 - 1} \vee \left(\bigvee_{i=3}^{k} I_{q_i} \right)$, and $\max_i\{q_i\} \leq \frac{j-1}{2}$. By Lemma 5.1, $\langle C \rangle$ is Hamiltonian. This contradicts the assumption that $\langle A \rangle$ is C_j-free. So $|A| \leq j+1$.

If $|A| = j+1$, then by Lemma 5.2, either $2I_{\frac{j+1}{2}} \subseteq \langle A \rangle$ or $\langle A \rangle$ contains a j-cycle. Because $\langle A \rangle$ cannot contain C_j, we must have $2I_{\frac{j+1}{2}} \subseteq \langle A \rangle$. Now, if $|A| = j$, then by Lemma 5.1, we have $I_{\frac{j+1}{2}} \subseteq \langle A \rangle$. Let a be the number of color classes of size $j+1$ and b be the number of color classes of size j in a $\neg C_j$-coloring of G. Since color classes are vertex disjoint, we get $2a+b \leq m$. Now, $\chi(G, \neg C_j) \geq \left\lceil \frac{n-(j+1)a-jb}{j-1} \right\rceil + a + b = \left\lceil \frac{n-2a-b}{j-1} \right\rceil \geq \left\lceil \frac{n-m}{j-1} \right\rceil$.

To show the inequality in the other direction, we split the proof into three cases. Recall that $G = \bigvee_{i=1}^{m+n-\left(\frac{j+1}{2}\right)m} S_i$ where $S_i = I_{\frac{j+1}{2}}$ for $i = 1, 2, \ldots, m$ and $S_i = I_1$ for $i = m+1, m+2, \ldots, n - \left(\frac{j+1}{2}\right) m$.

Case 1. m is even. Color G as follows: For $1 \leq l \leq m/2$, form a color class of size $j+1$ by using all the vertices from S_{2l-1} and S_{2l}. The subgraph induced by each of these color classes is isomorphic to $M = I_{\frac{j+1}{2}} \vee I_{\frac{j+1}{2}}$, which is bipartite. Since bipartite graphs only contain even cycles, M is C_j-free. With the remaining vertices, form as many other color classes of size $j-1$ as possible. With this coloring, we get $\chi(G, \neg C_j) \leq \left\lceil \frac{n-(j+1)\frac{m}{2}}{j-1} \right\rceil + \frac{m}{2} = \left\lceil \frac{n-m}{j-1} \right\rceil$.

Case 2. m is odd and $\mathbf{n \geq m\left(\frac{j+1}{2}\right) + \frac{j-1}{2}}$. Color G as follows: For $1 \leq l \leq \frac{m-1}{2}$, form a color class of size $j+1$ by using all the vertices from S_{2l-1} and S_{2l}. Again, since the subgraph induced by each of these color classes is bipartite, it is C_j-free. Observe that $n \geq m\left(\frac{j+1}{2}\right) + \frac{j-1}{2}$ guarantees that there are at least j vertices not yet assigned to color classes. Form one color class C of size j using all the vertices from S_m together with $\frac{j-1}{2}$ vertices from $V(G) - \bigcup_{i=1}^{m} V(S_i)$. Now, $\langle C \rangle = I_{\frac{j+1}{2}} \vee K_{\frac{j-1}{2}}$ and the largest cycle we can form is C_{j-1} obtained by alternately traversing between the vertices of $K_{\frac{j-1}{2}}$ and $I_{\frac{j+1}{2}}$ until we use all the vertices in $K_{\frac{j-1}{2}}$. Therefore, $\langle C \rangle$ is C_j-free. With the remaining vertices, form as many other color classes of size $j-1$ as possible. Then $\chi(G, \neg C_j) \leq \left\lceil \frac{n - \frac{m-1}{2}(j+1) - j}{j-1} \right\rceil + \frac{m-1}{2} + 1 = \left\lceil \frac{n-m}{j-1} \right\rceil$.

Case 3. m is odd and $\mathbf{n < m \left(\frac{j+1}{2}\right) + \frac{j-1}{2}}$. This means that $0 \leq n - m\left(\frac{j+1}{2}\right) < \frac{j-1}{2}$. We will show that $n-m$ is not divisible by $j-1$. Suppose that $n-m = z(j-1)$ for some nonnegative integer z. Then

$$\begin{array}{lrcl} & 0 & \leq & n - m\left(\dfrac{j+1}{2}\right) < \dfrac{j-1}{2} \\ \text{iff} & 0 & \leq & z(j-1) + m - m\left(\dfrac{j+1}{2}\right) < \dfrac{j-1}{2} \\ \text{iff} & 0 & \leq & 2z(j-1) + 2m - mj - m < j-1 \\ \text{iff} & 0 & \leq & 2z(j-1) - m(j-1) < j-1 \\ \text{iff} & 0 & \leq & (2z-m)(j-1) < j-1 \\ \text{iff} & 0 & \leq & 2z - m < 1 \\ \text{iff} & m & = & 2z. \end{array}$$

However, this contradicts the assumption that m is odd. Therefore, $n-m$ is not divisible by $j-1$. This implies that $\left\lceil \frac{n-m+1}{j-1} \right\rceil = \left\lceil \frac{n-m}{j-1} \right\rceil$.

Now back to the proof. Form a new graph F by taking G and adding back in all the missing edges in S_1. Now F is missing $m-1$ pairwise vertex disjoint copies of $K_{\frac{j+1}{2}}$ and we are back in **Case 1**. So, $\chi(F, \neg C_j) \leq \left\lceil \frac{n-(m-1)}{j-1} \right\rceil = \left\lceil \frac{n-m+1}{j-1} \right\rceil = \left\lceil \frac{n-m}{j-1} \right\rceil$, and by Theorem 2.4, $\chi(G, \neg C_j) \leq \chi(F, \neg C_j) = \left\lceil \frac{n-m}{j-1} \right\rceil$. This completes the proof when j is odd.

Next, let's assume j is even. In this case, $\left\lceil \frac{j+1}{2} \right\rceil = \frac{j+2}{2}$ and $G = \bigvee_{i=1}^{m+n-\left(\frac{j+2}{2}\right)m} S_i$ where $S_i = I_{\frac{j+2}{2}}$ for $i = 1, 2, \ldots, m$ and $S_i = I_1$ for $i = m+1, m+2, \ldots, n - \left(\frac{j+2}{2}\right)m$.

Let A be a color class in a $\neg C_j$-coloring of G. Suppose that $|A| > j$. Let $B \subseteq A$ such that $|B| = j + 1$. Then $\langle B \rangle = \bigvee_{i=1}^{k} I_{q_i}$ where $1 \leq q_1 \leq (j+2)/2$, and $1 \leq q_i \leq j/2$ for $2 \leq i \leq k$ since $|B| = j+1$ implies that $\langle B \rangle$ can contain at most one copy of $I_{\frac{j+2}{2}}$. By removing one vertex from I_{q_1}, form $C \subseteq B$ such that $|C| = j$. Now, $\langle C \rangle = I_{q_1 - 1} \vee \left(\bigvee_{i=2}^{k} I_{q_i} \right)$, and $\max_i\{q_i\} \leq \frac{j}{2}$. By Lemma 5.1, $\langle C \rangle$ is Hamiltonian. This contradicts the assumption that $\langle A \rangle$ is C_j-free. So $|A| \leq j$ and $\chi(G, \neg C_j) \geq \left\lceil \frac{n}{j} \right\rceil$.

If $|A| = j$, then since $\langle A \rangle$ is C_j-free, by Lemma 5.1 we have $I_{\frac{j+2}{2}} \subseteq \langle A \rangle$. Let a be the number of color classes of size j in a $\neg C_j$-coloring of G. Since color classes are vertex disjoint, we get $a \leq m$. Therefore, $\chi(G, \neg C_j) \geq \left\lceil \frac{n - jm}{j-1} \right\rceil + m = \left\lceil \frac{n-m}{j-1} \right\rceil$. Hence, $\chi(G, \neg C_j) \geq \max\left(\left\lceil \frac{n}{j} \right\rceil, \left\lceil \frac{n-m}{j-1} \right\rceil \right)$.

To show the inequality in the other direction, we produce a minimum $\neg C_j$-coloring. Let $U = V(G) - \bigcup_{i=1}^{m} V(S_i)$. Now, $n = (\frac{j+2}{2})m + s$. We consider two cases: $\frac{j-2}{2}m \leq s$ and $0 \leq s < \frac{j-2}{2}m$.

If $\frac{j-2}{2}m \leq s$, then $n \geq mj$ and $\left\lceil \frac{n-m}{j-1} \right\rceil \geq \left\lceil \frac{n}{j} \right\rceil$. For $1 \leq i \leq m$, form a color class of size j by using all the vertices from S_i together with $\frac{j-2}{2}$ vertices from U not yet assigned to a color class. Since $\frac{j-2}{2}m \leq s$, U contains a sufficient number of vertices to form these m color classes. Now, all the subgraphs induced by these color classes are isomorphic to $C = I_{\frac{j+2}{2}} \vee K_{\frac{j-2}{2}}$ and the largest cycle we can form is C_{j-2} obtained by alternately traversing between the vertices of $K_{\frac{j-2}{2}}$ and $I_{\frac{j+2}{2}}$ until we use all the vertices in $K_{\frac{j-2}{2}}$. Therefore, C is C_j-free. We partition the remaining $s - \frac{j-2}{2}m$ vertices into as many color classes of size $j-1$ as possible. Thus, we have $\chi(G, \neg C_j) \leq m + \left\lceil \frac{s - \left(\frac{j-2}{2}\right)m}{j-1} \right\rceil = \left\lceil \frac{mj - \left(\frac{j-2}{2}\right)m + s - m}{j-1} \right\rceil = \left\lceil \frac{\left(\frac{j+2}{2}\right)m + s - m}{j-1} \right\rceil = \left\lceil \frac{n-m}{j-1} \right\rceil$.

If $0 \le s < \frac{j-2}{2}m$, then $n < mj$ and $\left\lceil \frac{n}{j} \right\rceil \ge \left\lceil \frac{n-m}{j-1} \right\rceil$. Let $r = \left\lceil \frac{\left(\frac{j-2}{2}\right)m-s}{j} \right\rceil$ and $T = U \cup (\bigcup_{i=m-r+1}^{m} V(S_i))$. For $1 \le i \le m-r$, form a color class of size j by using all the vertices from S_i together with $\frac{j-2}{2}$ vertices from T not yet assigned to a color class. Again these subgraphs are C_j-free since their longest cycle is C_{j-2}. Note that $m - r = m - \left\lceil \frac{\left(\frac{j-2}{2}\right)m-s}{j} \right\rceil = \left\lfloor \frac{mj-\left(\frac{j-2}{2}\right)m+s}{j} \right\rfloor = \left\lfloor \frac{\left(\frac{j+2}{2}\right)m+s}{j} \right\rfloor = \left\lfloor \frac{n}{j} \right\rfloor$, and there are sufficient elements in T to form $\left\lfloor \frac{n}{j} \right\rfloor$ color classes since $|T| = s+r\left(\frac{j+2}{2}\right) = s + \left\lceil \frac{\left(\frac{j-2}{2}\right)m-s}{j} \right\rceil \left(\frac{j+2}{2}\right) \ge s + \frac{\left(\frac{j-2}{2}\right)m-s}{j}\left(\frac{j+2}{2}\right) = \left(\frac{j-2}{2}\right)\left(\frac{n}{j}\right) \ge \left(\frac{j-2}{2}\right)\left\lfloor \frac{n}{j} \right\rfloor$. Again, all the subgraphs induced by these color classes are isomorphic to $C = I_{\frac{j+2}{2}} \vee K_{\frac{j-2}{2}}$, and we have already shown that C is C_j-free.

If there are vertices remaining in T, form one more color class B containing those vertices. To show $\langle B \rangle$ is C_j-free, we will show that $|B| < j$. Now,

$$\begin{aligned}
|B| &= |T| - \left\lfloor \frac{n}{j} \right\rfloor \left(\frac{j-2}{2}\right) \\
&= s + r\left(\frac{j+2}{2}\right) - (m-r)\left(\frac{j-2}{2}\right) \\
&= s + rj - m\left(\frac{j-2}{2}\right) \\
&< s + \left(\frac{\left(\frac{j-2}{2}\right)m - s}{j} + 1\right) j - m\left(\frac{j-2}{2}\right) \\
&= j.
\end{aligned}$$

If $B = \emptyset$, then there were no remaining vertices after forming $m - r$ color classes of size j. Therefore, j divides n and $\chi(G, \neg C_j) \le \left\lfloor \frac{n}{j} \right\rfloor = \left\lceil \frac{n}{j} \right\rceil$. If $B \ne \emptyset$, then j does not divide n and we get $\chi(G, \neg C_j) \le \left\lfloor \frac{n}{j} \right\rfloor + 1 = \left\lceil \frac{n}{j} \right\rceil$. In either case, we have produced a $\neg C_j$-coloring with $\left\lceil \frac{n}{j} \right\rceil$ colors. □

5.2 Determining $\chi(\mathbf{G}, \neg \mathbf{C_j})$ when $\mathbf{e(G)} \geq \binom{\mathbf{n}}{\mathbf{2}} - (\mathbf{j} - \mathbf{1})$

To determine the $\neg C_j$-chromatic number when $j \geq 8$ for graphs given the size of the complement is at most $j-1$, we first determine which graphs of order $n \geq 8$ are Hamiltonian given the size of their complement is at most $n-1$.

Theorem 5.4 *Let G be a graph of order $n \geq 8$ with $e(\bar{G}) \leq n-1$. Then either $K_{1,n-2} \subseteq \bar{G}$ or G is Hamiltonian.*

Proof. Assume $K_{1,n-2} \not\subseteq \bar{G}$. If $G = K_n$, then G is obviously Hamiltonian. So assume G is not complete and let u and v be a pair of nonadjacent vertices. Since $K_{1,n-2} \not\subseteq \bar{G}$, we must have $d(u) \geq 2$ and $d(v) \geq 2$. Let $H = G - \{u, v\}$. We break the proof into cases based on the number of edges in $\bar{G}$ incident to u or v.

Case 1. Suppose that the number of edges in $\bar{\mathbf{G}}$ incident to u or v is $\mathbf{n-1}$. Since $e(\bar{G}) = n-1$, H is complete.

If $|N(u) \cap N_(v)| = 0$, then let $x, y \in N(u)$ and $a, b \in N(v)$ be distinct vertices. These vertices exist since $d(u) \geq 2$ and $d(v) \geq 2$. Now $xuybva$ together with a Hamiltonian (a,x)-path from $H - \{y, b\}$ forms C_n. So assume $|N(u) \cap N(v)| \geq 1$.

If $|N(u) \cap N(v)| = 1$, then let w, x, and y be distinct vertices in H such that $w \in N(u) \cap N(v)$, $x \in N(u)$ and $y \in N(v)$. Then $xuwvy$ together with an (y,x)-path from $H - \{w\}$ forms C_n.

So assume $|N(u) \cap N(v)| \geq 2$. Since there are $n-2$ edges missing from $\{u, v\}$ into H out of a possible $2(n-2)$, there must be $n-2$ edges from $\{u, v\}$ into H. Since $n \geq 8$, there are at least six edges from $\{u, v\}$ into H. Therefore, we can assume $d(u)$ or $d(v)$ is at least 3. Without loss of generality, assume $d(u) \geq 3$. Let $w, x, y \in N(u)$, and $w, x \in N(v)$. Then $xvwuy$ together with a Hamiltonian

(y,x)-path from $H - \{w\}$ forms C_n. Therefore, we can assume that every such pair of vertices can be incident to no more than $n-2$ edges in $\bar{G}$.

Case 2. Suppose that the number of edges in $\bar{\mathbf{G}}$ incident to u or v is exactly $\mathbf{n-2}$. Since there are $n-3$ missing edges from $\{u,v\}$ into H, H is missing at most one edge, and there are $2(n-2)-(n-3) = n-1$ edges from $\{u,v\}$ into H.

Suppose $|N(u) \cap N(v)| = 0$. Then in order to have $n-1$ edges from $\{u,v\}$ into H and $|N(u) \cap N(v)| = 0$, the subgraph H needs to have at least $n-1$ vertices. But H only contains $n-2$ vertices, and we reach a contradiction.

Therefore, we can assume $|N(u) \cap N(v)| \geq 1$. Let $w \in N(u) \cap N(v)$. Consider $J = G - \{u,v,w\}$. Then J is a graph of order $n-3$ missing at most one edge. Since $n \geq 8$, we have $e(J) \geq \binom{n-3}{2} - 1 \geq \binom{n-3}{2} - (n-7)$ and J is Hamiltonian connected by Theorem 2.9.

If $N(u) \neq N(v)$, then since $d(u) \geq 2$, $d(v) \geq 2$,and $n \geq 8$, there exist distinct vertices $x, y \in V(J)$ such that $x \in N(u)$ and $y \in N(v)$. Now $xuwvy$ together with a Hamiltonian (x,y)-path in J forms C_n.

So we can assume that $N(u) = N(v)$. Recall that there are $n-1$ edges from $\{u,v\}$ into H. Therefore, $d(u) = d(v) = \frac{n-1}{2}$. The integer n must be odd in order for $\frac{n-1}{2}$ to be an integer. Therefore, $n \geq 9$ and $d(u) = d(v) \geq 4$. This implies that there exist distinct vertices $x, y \in V(J)$ such that $x, y \in N(u) \cap N(v)$, $x \neq w$, and $y \neq w$. Then $xuwvy$ together with a Hamiltonian (x,y)-path in J forms C_n. Thus, we may assume that for every nonadjacent pair of vertices u and v, the number of edges in $\bar{G}$ incident to u or v is at most $n-3$.

Case 3. Suppose that the number of edges in $\bar{\mathbf{G}}$ incident to every nonadjacent pair of of vertices u, v is at most n − 3.

There are at most $n-4$ edges missing from $\{u, v\}$ into H. Therefore there are at least $2(n-2)-(n-4)=n$ edges from $\{u, v\}$ into H. So, $d(u)+d(v) \geq n$ and since u and v are arbitrary nonadjacent vertices, by Ore's Theorem (Theorem 2.6) G is Hamiltonian. □

Now, armed with Theorems 4.1, 4.2, 5.2, 5.3, and 5.4, we can proceed with determining $\chi(G, \neg C_j)$ when $e(G) \geq \binom{n}{2}-(j-1)$. We begin by determining $\chi(G, \neg C_j)$ when $j \geq 8$.

Theorem 5.5 *If $j \geq 8$ and $e(G) \geq \binom{n}{2}-(j-1)$, then*

$$\chi(G, \neg C_j)=\begin{cases}\left\lceil\frac{n-1}{j-1}\right\rceil & \text{if } K_{1,j-2} \subseteq \bar{G} \\ \left\lceil\frac{n}{j-1}\right\rceil & \text{otherwise.}\end{cases}$$

Proof. Suppose $K_{1,j-2} \subseteq \bar{G}$. We have $j-1 \leq 2j-6$ since $j \geq 8$. Therefore, $e(G)=\binom{n}{2}-(j-1) \geq \binom{n}{2}-(2j-6)$, and by Theorem 5.2, we have $\chi(G, \neg C_j)= \left\lceil\frac{n-1}{j-1}\right\rceil$.

Assume $K_{1,j-2} \nsubseteq \bar{G}$. Let A be a color class in a $\neg C_j$-coloring of G. We will show that $|A|<j$, which implies that $\chi(G, \neg C_j) \geq \left\lceil\frac{n}{j-1}\right\rceil$ and equality follows from Corollary 2.2. Suppose that $|A| \geq j$. Let $B \subseteq A$ such that $|B|=j$. Now $K_{1,j-2} \nsubseteq \bar{G}$ implies that $K_{1,j-2} \nsubseteq \overline{\langle B\rangle}$. By Theorem 5.4, $\langle B\rangle$ is Hamiltonian, a contradiction. □

We wish to extend the above result for all $j \geq 3$. To accomplish this, we need to determine which graphs of order n are Hamiltonian given the size of their complement is at most $n-1$.

Theorem 5.6 *Let G be a graph of order $n \geq 3$ with $e(\bar{G}) \leq n-1$. Then G is Hamiltonian except in the following cases:*

$$\begin{array}{lr} K_{1,n-2} \subseteq \bar{G} & (i) \\ K_3 \subseteq \bar{G} \text{ and } n=5 & (ii) \\ \langle E(\bar{G})\rangle = C_4 \text{ and } n=5 & (iii) \\ \langle E(\bar{G})\rangle = K_4 - e \text{ where } e \text{ is an edge and } n=6 & (iv) \\ \langle E(\bar{G})\rangle = K_4 \text{ and } n=7 & (v) \end{array}$$

Proof. Let G be a graph of order $n \geq 3$ with $e(\bar{G}) \leq n-1$. If $e(G) \geq \binom{n}{2}-(n-3)$, then G is Hamiltonian by Theorem 2.10. So assume $n-2 \leq e(\bar{G}) \leq n-1$. If $\delta(G) \leq 1$ (or equivalently $K_{1,n-2} \subseteq \bar{G}$), then clearly G is not Hamiltonian. This is exceptional case (i). So assume $\delta(G) \geq 2$. If $n=3$, then $\delta(G) \geq 2$ implies that $G = C_3$, which is Hamiltonian. If $n=4$, then $\delta(G) \geq 2$ implies that $G \in \{K_4, K_4-e, C_4\}$, all of which are Hamiltonian. If $n \geq 8$, then by Theorem 5.4, G is Hamiltonian . So, assume $5 \leq n \leq 7$.

Now $n-2 \leq e(\bar{G}) \leq n-1$ and $5 \leq n \leq 7$ imply G is not complete. So let u and v be a pair of nonadjacent vertices in G. Let $H = G - \{u,v\}$. Next, we break the proof into cases based on the number of edges in $\bar{G}$ incident to u or v.

Case 1. Suppose that the number of edges in $\bar{\mathbf{G}}$ incident to u or v is $\mathbf{n-1}$. Since the total number of edges in $\bar{G}$ is at most $n-1$, H is complete.

If $n=5$, then having four edges incident to u or v in $\bar{G}$ forces $\delta(G) \leq 1$, which contradicts the assumption that $\delta(G) \geq 2$.

If $n=6$, then there are four edges from $\{u,v\}$ to H in $\bar{G}$. Moreover, $\delta(G) \geq 2$ implies that $\Delta(\bar{G}) \leq 3$ and Therefore $\bar{G}$ is, up to isomorphism, one of the graphs in Figure 5.1. If $\bar{G}$ is graph (a) in Figure 5.1, then $uwyvzxu$ forms a Hamiltonian cycle. If $\bar{G}$ is graph (b) in Figure 5.1, then $uwyzvxu$ forms a Hamiltonian cycle. If $\bar{G}$ is graph (c) in Figure 5.1, then clearly G is not Hamiltonian since $N(u) =$

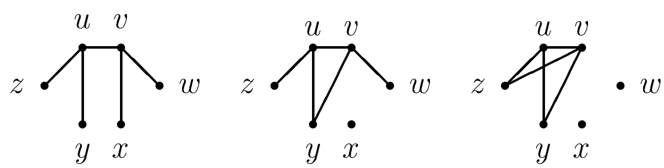

Figure 5.1. The nonisomorphic possibilities for $\bar{G}$ when the number of edges incident to u or v is 5 and $n = 6$.

$N(v) = \{w, x\}$ and the largest cycle containing vertices u and v is C_4. Further, $\langle E(\bar{G}) \rangle = K_4 - e$, where e is an edge, and we get exceptional case (iv).

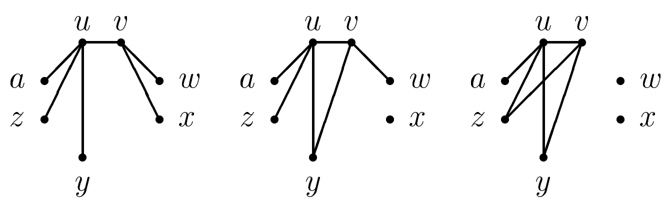

Figure 5.2. The nonisomorphic possibilities for $\bar{G}$ when the number of edges incident to u or v is 6 and $n = 7$.

If $n = 7$, then there are five edges from $\{u, v\}$ to H in $\bar{G}$. Moreover, $\delta(G) \geq 2$ implies that $\Delta(\bar{G}) \leq 4$. Therefore $\bar{G}$ is, up to isomorphism, one of the graphs in Figure 5.2. If $\bar{G}$ is graph (a) or (b) in Figure 5.2, then $uwavzyxu$ forms a Hamiltonian cycle. If $\bar{G}$ is graph (c) in Figure 5.2, then $uwvazyxu$ forms a Hamiltonian cycle. This addresses all the graphs where the number of missing edges incident to $\{u, v\}$ is $n - 1$.

Case 2. Suppose that the number of edges in $\bar{\mathbf{G}}$ incident to u or v is exactly $\mathbf{n-2}$. Since the total number of edges in $\bar{G}$ is at most $n - 1$, $H =$

$G - \{u, v\}$ is missing at most one edge.

If $n = 5$, then there are two edges from $\{u, v\}$ to H in $\bar{G}$. Moreover, $\delta(G) \geq 2$ implies $\Delta(\bar{G}) \leq 2$. Therefore $\bar{G}$ is, up to isomorphism, one of the graphs in Figure 5.3. The two nonisomorphic possibilities for $\bar{G}$ when $\bar{G}$ has three edges incident

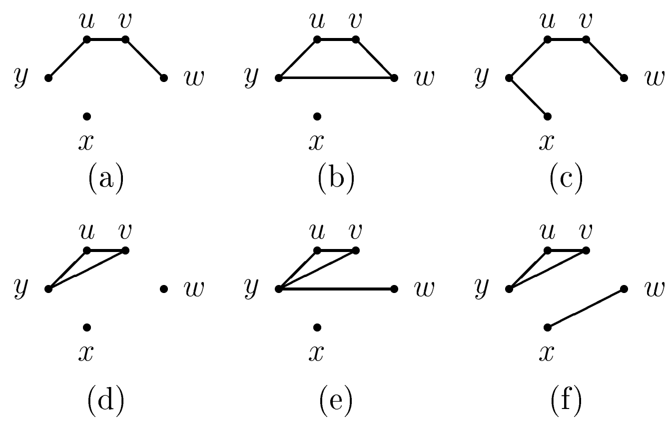

Figure 5.3. The nonisomorphic possibilities for $\bar{G}$ when the number of edges incident to u or v is 3 and $n = 5$.

to u or v and $e(\bar{H}) = 0$ are depicted in the first column in Figure 5.3. As stated previously, the subgraph H can be missing at most one edge. The graphs in the second and third column depict all possible nonisomorphic placements of that additional edge.

If $\bar{G}$ is graph (a) or (c) in Figure 5.3, then $uxvywu$ forms a Hamiltonian cycle. If $\bar{G}$ is graph (b) in Figure 5.3, then clearly G is not Hamiltonian since x is a cut vertex in G. Further, $\langle E(\bar{G}) \rangle = C_4$ and we get exceptional case (iii). If $\bar{G}$ is graph (d), (e), or (f) in Figure 5.3, then G is not Hamiltonian since $N(u) \cup N(v) = \{w, x\}$. Further, $K_3 \subseteq \bar{G}$ and we get exceptional case (ii).

If $n = 6$, then there are three edges from $\{u, v\}$ to H in $\bar{G}$. Moreover, $\delta(G) \geq 2$ implies $\Delta(\bar{G}) \leq 3$. Therefore $\bar{G}$ is, up to isomorphism, one of the graphs in Figure

5.4. The two nonisomorphic possibilities for $\bar{G}$ when $\bar{G}$ has four edges incident

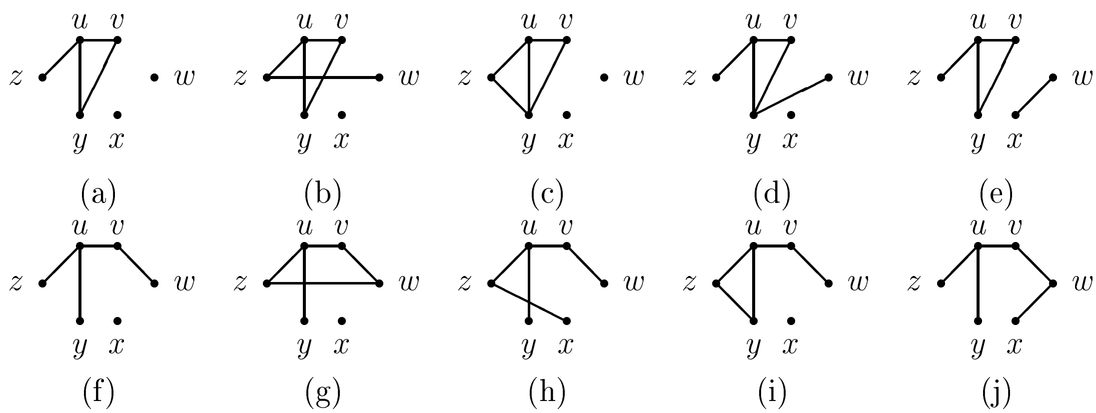

Figure 5.4. The nonisomorphic possibilities for $\bar{G}$ when the number of edges incident to u or v is 4 and $n = 6$.

to $\{u, v\}$ and $e(\bar{H}) = 0$ are depicted in the first column in Figure 5.4. As stated previously, the subgraph H can be missing at most one edge. The graphs in the second and subsequent columns depict all possible nonisomorphic placements of that additional edge.

If $\bar{G}$ is graph (c) in Figure 5.4, then this graph is isomorphic to graph (c) in Figure 5.1. We have already shown that this graph is not Hamiltonian and is exceptional case (iv). If $\bar{G}$ is graph (a), (b), (d), or (e) in Figure 5.4, then $uwvzyxu$ forms a Hamiltonian cycle. If $\bar{G}$ is graph (f), (g), or (h) in Figure 5.4, then $uwyzvxu$ forms a Hamiltonian cycle. If $\bar{G}$ is graph (i) or (j) in Figure 5.4, then $uwzvyxu$ forms a Hamiltonian cycle.

If $n = 7$, then there are four edges from $\{u, v\}$ to H in $\bar{G}$. Moreover, $\delta(G) \geq 2$ implies $\Delta(\bar{G}) \leq 4$. Therefore $\bar{G}$ is, up to isomorphism, one of the graphs in Figure 5.5.

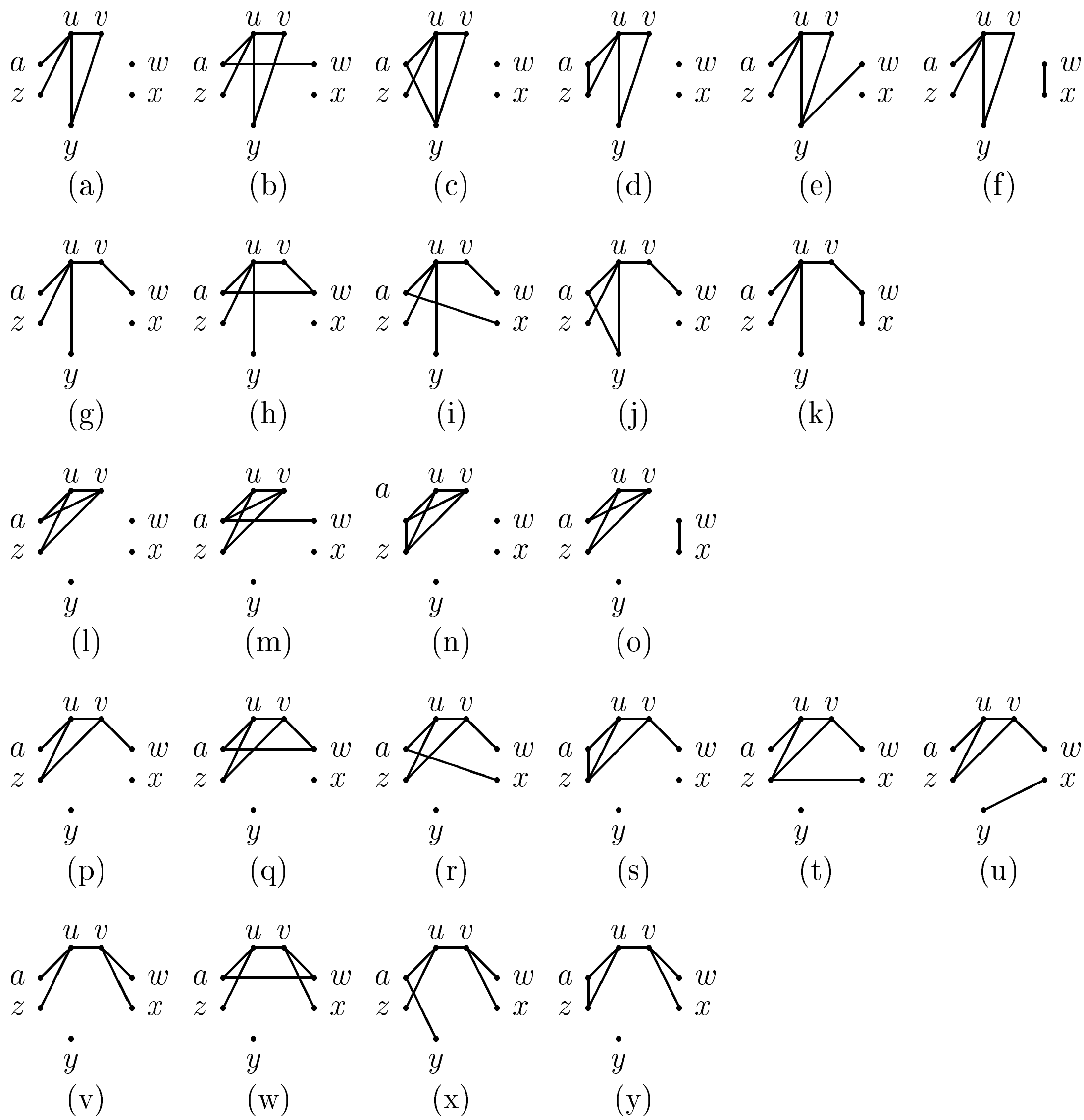

Figure 5.5. The nonisomorphic possibilities for $\bar{G}$ when the number of edges incident to u or v is 5 and $n = 7$.

The five nonisomorphic possibilities for $\bar{G}$ when $\bar{G}$ has 5 edges incident to u or v are depicted in the first column in Figure 5.5. Since the total number of edges in $\bar{G}$ is at most 6 and the number of edges in $\bar{H}$ incident to u or v is exactly 5, H is missing at most one edge. The graphs in the second and subsequent columns depict all possible nonisomorphic placements of that additional edge.

If $\bar{G}$ is graph (n) in Figure 5.5, then clearly $G = I_4 \vee K_3$ is not Hamiltonian since the largest cycle we can form is C_6 obtained by alternately traversing between the vertices of K_3 and I_4 until we use all the vertices in K_3. Further, $\langle E(\bar{G}) \rangle = K_4$ and we get exceptional case (v).

If $\bar{G}$ is graph (a), (b), (c), (d), or (j) in Figure 5.5, then $uwyzvaxu$ forms a Hamiltonian cycle. If $\bar{G}$ is graph (e) or (f) in Figure 5.5, then $uwvazyxu$ forms a Hamiltonian cycle. If $\bar{G}$ is graph (g), (h), (i), (k), (p), (q), (r), or (t) in Figure 5.5, then $uxvazywu$ forms a Hamiltonian cycle. If $\bar{G}$ is graph (l), (m), or (o) in Figure 5.5, then $uwvyzaxu$ forms a Hamiltonian cycle. If $\bar{G}$ is graph (s) or (u) in Figure 5.5, then $uwzyavxu$ forms a Hamiltonian cycle. If $\bar{G}$ is graph (v), (w), (x), or (y) in Figure 5.5, then $uwzxavyu$ forms a Hamiltonian cycle.

This addresses all the graphs where the number of missing edges incident to u or v is at least $n-2$. Therefore, we can assume for all nonadjacent vertices u and v in G that the number of edges in $\bar{G}$ incident to u or v is at most $n-3$.

Case 3. Suppose that the number of edges in $\bar{G}$ incident to every nonadjacent pair of of vertices u, v is at most n – 3.

There are at most $n-4$ edges missing from $\{u, v\}$ into H. Therefore, there are at least $2(n-2)-(n-4) = n$ edges from $\{u, v\}$ into H. So, $d_G(u) + d_G(v) \geq n$ and by Ore's Theorem (Theorem 2.6), G is Hamiltonian. □

Now we can proceed with determining $\chi(G, \neg C_j)$ when $e(G) \geq \binom{n}{2} - (j-1)$ for all $j \geq 3$.

Theorem 5.7 *If* $e(G) \geq \binom{n}{2} - (j-1)$*, then*

$$\chi(G, \neg C_j) = \begin{cases} \left\lceil \frac{n-2}{j-1} \right\rceil & \text{if } \langle E(\bar{G}) \rangle = 2K_2 \text{ and } j = 3, \\ \left\lceil \frac{n-1}{j-1} \right\rceil & \text{if } \langle E(\bar{G}) \rangle = P_3 \text{ and } j = 3, \\ & K_{1,j-2} \subseteq \bar{G} \text{ and } j \geq 4, \\ & K_3 \subseteq \bar{G} \text{ and } j = 5, \\ & \langle E(\bar{G}) \rangle = C_4 \text{ and } j = 5, \\ & \langle E(\bar{G}) \rangle = K_4 - e \text{ where } e \text{ is an edge and } j = 6, \\ & \text{or } \langle E(\bar{G}) \rangle = K_4 \text{ and } j = 7, \text{ and} \\ \left\lceil \frac{n}{j-1} \right\rceil & \text{otherwise.} \end{cases}$$

Proof. If $e(G) > \binom{n}{2} - (j-1)$, then the result follows from Theorem 5.1. So assume $e(G) = \binom{n}{2} - (j-1)$. Further, the result follows when $j \geq 8$ by Theorem 5.5. Therefore, we only need to determine $\chi(G, \neg C_j)$ when $j = 3$, 4, 5, 6 or 7.

If $j = 3$, then $\langle E(\bar{G}) \rangle \in \{2K_2, P_3\}$. If $\langle E(\bar{G}) \rangle = 2K_2$, then $\chi(G, \neg C_3) = \left\lceil \frac{n-2}{j-1} \right\rceil$ by Theorem 4.1. If $\langle E(\bar{G}) \rangle = P_3$, then $\chi(G, \neg C_3) = \left\lceil \frac{n-1}{j-1} \right\rceil$ by Theorem 4.1.

If $j = 4$, then $\langle E(\bar{G}) \rangle \in \{3K_2, P_3 + K_2, K_{1,3}, P_4, K_3\}$. If $\langle E(\bar{G}) \rangle = 3K_2$, then $P_3 \not\subseteq \bar{G}$ and, by Theorem 4.2, we get $\chi(G, \neg C_4) = \max\left(\left\lceil \frac{n}{j-1} \right\rceil, \left\lceil \frac{n}{j} \right\rceil\right) = \left\lceil \frac{n}{j-1} \right\rceil$. If $\langle E(\bar{G}) \rangle \in \{P_3 + K_2, K_{1,3}, P_4\}$, then by Theorem 4.2, $\chi(G, \neg C_4) = \left\lceil \frac{n-1}{j-1} \right\rceil$ if $n \geq j$, and $\chi(G, \neg C_4) = \left\lceil \frac{n}{j} \right\rceil$ if $n < j$ since $\left\lceil \frac{n-1}{j-1} \right\rceil \geq \left\lceil \frac{n}{j} \right\rceil$ if and only if $n \geq j$. But, when $n < j$, we know that $\left\lceil \frac{n}{j} \right\rceil = 1 = \left\lceil \frac{n-1}{j-1} \right\rceil$. Therefore, $\chi(G, \neg C_4) = \left\lceil \frac{n-1}{j-1} \right\rceil$. If $\langle E(\bar{G}) \rangle = K_3$, then by Theorem 5.3, $\chi(G, \neg C_4) = \left\lceil \frac{n-1}{j-1} \right\rceil$ if $n \geq j$, and $\chi(G, \neg C_4) = \left\lceil \frac{n}{j} \right\rceil$ if $n < j$ since $\left\lceil \frac{n-1}{j-1} \right\rceil \geq \left\lceil \frac{n}{j} \right\rceil$ if and only if $n \geq j$. But, when $n < j$, we know that $\left\lceil \frac{n}{j} \right\rceil = 1 = \left\lceil \frac{n-1}{j-1} \right\rceil$. Therefore, $\chi(G, \neg C_4) = \left\lceil \frac{n-1}{j-1} \right\rceil$.

Assume $5 \leq j \leq 7$. Let A be a color class in a $\neg C_j$-coloring of G. Suppose that $|A| \geq j+1$. Let $B \subseteq A$ with $|B| = j+1$. If $e(\langle B \rangle) \geq \binom{j+1}{2} - (j-2)$, then we get $C_j \subseteq \langle B \rangle$ by Theorem 2.10 and reach a contradiction. Therefore, $e(\langle B \rangle) \leq \binom{j+1}{2} - (j-1)$. But since G is missing $j-1$ edges, we must have $e(\langle B \rangle) = \binom{j+1}{2} - (j-1)$.

If $\Delta(\overline{\langle B \rangle}) \leq 1$, then $e(\overline{\langle B \rangle}) \leq (j+1)/2$. Now, $j - 1 = e(\overline{\langle B \rangle}) \leq (j+1)/2$, which implies $j \leq 3$. But this contradicts the assumption that $j \geq 5$. Therefore, $\Delta(\overline{\langle B \rangle}) \geq 2$, which implies $\delta(\langle B \rangle) \leq j-2$. Choose $v \in V(\langle B \rangle)$ so that $d_{\langle B \rangle}(v) \leq j-2$. Consider $\langle B \rangle - v$. Now, $e(\langle B \rangle - v) = e(\langle B \rangle) - d(v) \geq \binom{j+1}{2} - (j-1) - (j-2) = \binom{j}{2} - (j-3)$ and by Theorem 2.10, we have $C_j \subseteq \langle B \rangle$ and reach a contradiction. Therefore, $|A| \leq j$.

Let a be the number of color classes of size j in a $\neg C_j$-coloring of G and A be a color class of size j. By Theorem 2.10, $\langle A \rangle$ must be missing at least $j-2$ edges. Since G is missing at most $j-1$ edges, we must have $a(j-2) \leq j-1$, or simply $a \leq 1$. Therefore, all $\neg C_j$-colorings of G can contain at most one color class of size j, which implies that $\chi(G, \neg C_j) \geq \left\lceil \frac{n-j}{j-1} \right\rceil + 1 = \left\lceil \frac{n-1}{j-1} \right\rceil$.

By Theorem 5.6, we know which graphs missing at most $j-1$ edges are non-Hamiltonian. If G contains one of these non-Hamiltonian graphs as an induced subgraph, then we can color G so that there is a color class of size j and get $\chi(G, \neg C_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$, and if not, then $\chi(G, \neg C_j) = \left\lceil \frac{n}{j-1} \right\rceil$. □

The previous theorem produces an upper bound for determining how many edges need to removed from a complete graph to obtain a graph with $\chi(G, \neg C_j) = \left\lceil \frac{n-2}{j-1} \right\rceil$. Determining $\chi(G, \neg C_j)$ when G is missing j edges can be accomplished, we believe, using the above methods. However, extrapolating on the complexity of the proofs for removing $j-1$ edges versus the complexity of the proof for removing only $j-2$ edges leads us to believe that the proof for removing j edges will be extremely long and complicated. We have already determined the minimum number of edges to be removed to obtain a conditional chromatic number of $\left\lceil \frac{n-2}{j-1} \right\rceil$ in Theorem 5.7. Removing j edges from a complete graph will probably not allow the conditional chromatic number to decrease again. Thus, there is little to be gained by proving the above type of result for graphs missing j edges.

In fact, further research in the area of determining $\chi(G, \neg C_j)$ should focus on determining general upper and lower bounds on the number of edges for a graph to attain a particular conditional chromatic number.

6 Determining $\chi(\mathbf{G}, \neg\mathbf{P_j})$

We discovered in Theorem 5.7 that it is necessary to remove $j-2$ edges from a complete graph to decrease the $\neg C_j$-chromatic number from $\left\lceil \frac{n}{j-1} \right\rceil$ to $\left\lceil \frac{n-1}{j-1} \right\rceil$ and that it is necessary to remove $j-1$ edges to attain a value of $\left\lceil \frac{n-2}{j-1} \right\rceil$ for the $\neg C_j$-chromatic number when $j = 3$. We would like to determine the minimum number of edges to remove from a complete graph to attain a value of $\left\lceil \frac{n-1}{j-1} \right\rceil$ for the $\neg P_j$-chromatic number and determine the minimum number of edges to remove from a complete graph to attain a value of $\left\lceil \frac{n-2}{j-1} \right\rceil$ for the $\neg P_j$-chromatic number. While attempting to find these numbers, we will obtain information about the $\neg P_j$-chromatic number for a large set of graphs. We begin by determining which graphs of large size have a Hamiltonian path, then determine the minimum number of edges which need to be removed from a complete graph for $\chi(G, \neg P_j)$ to decrease from $\left\lceil \frac{n}{j-1} \right\rceil$ to $\left\lceil \frac{n-1}{j-1} \right\rceil$, and then determine the chromatic number for all graphs missing $2j-5$ or fewer edges. Finally, we will determine a lower bound on $\chi(G, \neg P_j)$ in terms of the size of G.

6.1 Determining which graphs of large size have a Hamiltonian path

To determine the $\neg P_j$-chromatic number for graphs whose complements have size at most $2j-5$, we need to determine which graphs of order n missing at most $2n-5$ edges have a Hamiltonian path. The following is an easy and useful result.

Lemma 6.1 *Let G be a graph of order n and $S \subseteq V(G)$. If G has a Hamiltonian path, then the number of components in $G - S$ is at most $|S| + 1$.*

Proof. Let $S \subseteq V(G)$ with $0 \leq |S| \leq n$, and let H be a Hamiltonian path in G. If we remove S from G, then H breaks apart into at most $|S| + 1$ subpaths. Therefore, $G - S$ contains at most $|S| + 1$ components. □

The following sets will be used repeatedly in this section. Let

$\mathcal{F} = \{K_2 \vee (K_2 + I_3),\ K_1 \vee (K_{n-3} + I_2),\ I_r \vee K_{r-2}$, where $r \geq 4$ and $n \geq 4\}$,

$\mathcal{G} = \{G \,|\, G$ is a spanning subgraph of a graph in $\mathcal{F}\}$, and

$\mathcal{H} = \{K_2 \vee (K_2 + I_3),\ K_1 \vee (K_{n-3} + I_2),\ I_4 \vee K_2,\ (I_4 \vee K_2) - e,\ I_5 \vee K_3,\ (I_5 \vee K_3) - e,\ I_6 \vee K_4$, where e is an edge and $n \geq 4\}$.

Lemma 6.2 *If a graph $G \in \mathcal{G}$, then G does not have a Hamiltonian path.*

Proof. Let $G = F \vee (K_2 + I_3)$ where $F = K_2$. Let $S = V(F)$. Then the number of components in $G - S$ is 4 and $4 > 3 = |S| + 1$. By Lemma 6.1, G does not have a Hamiltonian path.

Let $G = F \vee (K_{n-3} + I_2)$ where $F = K_1$. Let $S = V(F)$. Then the number of components in $G - S$ is 3 and $3 > 2 = |S| + 1$. By Lemma 6.1, G does not have a Hamiltonian path.

Let $G = I_r \vee F$ where $F = K_{r-2}$ and $r \geq 4$. Let $S = V(F)$. Then the number of components in $G - S$ is r and $r > (r - 2) + 1 = |S| + 1$. By Lemma 6.1, G does not have a Hamiltonian path.

Let $G = F$, where F is a spanning subgraph of a graph in $\mathcal{F}$. Since each graph in $\mathcal{F}$ does not contain a Hamiltonian path, F cannot contain a Hamiltonian path. □

Now $\mathcal{G}$ contains more graphs than just those graphs of order n missing at most $2n-5$ edges. Since we are interested in determining which graphs missing at most $2n-5$ edges have no Hamiltonian path, we will determine the largest subset of graphs of order n in $\mathcal{G}$ that are missing at most $2n-5$ edges.

Lemma 6.3 *Let G be a graph of order $n \geq 1$ with $e(\bar{G}) \leq 2n-5$. Then $G \in \mathcal{G}$ if and only if $G \in \mathcal{H}$.*

Proof. Let G be a graph of order $n \geq 1$ with $e(\bar{G}) \leq 2n-5$. Assume that $G \in \mathcal{G}$. If G is a spanning subgraph of $K_2 \vee (K_2 + I_3)$, then since $K_2 \vee (K_2 + I_3)$ is missing $9 = 2(7) - 5$ edges, we must have that $G = K_2 \vee (K_2 + I_3) \in \mathcal{H}$. If G is a spanning subgraph of $K_1 \vee (K_{n-3} + I_2)$, then since $K_1 \vee (K_{n-3} + I_2)$ is missing $2n-5$ edges, we must have that $G = K_1 \vee (K_{n-3} + I_2) \in \mathcal{H}$. So assume that G is a spanning subgraph of $I_r \vee K_{r-2}$, where $r \geq 4$. Now $\binom{n}{2} - (2n-5) \leq e(G) \leq e(I_r \vee K_{r-2}) = \binom{n}{2} - \binom{r}{2}$, or $\binom{r}{2} \leq 2n-5 = 2(2r-2)-5$. Simplifying, we get $r^2 - 9r + 18 \leq 0$ or $(r-6)(r-3) \leq 0$. This implies that $3 \leq r \leq 6$. We have already assumed that $r \geq 4$. If G is a spanning subgraph of $I_4 \vee K_2$, then since $I_4 \vee K_2$ is missing $6 = 2(6) - 6$ edges, G can be missing up to one more edge. Thus, $G = I_4 \vee K_2$ or $G = (I_4 \vee K_2) - e$ where e is an edge. Both of these graphs are in $\mathcal{H}$. If G is a spanning subgraph of $I_5 \vee K_3$, then since $I_5 \vee K_3$ is missing $10 = 2(8) - 6$ edges, G can be missing up to one more edge. Thus, $G = I_5 \vee K_3$ or $G = (I_5 \vee K_3) - e$ where e is an edge. Both of these graphs are in $\mathcal{H}$. If G is a spanning subgraph of $I_6 \vee K_4$, then since $I_6 \vee K_4$ is missing $15 = 2(10) - 5$ edges, we must have that $G = I_6 \vee K_4 \in \mathcal{H}$. Therefore, if $\mathcal{G} \subseteq \mathcal{H}$. Clearly, $\mathcal{H} \subseteq \mathcal{G}$. □

By Lemma 6.2, $\mathcal{H}$ consists of graphs of order n missing at most $2n-5$ edges with no Hamiltonian path. The next step is to prove that a connected graph missing at most $2n-5$ edges is either in $\mathcal{H}$ or has a Hamiltonian path. To prove this, we need the following theorem which appears in Chartrand and Lesniak [17].

Theorem 6.1 *Let G be a connected graph of order 3 or more that is not Hamiltonian. If for all distinct nonadjacent vertices u and v, $d(u) + d(v) \geq m$, where m is a positive integer, then $P_{m+1} \subseteq G$.*

The following theorem completely characterizes which graphs of order n missing at most $2n-5$ edges have a Hamiltonian path.

Theorem 6.2 *If G is a connected graph of order $n \geq 1$ and $e(\bar{G}) \leq 2n-5$, then either $G \in \mathcal{H}$ or G has a Hamiltonian path.*

Proof. Let G be a connected graph of order n with $e(\bar{G}) \leq 2n-5$. If $n \leq 2$, then the theorem is vacuously true. If $n = 3$, then $G \in \{P_3, K_3\}$, both of which have a Hamiltonian path. If $n = 4$, then since $e(\bar{G}) \leq 3$ and G is connected, $G \in \{K_{1,3}, P_4, K_1 \vee (K_1 + K_2), C_4, K_4 - e, K_4\}$. If $G = K_{1,3}$, then $G = K_1 \vee (K_1 + I_2) \in \mathcal{H}$. Otherwise, G has a Hamiltonian path. So assume $n \geq 5$. If $G = K_n$, then obviously G has a Hamiltonian path. So assume G is not complete. Let u and v be a pair of nonadjacent vertices in G and $H = G - \{u, v\}$. Since G is connected, each of u and v has a neighbor. If $N(u) = N(v) = \{w\}$, then the number of edges missing between u and H is $n-3$ and the number of edges missing between v and H is $n-3$. Since we assumed that $uv \notin E(G)$, we have a total of $2n-5$ missing edges incident to u or v. This implies that $G = K_1 \vee (K_{n-3} + I_2) \in \mathcal{H}$. Therefore, we can assume that there exist $a, b \in V(H)$ such that $a \in N(u)$, $b \in N(v)$ and $a \neq b$. We break the proof into cases based on the number of edges in $\bar{G}$ incident to u or v.

Case 1. Suppose that the number of edges in $\bar{\mathrm{G}}$ incident to u or v is $\mathbf{n+1}$ or more. Then H is a graph on $n-2$ vertices missing at most $(2n-5)-(n+1) = (n-2)-4$ edges. By Theorem 2.9, H is Hamiltonian connected, and hence there is a Hamiltonian (a,b)-path in H. So, ua together with the Hamiltonian (a,b)-path in H together with bv is a Hamiltonian path in G. Therefore, we can assume that every pair of nonadjacent vertices is incident to no more than n edges in $\bar{G}$.

Case 2. Suppose the number of edges in $\bar{\mathrm{G}}$ incident to u or v is exactly n. Then H is missing at most $(2n-5)-n = (n-2)-3$ edges. By Theorem 2.9, the subgraph H has a Hamiltonian cycle C. Let $C = u_1u_2\ldots u_{n-2}u_1$. If u (or v) is adjacent to consecutive vertices on C, then we could construct a $(n-1)$-cycle using a detour through u (or v) and could then attach v (or u) to this cycle and produce a Hamiltonian path in G. So assume neither u nor v is adjacent to consecutive vertices on C. Thus, $d(u) \leq \frac{n-2}{2}$ and $d(v) \leq \frac{n-2}{2}$; for otherwise, by the pigeonhole principle, u (or v) would be adjacent to consecutive vertices on C.

Next, we determine all possible values for the degrees of u and v in G. There are $2(n-2)$ possible edges from $\{u,v\}$ to H and since $n-1$ of these edges are missing, there are exactly $n-3$ edges from $\{u,v\}$ to H. Now $d(u)+d(v) = n-3$ and the fact that each of u and v has degree at most $\frac{n-2}{2}$ implies that either n is even, $d(u) = \frac{n-4}{2}$, and $d(v) = \frac{n-2}{2}$ (**Case 2A**); or n is odd and $d(u) = d(v) = \frac{n-3}{2}$ (**Case 2B**).

Case 2A. Assume n is even, neither u nor v is adjacent to a pair of consecutive vertices on C and $d(u) = \frac{n-4}{2}$ and $d(v) = \frac{n-2}{2}$.

For v to be adjacent to half of the vertices in H and not have two of its neighbors adjacent on C, the vertex v must be adjacent to alternating vertices on C, say the vertices on C with odd subscripts. Since $d(u) = \frac{n-4}{2}$, we can say without loss of generality that the vertex u is not adjacent to u_1. Now we break this case into subcases based on whether or not $N(u) \subseteq N(v)$.

Case 2A1. $|N(u) \cap N(v)| = \frac{n-4}{2}$. First we will show that if there is an edge in H between a pair of vertices with even indices, then we can construct a Hamiltonian path in G. If $u_2u_4 \in E(G)$, then we can construct a Hamiltonian path in G as follows: $u_4u_2u_1vu_3u$ if $n = 6$ and $u_4u_2u_1vu_3uu_5 \ldots u_{n-2}$ if $n > 6$. If $u_2u_{n-2} \in E(G)$, then we can construct a Hamiltonian path in G as follows: $uu_3u_4 \ldots u_{n-2}u_2u_1v$. If $u_2u_{2k} \in E(G)$ for some $2 < k < (n-2)/2$, then we can construct a Hamiltonian path in G as follows (see Figure 6.1): $u_4u_5 \ldots u_{2k}u_2u_1vu_3uu_{2k+1} \ldots u_{n-2}$. If $u_{n-4}u_{n-2} \in E(G)$, then we can construct a Hamiltonian path in G as follows:

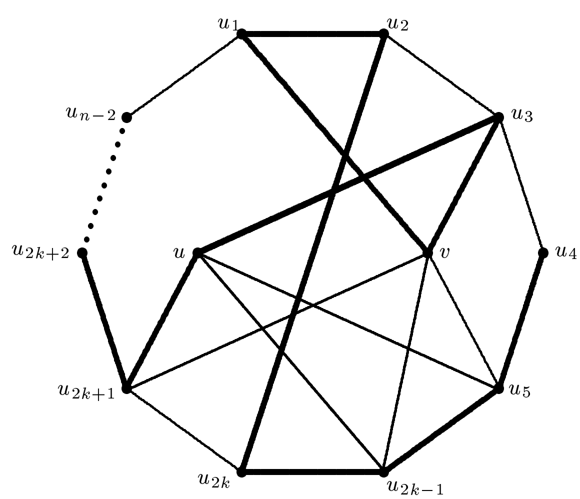

Figure 6.1. The graph G when n is even and $u_2u_{2k} \in E(G)$ for some $2 < k < \frac{n-2}{2}$.

$u_{n-4}u_{n-2}u_{n-3}vu_1u_2 \ldots u_{n-5}u$. If $u_{2k}u_{n-2} \in E(G)$ for some $1 < k < (n-4)/2$, then we can construct a Hamiltonian path in G as follows:

$$u_{2k}u_{n-2}u_{n-3}vu_1u_2\ldots u_{2k-1}uu_{2k+1}u_{2k+2}\ldots u_{n-4}$$

If $u_{2k}u_{2k+2} \in E(G)$ for some $1 < k < \frac{n-4}{2}$, then we can construct a Hamiltonian path in G as follows: $u_{2k}u_{2k+2}u_{2k+1}uu_{2k+3}u_{2k+4}\ldots u_{n-2}u_1\ldots u_{2k-1}v$. If $u_{2k}u_{2l} \in E(G)$ where $1 < k < l < \frac{n-2}{2}$ and $l > k+1$, then we can construct a Hamiltonian path in G as follows (see Figure 6.2):

$$u_{2k}u_{2l}u_{2l-1}uu_{2l+1}u_{2l+2}\ldots u_{n-2}u_1\ldots u_{2k-1}vu_{2k+1}u_{2k+2}\ldots u_{2l-2}$$

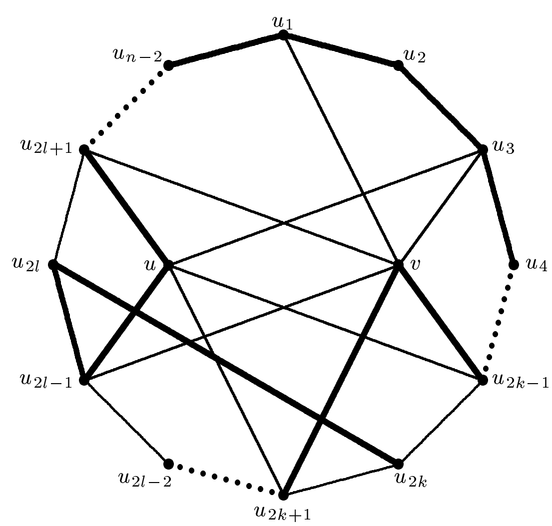

Figure 6.2. The graph G when n is even and $u_{2k}u_{2l} \in E(G)$, $1 < k < l \leq \frac{n-2}{2}$ and $l > k+1$.

Next, assume that there are no edges between any pair of vertices in H with even subscripts. Thus $\{u_2, u_4, \ldots u_{n-2}, u, v\}$ is an independent set of size $\frac{n+2}{2}$. Therefore, G is a spanning subgraph of $I_{\frac{n+2}{2}} \vee K_{\frac{n-2}{2}} = I_r \vee K_{r-2}$, where $r = (n+2)/2$. Now $n \geq 5$ and even implies that $r \geq 4$. Therefore, $G \in \mathcal{G}$ and by Lemma 6.3, $G \in \mathcal{H}$.

Case 2A2. $|N(u) \cap N(v)| < \frac{n-4}{2}$. Since v is adjacent to all vertices in H with odd subscripts, the vertex u cannot be adjacent to all vertices in H with odd subscripts excluding u_1, or else $|N(u) \cap N(v)| = \frac{n-4}{2}$. Therefore, u must be adjacent to u_{2k} for some $1 \leq k \leq \frac{n-2}{2}$. If $uu_{n-2} \in E(G)$, then $uu_{n-2}u_{n-3} \ldots u_1 v$ is a Hamiltonian path in G. If $uu_{2k} \in E(G)$ for some $1 \leq k < \frac{n-2}{2}$, then $uu_{2k}u_{2k-1} \ldots u_1 u_{n-2} u_{n-3} \ldots u_{2k+1} v$ is a Hamiltonian path in G.

Case 2B. Assume n is odd, neither u nor v is adjacent to a pair of consecutive vertices on C, and $d(u) = d(v) = \frac{n-3}{2}$. Recall that $n \geq 5$ and there exist $a, b \in V(H)$ such that $a \in N(u)$, $b \in N(v)$ and $a \neq b$.

If $n = 5$, then $H = K_3$ and G is K_3 with two pendant vertices, each joined to a different vertex in K_3. This graph contains a Hamiltonian path. So, assume $n \geq 7$.

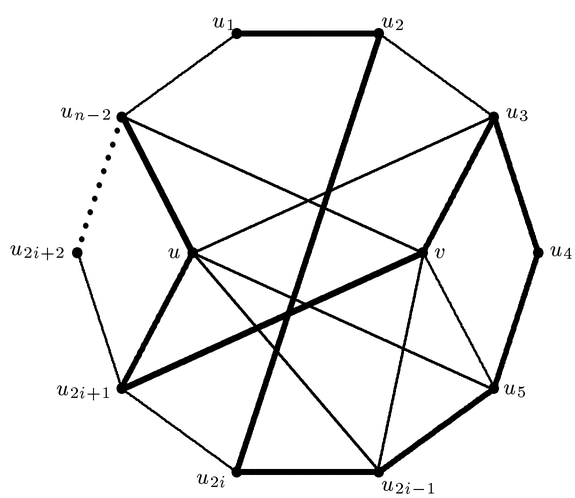

Figure 6.3. The graph G when n is odd and $u_2u_{2i} \in E(G)$ for some $2 \leq i \leq \frac{n-3}{2}$ and $n > 7$.

Since $d(u) = \frac{n-3}{2}$ and there are $n-2$ vertices on C, the vertex u is not adjacent to two consecutive vertices on C, say $uu_1 \notin E(G)$ and $uu_2 \notin E(G)$. For u to have degree $\frac{n-3}{2}$ and not have two adjacent neighbors on C, the vertex u must be adjacent to the remaining vertices on C with odd (or even) subscripts, say odd. Thus, we can assume that $N(u) = \{u_3, u_5, \ldots, u_{n-2}\}$. Next we break this case into subcases based on whether or not $N(u) = N(v)$.

Case 2B1. $N(u) = N(v)$. First we will show that if there is an edge in H between a pair of vertices with even indices, then we can construct a Hamiltonian path in G. If $u_2u_{2i} \in E(G)$ for some $2 \leq i \leq (n-3)/2$, then we can construct a Hamiltonian path in G as follows $u_1u_2u_4u_3vu_5u$ if $n = 7$ and

$$u_1u_2u_{2i}u_{2i-1}u_{2i-2}\ldots u_3vu_{2i+1}uu_{n-2}u_{n-3}\ldots u_{2i+2}$$

if $n > 7$ (see Figure 6.3).

If $u_4u_{2l} \in E(G)$ for some $2 < l \leq \frac{n-3}{2}$, then we can construct a Hamiltonian path in G as follows: $u_1u_2u_3vu_5\ldots u_{2l-1}uu_{n-2}\ldots u_{2l+1}u_{2l}u_4$. If $u_{2k}u_{2l} \in E(G)$ for some $2 < k < l \leq \frac{n-3}{2}$, then we can construct a Hamiltonian path in G as follows (see Figure 6.4):

$$u_1u_2u_3vu_{2k+1}\ldots u_{2l-1}uu_{n-2}\ldots u_{2l+1}u_{2l}u_{2k}u_{2k-1}\ldots u_4$$

Next, assume that there are no edges between any pair of vertices in H with even subscripts. Thus, $e(\bar{H}) \geq \binom{\frac{n-3}{2}}{2}$. Recall that $e(\bar{H}) \leq n-5$. Therefore, $n-5 \geq e(\bar{H}) \geq \binom{\frac{n-3}{2}}{2}$, which implies that $n \leq 11$. Recall that the number of missing edges incident to u or v is n.

If $n = 7$, then $N(u) = N(v) = \{u_3, u_5\}$. We have assumed that $u_2u_4 \notin E(G)$. If $u_1u_4 \in E(G)$, then $uu_5vu_3u_4u_1u_2$ is a Hamiltonian path in G. If $u_1u_4 \notin E(G)$,

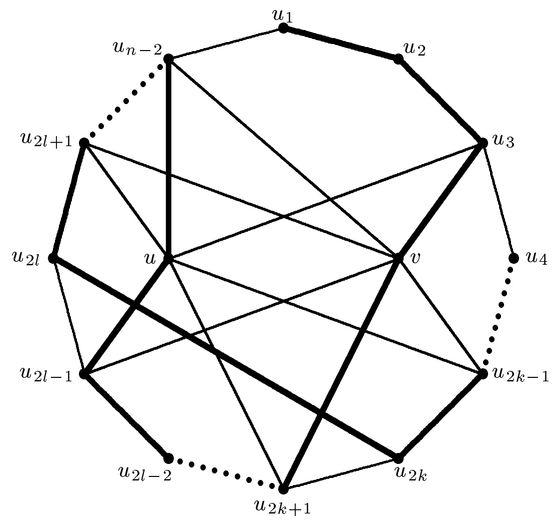

Figure 6.4. The graph G when n is odd and $u_{2k}u_{2l} \in E(G)$ for some $2 < k < l \leq \frac{n-3}{2}$.

then we have accounted for all nine missing edges in G. Therefore, $G = \langle u_3, u_5 \rangle \vee (\langle u_1, u_2 \rangle + u + v + u_4) = K_2 \vee (K_2 + I_3) \in \mathcal{H}$.

If $n = 9$, then $N(u) = N(v) = \{u_3, u_5, u_7\}$. We have already assumed that $\{u_2u_4, u_2u_6, u_4u_6\} \subseteq E(\bar{G})$. This leaves at most one additional edge in $\bar{H}$. If $u_1u_4 \in E(G)$, then $u_6u_7uu_5vu_3u_4u_1u_2$ is a Hamiltonian path in G. If $u_1u_4 \notin E(G)$, then we have accounted for all thirteen missing edges in G. Thus, $u_6u_1 \in E(G)$ and $u_4u_3uu_5vu_7u_6u_1u_2$ is a Hamiltonian path in G.

If $n = 11$, then $N(u) = N(v) = \{u_3, u_5, u_7, u_9\}$. Since we have assumed that there are no edges between any pair of vertices in H with even subscripts, the graph induced by $\{u_2, u_4, u_6, u_8\}$ is a copy of I_4 and we have accounted for all six missing edges in H. Therefore, $u_1u_8 \in E(G)$ and $uu_9vu_3u_2u_1u_8u_7u_6u_5u_4$ is a Hamiltonian path in G.

Case 2B2. $N(u) \neq N(v)$. Recall that $N(u) = \{u_3, u_5, \ldots, u_{n-2}\}$. Now either $vu_1 \in E(G)$ or $vu_2 \in E(G)$, otherwise, by the pigeonhole principle, v would be adjacent to consecutive vertices on C. If $vu_1 \in E(G)$, then $vu_1u_2 \ldots u_{n-2}u$ is a Hamiltonian path in G. If $vu_2 \in E(G)$, then $vu_2u_1u_{n-2} \ldots u_4u_3u$ is a Hamiltonian path in G. Therefore, we can assume that every pair of nonadjacent vertices is incident to no more than $n-1$ edges in $\bar{G}$.

Case 3. Suppose that the number of edges in $\bar{\mathbf{G}}$ incident to u or v is exactly $\mathbf{n-1}$. There are $2(n-2)$ possible edges from $\{u, v\}$ to H, and since $n-2$ of these edges are missing, we get $d(u) + d(v) = n - 2$. The graph H is missing at most $(2n-5)-(n-1) = n-4$ edges. Therefore, $e(H) \geq \binom{n-2}{2} - (n-4) = \frac{(n-2)^2 - 3(n-2)+4}{2}$ edges. By Theorem 2.8, either H is Hamiltonian, $H = K_1 \vee (K_1 + K_{n-4})$ for $n \geq 5$, or $H = K_2 \vee I_3$.

Case 3A. Assume H is Hamiltonian. Again, let $C = u_1u_2 \ldots u_{n-2}u_1$ be a Hamiltonian cycle in H. If u (or v) is adjacent to consecutive vertices on C, then we could construct a $(n-1)$-cycle using a detour through u (or v) and could then attach v (or u) to this cycle and produce a Hamiltonian path in G. So assume neither u nor v is adjacent to consecutive vertices on C. Now n must be even, for otherwise either u or v would be adjacent to at least $\frac{n-1}{2}$ vertices in H, say u, and by the pigeonhole principle, u would be adjacent to consecutive vertices on C.

So assume that n is even, neither u nor v is adjacent to a pair of consecutive vertices on C, and $d(u) = d(v) = \frac{n-2}{2}$. Since u is adjacent to exactly half of the vertices on C and u is not adjacent to a pair of consecutive vertices on C, the vertex u is adjacent to either all the vertices on C with odd subscripts or all the vertices on C with even subscripts. The same is true for v. By symmetry, there are two cases to consider: $N(u) = \{u_2, u_4, \ldots, u_{n-2}\}$ and $N(v) = \{u_1, u_3, u_5, \ldots, u_{n-3}\}$ (**Case 3A1**); and $N(u) = N(v) = \{u_1, u_3, u_5, \ldots, u_{n-3}\}$ (**Case 3A2**).

Case 3A1. $N(u) = \{u_2, u_4, \ldots, u_{n-2}\}$ and $N(v) = \{u_1, u_3, u_5, \ldots, u_{n-3}\}$. In this case, $vu_1u_2 \ldots u_{n-2}u$ is a Hamiltonian path in G.

Case 3A2. $N(u) = N(v) = \{u_1, u_3, u_5, \ldots, u_{n-3}\}$. First we will show that if there is an edge in H between a pair of vertices with even indices, then we can construct a Hamiltonian path in G. If $u_{n-4}u_{n-2} \in E(G)$, then $u_{n-4}u_{n-2}u_{n-3}uu_1 \ldots u_{n-5}v$ is a Hamiltonian path in G. If $u_{2k}u_{n-2} \in E(G)$ for some $1 \leq k < (n-4)/2$, then

$$u_{2k}u_{n-2}u_{n-3}uu_1 \ldots u_{2k-1}vu_{2k+1}u_{2k+2} \ldots u_{n-4}$$

is a Hamiltonian path in G. If $u_{2k}u_{2k+2} \in E(G)$ for some $1 \leq k < (n-4)/2$, then $u_{2k}u_{2k+2}u_{2k+1}uu_{2k+3}u_{2k+4} \ldots u_{n-2}u_1 \ldots u_{2k-1}v$ is a Hamiltonian path in G. If $u_{2k}u_{2l} \in E(G)$ for some $1 \leq k < l < \frac{n-2}{2}$ and $l > k+1$, then

$$u_{2k}u_{2l}u_{2l-1}uu_{2l+1}u_{2l+2} \ldots u_{n-2}u_1 \ldots u_{2k-1}vu_{2k+1}u_{2k+2} \ldots u_{2l-2}$$

is a Hamiltonian path in G (see Figure 6.5). Next, assume that there are no edges between any pair of vertices in H with even subscripts. Thus $\{u_2, u_4, \ldots u_{n-2}, u, v\}$ is an independent set of size $\frac{n+2}{2}$. Therefore, G is a spanning subgraph of $I_{\frac{n+2}{2}} \vee K_{\frac{n-2}{2}} = I_r \vee K_{r-2}$, where $r = (n+2)/2$. Now $n \geq 5$ and even implies that $r \geq 4$. By Lemma 6.3, $G \in \mathcal{H}$.

Case 3B. Assume that $H = K_1 \vee (K_1 + K_{n-4})$ and $n \geq 5$. Let w be the vertex of degree 1 in H, let x be the neighbor of w in H, and let $U = \{u_1, u_2, \ldots u_{n-4}\}$ be the set of remaining vertices in H. Since G is connected, there are two cases to consider: $d(u) \geq 2$ and $d(v) \geq 2$ (**Case 3B1**); and $d(u) = 1$ or $d(v) = 1$, without loss of generality, say $d(u) = 1$ (**Case 3B2**).

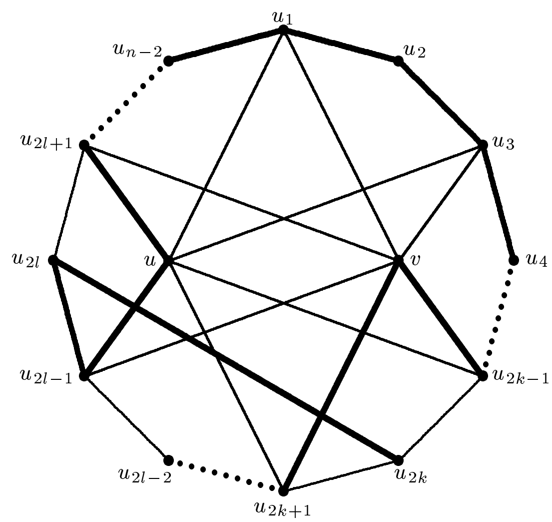

Figure 6.5. The graph G when n is even and $u_{2k}u_{2l} \in E(G)$ for some $1 \leq k < l < \frac{n-2}{2}$ and $l > k+1$.

Case 3B1. Assume that $d(u) \geq 2$ and $d(v) \geq 2$. Suppose that $w \in N(u)$. If there exists $u_i \in U$ such that $u_iv \in E(G)$, without loss of generality, say $u_{n-4}v \in E(G)$, then $uwxu_1 \ldots u_{n-4}v$ is a Hamiltonian path in G. If v is not adjacent to any vertex in U, then $N(v) = \{w, x\}$ and $uwvxu_1 \ldots u_{n-4}$ forms a Hamiltonian path in G. So we can assume that $w \notin N(u)$. By symmetry, we can assume that $w \notin N(v)$.

Suppose that $x \in N(u)$. If u and v were each adjacent to only one $u_i \in U$, say u_1, then since $d(v) \geq 2$, we get $N(u) = N(v) = \{u_1, x\}$. Since $d(u) = 2$, $d(v) = 2$, and $d(u) + d(v) = n - 2$, we must have $n = 6$. Thus, $G = ((u + v + w + u_2) \vee \{u_1, x\}) - u_1w = (I_4 \vee K_2) - e \in \mathcal{H}$. So we can assume that there exist distinct vertices u_i and u_k such that $u_k \in N(u)$ and $u_i \in N(v)$. Without loss of generality, say $u_1 \in N(u)$ and $u_{n-4} \in N(v)$. Now $wxuu_1 \ldots u_{n-4}v$ is a Hamiltonian path in G. So we can assume that $x \notin N(u)$. Further, by symmetry, we can assume that $x \notin N(v)$. Therefore, the only remaining case to consider is when $N(u) \subseteq U$ and

$N(v) \subseteq U$.

Suppose that $N(u) \subseteq U$ and $N(v) \subseteq U$. Since $d(u) \geq 1$, the vertex u is adjacent to a vertex in U, say u_1. Since $d(v) \geq 2$, v is adjacent to a vertex u_i in U where $1 < i \leq n-2$. Without loss of generality, say $uu_2 \in E(G)$. If $vu_1 \in E(G)$, then $uu_1vu_2u_3 \ldots u_{n-4}xw$ forms a Hamiltonian path in G. If $vu_1 \notin E(G)$, then v must be adjacent to another vertex in U, without loss of generality, say u_3. Now $uu_1u_2vu_3 \ldots u_{n-4}xw$ forms a Hamiltonian path in G.

Case 3B2. Assume that $d(u) = 1$. Since $d(u)+d(v) = n-2$, we get $d(v) = n-3$, which implies that v is adjacent to all but one of the vertices in H. Therefore, either $\{x, w\} \subseteq N(v)$, $x \notin N(v)$, or $w \notin N(v)$.

Suppose that $N(u) = \{w\}$. If $\{x, w\} \subseteq N(v)$, then $uwvxu_1 \ldots u_{n-4}$ forms a Hamiltonian path in G. If $x \notin N(v)$ or $w \notin N(v)$, then v is adjacent to a vertex in U, say u_{n-4}. Now $uwxu_1 \ldots u_{n-4}v$ forms a Hamiltonian path in G.

Suppose that $N(u) = \{x\}$. If v has no neighbor in U, then since v is adjacent to all but one vertex in H, $N(v) = \{x, w\}$ and $U = \{u_1\}$. Thus, $G = \{x\} \vee (\langle v, w\rangle + u + u_1) = K_1 \vee (K_2 + I_2) \in \mathcal{H}$. So assume that v has a neighbor in U, say u_1. If $w \in N(v)$, then $uxwvu_1 \ldots u_{n-4}$ forms a Hamiltonian path in G. If $w \notin N(v)$, then $x \in N(v)$ and $G = \{x\} \vee (\langle u_1, u_2, \ldots, u_{n-4}, v\rangle + u + w) = K_1 \vee (K_{n-3} + I_2) \in \mathcal{H}$.

The last case to consider is when $N(u) \subseteq U$. Without loss of generality, suppose that $N(u) = \{u_1\}$. If $w \in N(v)$, then $uu_1 \ldots u_{n-4}xwv$ forms a Hamiltonian path in G. If $w \notin N(v)$, then $N(v) = \{u_1, u_2 \ldots, u_{n-4}, x\}$ and $uu_1 \ldots u_{n-4}vwx$ forms a Hamiltonian path in G.

Case 3C. Suppose that $H = K_2 \vee I_3$. Let $U = \{u_1, u_2\}$ be the set of vertices of degree 4 in H and $V = \{u_3, u_4, u_5\}$ be the set of vertices of degree 2 in H. In this case there are five edges between $\{u, v\}$ and the vertices of H. Without loss of generality, either $d(u) = 1$ and $d(v) = 4$, or $d(u) = 2$ and $d(v) = 3$.

Case 3C1. Suppose that $d(u) = 1$ and $d(v) = 4$. Then by the symmetry of H, we can assume that either $vu_5 \notin E(G)$ or $vu_1 \notin E(G)$. If $vu_5 \notin E(G)$, then $u_1u_5u_2u_3vu_4u_1$ forms C_6 and we can attach u to this cycle to form $P_7 \subseteq G$. If $vu_1 \notin E(G)$, then $u_1u_4vu_3u_2u_5u_1$ forms C_6 and we can attach u to this cycle to form $P_7 \subseteq G$.

Case 3C2. Suppose that $d(u) = 2$ and $d(v) = 3$. Then either v is adjacent to at least two of the vertices in V or v is adjacent to at exactly one vertex in V.

If v is adjacent to at least two of the vertices in V, say u_4 and u_5, then $u_1u_5vu_4u_2u_3u_1$ forms C_6 and we can add u to form $P_7 \subseteq G$.

If v is adjacent to exactly one vertex in V, say u_5, then $N(v) = \{u_1, u_2, u_5\}$. Now, if u is adjacent to at least one of u_3 or u_4, say u_3 (a similar construction works for u_4) then $uu_3u_1u_4u_2u_5v$ is a Hamiltonian path in G. On the other hand, if u is not adjacent to at least one of u_3 or u_4, then without loss of generality, either $N(u) = \{u_1, u_2\}$ or $N(u) = \{u_1, u_5\}$. If $N(u) = \{u_1, u_2\}$, then $G = \langle u_1, u_2 \rangle \vee (\langle v, u_5 \rangle + u + u_3 + u_4) = K_2 \vee (K_2 + I_3) \in \mathcal{H}$. If $N(u) = \{u_1, u_5\}$, then $uu_5vu_1u_4u_2u_3$ is a Hamiltonian path in G.

Case 4. Suppose that the number of edges in $\bar{G}$ incident to u or v is at most $n - 2$. If G is Hamiltonian, then obviously G has a Hamiltonian path. So assume G is not Hamiltonian. There are at most $n - 3$ edges missing from $\{u, v\}$ into H. Therefore, there are at least $2(n - 2) - (n - 3) = n - 1$ edges from $\{u, v\}$ into H. So, $d(u) + d(v) \geq n - 1$. By Theorem 6.1, $P_n \subseteq G$. □

The next theorem builds on the previous theorem by stating which graphs of order n missing at most $2n-5$ edges have a Hamiltonian path.

Theorem 6.3 *Let G be a graph of order $n \geq 1$ with $e(\bar{G}) \leq 2n-5$. Then G has a Hamiltonian path if and only if $\delta(G) > 0$ and $G \notin \mathcal{H}$.*

Proof. Let G be a graph of order $n \geq 1$ with $e(\bar{G}) \leq 2n-5$. Assume that $\delta(G) > 0$ and $G \notin \mathcal{H}$. To apply Theorem 6.2, we must show that G is connected. Suppose that G is not connected. Let A be a component of G with $|A| = a$ and $B = G - A$. Since $\delta(G) > 0$, we have $2 \leq a \leq n-2$. Now, $e(\bar{G}) \geq a(n-a) = an - a^2$. The minimum for this quadratic function occurs when $a = 2$ or $a = n-2$ and we get $e(\bar{G}) \geq an - a^2 \geq 2n-4$. Thus, the size of $\bar{G}$ must be at least $2n-4$, which contradicts the assumption that $e(\bar{G}) \leq 2n-5$. Therefore, G is connected. By Theorem 6.2, G has a Hamiltonian path.

Assume that G has a Hamiltonian path. Now $\delta(G) > 0$ or else G would not have a Hamiltonian path. By Lemma 6.2 and Lemma 6.3, $G \notin \mathcal{H}$. □

Now that we know which graphs of large size have a Hamiltonian path, we will address the problem of determining $\chi(G, \neg P_j)$ when $e(\bar{G}) \leq 2j-5$.

6.2 Graphs missing complete subgraphs or a star

As we have seen in the previous chapter, graphs of large size missing a star or complete subgraphs are the graphs which need to handled as special cases when determining $\chi(G, \neg C_j)$. The same will prove to be true when determining $\chi(G, \neg P_j)$ for graphs of large size. First, let's prove a result for a graph whose complement contains a star as a subgraph.

Theorem 6.4 *Let G be a graph of order n. If $j \geq 2$, $e(G) \geq \binom{n}{2} - (2j-5)$, and $K_{1,j-1} \subseteq \bar{G}$, then $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$.*

Proof. If $j = 2$, then the theorem is vacuously true. So assume $j \geq 3$. Let X be a set of vertices which induce $K_{1,j-1} \subseteq \bar{G}$. Since $K_{1,j-1} \subseteq \bar{G}$ and $e(\bar{G}) \leq 2j-5$, there is exactly one vertex v_0 of degree at least $j-1$ in $\bar{G}$. Therefore, $v_0 \in X$. Let A be a color class in a $\neg P_j$-coloring of G.

Suppose $v_0 \notin A$. If $|A| \geq j$, then $\langle A \rangle$ would be missing at most $2j-5-(j-1) = j-4$ edges, and by Theorem 2.9, we know that $\langle A \rangle$ has a Hamiltonian path. Therefore, for $\langle A \rangle$ to be P_j-free, we must have $|A| \leq j-1$. Suppose $v_0 \in A$, and consider $A - v_0$. If $|A - v_0| \geq j$, then by the same argument, we get $P_j \subseteq \langle A - v_0 \rangle$. Therefore, $|A - v_0| \leq j-1$, which implies $|A| \leq j$. Thus, there can be at most one color class of size j (one containing v_0), and $\chi(G, \neg P_j) \geq \left\lceil \frac{n-j}{j-1} \right\rceil + 1 = \left\lceil \frac{n-1}{j-1} \right\rceil$.

To show the inequality in the other direction, form one color class B of size j by using the vertices in X together with one vertex from $G - X$ and form as many other color classes as possible of size $j-1$ with the remaining vertices in $G - X$. Now $\langle B \rangle$ is P_j-free since v_0 is not adjacent to any other vertex in $\langle B \rangle$. Therefore, $\chi(G, \neg P_j) \leq \left\lceil \frac{n-j}{j-1} \right\rceil + 1 = \left\lceil \frac{n-1}{j-1} \right\rceil$. □

The previous theorem along with most of the following theorems will be used in the next section. The following theorems are more general than we need, but we wish to present them in their entire generality. Their proofs are straightforward and use the same proof methodology. Therefore, to quickly reach the main results of this chapter, the remaining theorems in this section will be presented here without proof. Their proofs can be found in Appendix A.

Theorem 6.5 *Let G be a graph of order $n \geq 1$. If $j \geq 2$ is even, $m \geq 0$, and $\langle E(\bar{G})\rangle = mK_{\frac{j+2}{2}}$, then $\chi(G, \neg P_j) = \max\left(\left\lceil \frac{n-m}{j-1} \right\rceil, \left\lceil \frac{n}{j} \right\rceil\right)$.*

Theorem 6.6 *Let G be a graph of order n. If $j \geq 3$ is odd and $\langle E(\bar{G})\rangle = K_{\frac{j+3}{2}} - e$, where e is an edge, then $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$.*

Theorem 6.7 *Let G be a graph of order n. If $j \geq 4$ and $\langle E(\bar{G})\rangle = I_{j-3} \vee K_2$, then $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$.*

Theorem 6.8 *Let G be a graph of order $n \geq 1$. If $j \geq 4$ is even, $m \geq 0$, and $\langle E(\bar{G})\rangle = mK_{\frac{j+4}{2}}$, then $\chi(G, \neg P_j) = \max\left(\left\lceil \frac{n-2m}{j-1} \right\rceil, \left\lceil \frac{n}{j+1} \right\rceil\right)$.*

Theorem 6.9 *Let G be a graph of order n. If $j \geq 2$ is even and $\langle E(\bar{G})\rangle = K_{\frac{j+4}{2}} - e$, where e is an edge, then $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$.*

Theorem 6.10 *Let G be a graph of order n. If $j \geq 6$ is even and $\langle E(\bar{G})\rangle = K_1 \vee (K_{\frac{j}{2}} + K_1)$, then $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$.*

Theorem 6.11 *Let G be a graph of order n. If $j \geq 6$ is even and $\langle E(\bar{G})\rangle = K_{\frac{j+2}{2}} + K_2$, then $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$.*

We will use most of these theorems to determine $\chi(G, \neg P_j)$ when $e(G)$ is large.

6.3 Determining $\chi(\mathbf{G}, \neg \mathbf{P_j})$ when $\mathbf{e(G)}$ is large

First of all, we can remove $j-2$ edges and show that $\chi(G, \neg P_j)$ does not decrease by applying a well known result about Hamiltonian paths. As an aside, if $j \geq 3$, then Theorem 6.12 is an immediate result of Theorem 6.13. Thus we did not need to prove Theorem 6.12 in full generality. However, it was easier to prove Theorem 6.12 this way.

Theorem 6.12 *Let G be a graph of order n. If $j \geq 2$ and $e(G) \geq \binom{n}{2} - (j-2)$, then $\chi(G, \neg P_j) = \left\lceil \frac{n}{j-1} \right\rceil$.*

Proof. Let A be a color class in a $\neg P_j$-coloring of G. Suppose that $|A| \geq j$. Let $B \subseteq A$ with $|B| = j$. Now $\langle B \rangle$ is missing at most $j-2$ edges. By Theorem 2.9, we get $P_j \subseteq \langle B \rangle$, a contradiction. Therefore, $|A| < j$, which implies that $\chi(G, \neg P_j) \geq \left\lceil \frac{n}{j-1} \right\rceil$. Equality follows from Theorem 2.1. □

We can see that this theorem is best possible since, if we remove the edges of a copy of $K_{1,j-1}$ from a complete graph of order $n \geq j$, then by Corollary 4.1, we can obtain a graph whose $\neg P_j$-chromatic number is $\left\lceil \frac{n-1}{j-1} \right\rceil$. What is the minimum number of edges we need to remove from a complete graph to obtain $\chi(G, \neg P_j) = \left\lceil \frac{n-2}{j-1} \right\rceil$? One way to obtain $\chi(G, \neg P_j) = \left\lceil \frac{n-2}{j-1} \right\rceil$ is to remove the edges of $2K_{1,j-1}$ from a complete graph, i.e., remove $2j-2$ edges. With this in mind, we will attempt to close in on graphs missing $2j-2$ edges by examining graphs missing up to $2j-5$ edges.

Theorem 6.13 *Let G be a graph of order $n \geq 1$ and e be an edge. If $j \geq 2$ and $e(G) \geq \binom{n}{2} - (2j-5)$, then*

$$\chi(G, \neg P_j) = \begin{cases} \left\lceil \frac{n-1}{j-1} \right\rceil & \text{if } G = K_{n-5} \vee (K_2 + I_3) \text{ and } j = 7, \\ & K_{1,j-1} \subseteq \bar{G}, \\ & G = K_{n-(j-1)} \vee (K_{j-3} + I_2) \text{ and } j \geq 4, \\ & G = K_{n-4} \vee I_4 \text{ and } j = 6, \\ & G = (K_{n-4} \vee I_4) - e \text{ and } j = 6, \\ & G = K_{n-5} \vee I_5 \text{ and } j = 8, \\ & G = (K_{n-5} \vee I_5) - e \text{ and } j = 8, \\ & \text{or } G = K_{n-6} \vee I_6 \text{ and } j = 10, \text{ and} \\ \left\lceil \frac{n}{j-1} \right\rceil & \text{otherwise.} \end{cases}$$

Proof. Let $j \geq 2$ and G be a graph of order n with $e(G) \geq \binom{n}{2} - (2j-5)$. If $j = 2$, then the theorem is vacuously true. So assume $j \geq 3$.

Suppose that $G = K_{n-5} \vee (K_2 + I_3)$ and $j = 7$. Then $\langle E(\bar{G}) \rangle = K_5 - e$, where e is an edge, and by Theorem 6.6, we get $\chi(G, \neg P_7) = \left\lceil \frac{n-1}{j-1} \right\rceil$.

Suppose that $K_{1,j-1} \subseteq \bar{G}$. Then by Theorem 6.4, we get $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$.

Suppose that $G = K_{n-(j-1)} \vee (K_{j-3} + I_2)$ and $j \geq 4$. Then $\langle E(\bar{G}) \rangle = I_{j-3} \vee K_2$, and by Theorem 6.7, we get $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$.

Suppose that $G = K_{n-4} \vee I_4$ and $j = 6$, or $G = K_{n-5} \vee I_5$ and $j = 8$, or $G = K_{n-6} \vee I_6$ and $j = 10$. Then $\langle E(\bar{G}) \rangle = K_{\frac{j+2}{2}}$, and by Theorem 6.5, we get $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$ if $n \geq j$, and $\chi(G, \neg P_j) = \left\lceil \frac{n}{j} \right\rceil$ if $n < j$ since $\left\lceil \frac{n-1}{j-1} \right\rceil \geq \left\lceil \frac{n}{j} \right\rceil$ if and only if $n \geq j$. But, when $n < j$, we know that $\left\lceil \frac{n}{j} \right\rceil = 1 = \left\lceil \frac{n-1}{j-1} \right\rceil$. Therefore, $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$.

Suppose that $G = (K_{n-4} \vee I_4) - e$ and $j = 6$ or that $G = (K_{n-5} \vee I_5) - e$ and $j = 8$, where e is an edge. Then $G = K_{n-5} \vee (K_{1,3} + I_1)$ and $j = 6$; or $G = K_{n-6} \vee (I_4 \vee I_2)$ and $j = 6$; or $G = K_{n-6} \vee (K_{1,4} + I_1)$ and $j = 8$; or $G = K_{n-7} \vee (I_5 \vee I_2)$ and $j = 8$.

If $G = K_{n-5} \vee (K_{1,3} + I_1)$ and $j = 6$ or if $G = K_{n-6} \vee (K_{1,4} + I_1)$ and $j = 8$, then $\langle E(\bar{G}) \rangle = K_1 \vee (K_{\frac{j}{2}} + K_1)$. By Theorem 6.10, we get $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$.

If $G = K_{n-6} \vee (I_4 \vee I_2)$ and $j = 6$ or if $G = K_{n-7} \vee (I_5 \vee I_2)$ and $j = 8$, then $\langle E(\bar{G}) \rangle = K_{\frac{j+2}{2}} + K_2$. By Theorem 6.11, we get $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$.

So assume that G and j do not meet one of the conditions specifically listed in the statement of this theorem. In this case we will show that $\chi(G, \neg P_j) = \left\lceil \frac{n}{j-1} \right\rceil$. Let A be a color class in a $\neg P_j$-coloring of G. If we can show that $|A| < j$, then $\chi(G, \neg P_j) \geq \left\lceil \frac{n}{j-1} \right\rceil$ and equality follows from Theorem 2.1. Suppose that $|A| \geq j$.

Let $B \subseteq A$ such that $|B| = j$. We want to use Theorem 6.3 to show that $\langle B \rangle$ has a Hamiltonian path. Since $K_{1,j-1} \not\subseteq \bar{G}$, $\delta(\langle B \rangle) > 0$.

If $\langle B \rangle = K_2 \vee (K_2 + I_3)$, then $j = 7$. Now $9 = 2(7) - 5 = e(\overline{\langle B \rangle})$ and $e(\bar{G}) \leq 2j - 5$ imply that $G = K_{n-5} \vee (K_2 + I_3)$, a contradiction.

If $\langle B \rangle = K_1 \vee (K_{j-3} + I_2)$, then $j \geq 4$. Now $2j - 5 = e(\overline{\langle B \rangle})$, which implies that $G = K_{n-(j-1)} \vee (K_{j-3} + I_2)$, a contradiction.

If $\langle B \rangle = I_4 \vee K_2$, then $j = 6$ and $2j - 5 = 7 = e(\overline{\langle B \rangle}) + 1$. If $\langle B \rangle = (I_4 \vee K_2) - e$, then $j = 6$ and $2j - 5 = 7 = e(\overline{\langle B \rangle})$. Now either $G = K_{n-4} \vee I_4$ or $G = (K_{n-4} \vee I_4) - e$, and we reach a contradiction.

If $\langle B \rangle = I_5 \vee K_3$, then $j = 8$ and $2j - 5 = 11 = e(\overline{\langle B \rangle}) + 1$. If $\langle B \rangle = (I_5 \vee K_3) - e$, then $j = 8$ and $2j - 5 = 11 = e(\overline{\langle B \rangle})$. Now either $G = K_{n-5} \vee I_5$ or $G = (K_{n-5} \vee I_5) - e$, and we reach a contradiction.

If $\langle B \rangle = I_6 \vee K_4$, then $j = 10$. Now $2(10) - 5 = 15 = e(\overline{\langle B \rangle})$, which implies that $G = K_{n-4} \vee I_6$, a contradiction.

Therefore $\langle B \rangle \notin \mathcal{H}$ and by Theorem 6.3, $\langle B \rangle$ has a Hamiltonian path, a contradiction. Thus $|A| < j$. □

The previous theorem has instances when $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$ and instances when $\chi(G, \neg P_j) = \left\lceil \frac{n}{j-1} \right\rceil$. We can see that $\left\lceil \frac{n-1}{j-1} \right\rceil = \left\lceil \frac{n}{j-1} \right\rceil$ in many cases as follows: if $n = p(j-1) + q$, where $p \geq 0$, $j \geq 2$, $0 \leq q < j - 1$ and $q \neq 1$, then $\left\lceil \frac{n-1}{j-1} \right\rceil = \left\lceil \frac{p(j-1)+(q-1)}{j-1} \right\rceil = \left\lceil \frac{n}{j-1} \right\rceil$.

To continue with the approach of using a Hamiltonian cycle on $n-2$ vertices and attaching two vertices to this cycle to create a Hamiltonian path requires us to know which graphs of a fixed size are Hamiltonian. This is labor intensive and, even if we obtain this knowledge, will not yield a general result for conditional coloring. Therefore, we abandon this approach to find $\chi(G, \neg P_j)$ for all graphs of large size with the knowledge that in order to produce a graph whose $\neg P_j$-chromatic number is $\left\lceil \frac{n-2}{j-1} \right\rceil$, we need to remove somewhere between $2j-4$ and $2j-2$ edges.

6.4 Determining bounds on $\chi(\mathbf{G}, \neg \mathbf{P_j})$ given the number of edges in a graph

In this section, we find an upper bound on the size of a graph given the constraint that the $\neg P_j$-chromatic number is at most $\left\lfloor \frac{n-k}{j-1} \right\rfloor$.

To find the upper bound, we first need to determine the maximum size of a P_j-free graph of order n. The first upper bound on the size of a P_j-free graph was discovered in 1959, by P. Erdös and T. Gallai [24]. Their theorem states if G is a P_j-free graph, then $e(G) \leq (\frac{j-2}{2})n$. In 1975, R. Faudree and R. Schelp [25] improved on this bound for the maximum size of a P_j-free graph. In fact, they found the least upper bound and determined which graphs met that bound. The theorem is restated here for completeness and reference.

Theorem 6.14 *(Faudree and Schelp, [25]) If G is a graph of order $n = q(j-1)+r$ $(0 \leq q,\ 0 \leq r < j-1)$ and G contains no P_j, then $e(G) \leq q\binom{j-1}{2} + \binom{r}{2}$ with equality if and only if $G = qK_{j-1} + K_r$ or (when j is even, $q > 0$, and $r = j/2$ or $(j-2)/2$), $G = lK_{j-1} + \left(K_{\frac{j-2}{2}} \vee I_{\frac{j}{2}+(q-l-1)(j-1)+r}\right), 0 \leq l \leq q-1$.*

We would like to know how many edges we need to remove from a complete graph of order $j+i$ $(i \geq 0,\ j \geq 2)$ to guarantee that a graph missing at least this number of edges is P_j-free. To accomplish this, we tailor Faudree and Schelp's result to our needs with the following corollary.

Corollary 6.1 *If G is a P_j-free graph of order $n \geq j \geq 2$, then $e(\bar{G}) \geq (n - j + 1)(j - 1)$.*

Proof. Let m_n be the minimum number of edges in the complement of a P_j-free graph of order $n = q(j-1) + r$, where $q \geq 0$ and $0 \leq r \leq j - 2$. We will show by induction that for $n \geq j$, we have $m_n \geq (n - j + 1)(j - 1)$, which establishes the statement of the corollary. Now $n \geq j$ implies that $q \geq 1$. By Theorem 2.9 *(iii)*, $m_j \geq j - 1$. By Theorem 6.14,

$$\begin{aligned}
m_n &= \binom{n}{2} - q\binom{j-1}{2} - \binom{r}{2} \\
&= \binom{q(j-1)+r}{2} - q\binom{j-1}{2} - \binom{r}{2} \\
&= \frac{(q(j-1)+r)(q(j-1)+r-1) - q(j-1)(j-2) - r(r-1)}{2} \\
&= \frac{q^2(j-1)^2 + rq(j-1) + q(j-1)(r-1) - q(j-1)(j-2)}{2} \\
&= \frac{q(j-1)[q(j-1) + 2r - 1 - j + 2]}{2} \\
&= \frac{q(j-1)[(q-1)(j-1) + 2r]}{2}.
\end{aligned}$$

When $r < j - 2$, we have $n + 1 = q(j-1) + (r+1)$. Thus,

$$\begin{aligned}
m_{n+1} - m_n &= \frac{q(j-1)[(q-1)(j-1) + 2(r+1)]}{2} - \frac{q(j-1)[(q-1)(j-1) + 2r]}{2} \\
&= q(j-1) \\
&\geq j - 1.
\end{aligned}$$

When $r = j - 2$, we have $n + 1 = (q+1)(j-1)$, so that

$$\begin{aligned}
m_{n+1} - m_n &= \frac{(q+1)(j-1)q(j-1)}{2} - \frac{q(j-1)[(q-1)(j-1) + 2(j-2)]}{2} \\
&= \frac{q^2(j-1)^2 + q(j-1)^2 - q^2(j-1)^2 + q(j-1)^2 - 2q(j-1)(j-2)}{2}
\end{aligned}$$

$$
\begin{aligned}
&= q(j-1)^2 - q(j-1)(j-2) \\
&= q(j-1) \\
&\geq j-1.
\end{aligned}
$$

Now by the induction hypothesis, $(m_{n+1} - m_n) + m_n \geq j - 1 + (n - j + 1)(j - 1) = (n + 1 - j + 1)(j - 1)$. □

We get the following immediate result for the $\neg P_j$-chromatic number from Theorem 6.14.

Theorem 6.15 *Let $j \geq 2$. If G is a graph of order $n = k(j-1) + r$ ($0 \leq k$, $0 \leq r < j - 1$) and $e(G) > k\binom{j-1}{2} + \binom{r}{2}$, then $\chi(G, \neg P_j) \geq 2$.*

Proof. Let $j \geq 2$ and let G be a graph of order $n = k(j-1) + r$ ($0 \leq k$, $0 \leq r < j-1$) with $e(G) > k\binom{j-1}{2} + \binom{r}{2}$. By Theorem 6.14, we get $P_j \subseteq G$. Therefore, $\chi(G, \neg P_j) \geq 2$. □

Next we use Corollary 6.1 to derive an upper bound on the size of G given $\chi(G, \neg P_j)$.

Theorem 6.16 *Let G be a graph of order $n \geq 1$, $j \geq 2$ and $k \geq 0$. If $\chi(G, \neg P_j) \leq \left\lfloor \frac{n-k}{j-1} \right\rfloor$, then $e(G) \leq \binom{n}{2} - k(j-1)$.*

Proof. Let G be a graph of order n satisfying $\chi(G, \neg P_j) \leq \left\lfloor \frac{n-k}{j-1} \right\rfloor$ with the maximum number of edges. Now G must contain all edges between any two color classes. For, if not, we could add an edge between two color classes to form a new graph H with $\chi(H, \neg P_j) \leq \left\lfloor \frac{n-k}{j-1} \right\rfloor$ using the same coloring we used for G. But this contradicts the maximality of G. Therefore, we can assume all missing edges reside within the color classes. Furthermore, by maximality, any color class of size at most $j-1$ is not missing any edges.

Let a_{j+i} be the number of color classes of size $j+i$ for $i = 0, \ldots n-j$ and b_i be the number of color classes of size i for $i = 1, \ldots, j-1$. To minimize the number of missing edges in our graph G, we need to minimize the number of missing edges in all color classes of size at least j. By Corollary 6.1, a color class of size $j+i$ must be missing at least $(i+1)(j-1)$ edges for all $i = 0, \ldots, n-j$.

Therefore, we want to find a lower bound for the function $(j-1)a_j + 2(j-1)a_{j+1} + \ldots + (n-j+1)(j-1)a_{j+n-j} = (j-1)\sum_{i=0}^{n-j}(i+1)a_{j+i}$. Let's leave this for a moment and derive another inequality.

$$\begin{aligned} n &= \sum_{i=1}^{j-1} ib_i + \sum_{i=0}^{n-j}(j+i)a_{j+i} \\ &= \sum_{i=1}^{j-1}(j-1)b_i - \sum_{i=1}^{j-1}[(j-1)-i]b_i + \sum_{i=0}^{n-j}(i+1)a_{j+i} + \sum_{i=0}^{n-j}(j-1)a_{j+i} \\ &= (j-1)(\sum_{i=1}^{j-1} b_i + \sum_{i=0}^{n-j} a_{j+i}) - \sum_{i=1}^{j-1}[(j-1)-i]b_i + \sum_{i=0}^{n-j}(i+1)a_{j+i} \\ &= (j-1)\left\lfloor \frac{n-k}{j-1} \right\rfloor - \sum_{i=1}^{j-1}[(j-1)-i]b_i + \sum_{i=0}^{n-j}(i+1)a_{j+i} \\ &\leq n-k - \sum_{i=1}^{j-1}[(j-1)-i]b_i + \sum_{i=0}^{n-j}(i+1)a_{j+i}. \end{aligned}$$

Now, rearranging the above equation, we get

$$\sum_{i=0}^{n-j}(i+1)a_{j+i} \geq k + \sum_{i=1}^{j-1}[(j-1)-i]b_i \geq k$$

since $\sum_{i=1}^{j-1}[(j-1)-i]b_i \geq 0$.

Using the above inequality, we get $(j-1)\sum_{i=0}^{n-j}(i+1)a_{j+i} \geq (j-1)k$. Therefore, the number of missing edges in all color classes is at least $k(j-1)$, which implies that $e(G) \leq \binom{n}{2} - k(j-1)$. □

It is immediately seen from the following example that this bound is attainable. Let G be a graph of order n with $\bar{G} = kK_{1,j-1}$, where $k \geq 0$ and $j \geq 2$. Since $n = kj = k(j-1) + k$, we have that $j - 1$ divides $n - k$ and by Theorem 4.3, we get $\chi(G, \neg P_j) = \left\lceil \frac{n-k}{j-1} \right\rceil = \left\lfloor \frac{n-k}{j-1} \right\rfloor$. We have found an upper bound on the size of G given a bound for $\chi(G, \neg P_j)$ for $j \geq 2$. The next theorem gives a lower bound on the size of G given a particular $\neg P_j$-chromatic number.

Theorem 6.17 *Let G be a graph of order n and $j \geq 2$. If $\chi(G, \neg P_j) > \left\lceil \frac{n-1}{j-1} \right\rceil$, then $e(G) \geq \binom{n}{2} - \frac{n}{2}(j-2)$.*

Proof. We will prove the contrapositive. Let G be a graph with $e(G) < \binom{n}{2} - \frac{n}{2}(j-2)$. Suppose $\delta(G) \geq n-j+1$. Now, $e(G) = \frac{1}{2}\sum d(v) \geq \frac{1}{2}\sum \delta(G) \geq \frac{n}{2}(n-j+1) \geq \binom{n}{2} - \frac{n}{2}(j-2)$. This contradicts the assumption that $e(G) < \binom{n}{2} - \frac{n}{2}(j-2)$. Therefore, there exists a vertex v such that $d(v) \leq n-j$. In other words, the vertex v is nonadjacent to at least $j-1$ other vertices in G. Color G as follows: Form a color class of size j by using v and $j-1$ non-neighbors of v. With the remaining $n-j$ vertices, create as many color classes of size $j-1$ as possible. With this coloring, we get $\chi(G, \neg P_j) \leq \left\lceil \frac{n-j}{j-1} \right\rceil + 1 = \left\lceil \frac{n-1}{j-1} \right\rceil$. □

It is still an open question whether or not the lower bound is attainable. In other words, can we find a graph with $\binom{n}{2} - \frac{n}{2}(j-2)$ edges with $\chi(G, \neg P_j) = \left\lceil \frac{n}{j-1} \right\rceil$? It is also open as to what the lower bounds are for other values of $\chi(G, \neg P_j)$. Finally, given $\chi(G, \neg P_j)$, we would like to find an upper bound on the size of G that is tight when $j-1$ does not divide $n-k$.

A Appendix

All these theorems determine the $\neg P_j$-chromatic number for a graph when its complement contains a very specific set of edges. The first three theorems and the final two theorems are used in the proof of Theorem 6.13.

Theorem 6.5 *Let G be a graph of order $n \geq 1$. If $j \geq 2$ is even, $m \geq 0$, and $\langle E(\bar{G})\rangle = mK_{\frac{j+2}{2}}$, then $\chi(G, \neg P_j) = \max\left(\left\lceil \frac{n-m}{j-1} \right\rceil, \left\lceil \frac{n}{j} \right\rceil\right)$.*

Proof. If $j = 2$, then $\langle E(\bar{G})\rangle = mK_2$, which implies $n \geq 2m$, or $n - m \geq \left\lceil \frac{n}{2} \right\rceil$. We will show there is no color class of size at least 3. If A were a color class of size at least 3, then the graph induced by any three vertices in A would be missing at most one edge and $P_3 \subseteq \langle A \rangle$, a contradiction. Also, a color class of size 2 must consist of the endpoints of an edge in $\bar{G}$. Therefore, $\chi(G, \neg P_2) \geq n - m$. Since $n \geq 2m$, we get $n-m \geq \left\lceil \frac{n}{2} \right\rceil$. Thus, $\chi(G, \neg P_2) = \max\left(n - m, \left\lceil \frac{n}{2} \right\rceil\right)$. If $j \geq 4$, then by Theorems 5.3 and 2.2, we get $\chi(G, \neg P_j) \geq \chi(G, \neg C_j) = \max\left(\left\lceil \frac{n-m}{j-1} \right\rceil, \left\lceil \frac{n}{j} \right\rceil\right)$.

To show the inequality in the other direction, we produce a minimum $\neg P_j$-coloring. Observe that $G = \bigvee_{i=1}^{m+n-\left(\frac{j+2}{2}\right)m} S_i$ where $S_i = I_{\frac{j+2}{2}}$ for $i = 1, 2, \ldots, m$ and $S_i = I_1$ for $i = m+1, m+2, \ldots, n - \left(\frac{j+2}{2}\right) m$. Let $S = V(G) - \bigcup_{i=1}^{m} V(S_i)$. Now, $n = \left(\frac{j+2}{2}\right) m + s$. We consider two cases: $\frac{j-2}{2}m \leq s$ and $0 \leq s < \frac{j-2}{2}m$.

If $\frac{j-2}{2}m \leq s$, then $n \geq mj$ and $\left\lceil \frac{n-m}{j-1} \right\rceil \geq \left\lceil \frac{n}{j} \right\rceil$. For each $i = 1, \ldots, m$, form a color class of size j by using all the vertices from S_i together with $\frac{j-2}{2}$ vertices from S not yet assigned to a color class. Since $\frac{j-2}{2}m \leq s$, S contains a sufficient number of vertices to form these m color classes. Now, all the subgraphs induced by these color classes are isomorphic to $I_{\frac{j+2}{2}} \vee K_{\frac{j-2}{2}}$, which by Lemma 6.2, is P_j-free. We partition the remaining $s - \frac{j-2}{2}m$ vertices into as many color classes of size $j-1$ as possible. Thus, we have $\chi(G, \neg P_j) \leq m + \left\lceil \frac{s - \left(\frac{j-2}{2}\right)m}{j-1} \right\rceil = \left\lceil \frac{\left(\frac{j+2}{2}\right)m + s - m}{j-1} \right\rceil = \left\lceil \frac{n-m}{j-1} \right\rceil$.

If $0 \leq s < \frac{j-2}{2}m$, then $n < mj$ and $\left\lceil \frac{n}{j} \right\rceil \geq \left\lceil \frac{n-m}{j-1} \right\rceil$. Let $r = \left\lceil \frac{\left(\frac{j-2}{2}\right)m - s}{j} \right\rceil$ and $T = S \cup \left(\bigcup_{i=m-r+1}^{m} V(S_i) \right)$. For each $i = 1, \ldots, m-r$, form a color class of size j by using all the vertices from S_i together with $\frac{j-2}{2}$ vertices from T not yet assigned to a color class. Note that $m - r = m - \left\lceil \frac{\left(\frac{j-2}{2}\right)m - s}{j} \right\rceil = \left\lfloor \frac{mj - \left(\frac{j-2}{2}\right)m + s}{j} \right\rfloor = \left\lfloor \frac{\left(\frac{j+2}{2}\right)m + s}{j} \right\rfloor = \left\lfloor \frac{n}{j} \right\rfloor$ and there are sufficient elements in T to form $\left\lfloor \frac{n}{j} \right\rfloor$ color classes since $|T| = s + r\left(\frac{j+2}{2}\right) = s + \left\lceil \frac{\left(\frac{j-2}{2}\right)m - s}{j} \right\rceil \left(\frac{j+2}{2}\right) \geq s + \frac{\left(\frac{j-2}{2}\right)m - s}{j} \left(\frac{j+2}{2}\right) = \left(\frac{j-2}{2}\right)\left(\frac{n}{j}\right) \geq \left(\frac{j-2}{2}\right) \left\lfloor \frac{n}{j} \right\rfloor$. Again, all the subgraphs induced by these color classes are isomorphic to $I_{\frac{j+2}{2}} \vee K_{\frac{j-2}{2}}$, which by Lemma 6.2, is P_j-free.

If there are vertices remaining in T, form one more color class B containing those vertices. To show $\langle B \rangle$ is P_j-free, we will show that $|B| < j$. Now,

$$\begin{aligned} |B| &= |T| - \left\lfloor \frac{n}{j} \right\rfloor \left(\frac{j-2}{2}\right) \\ &= s + r\left(\frac{j+2}{2}\right) - (m-r)\left(\frac{j-2}{2}\right) \\ &= s + rj - m\left(\frac{j-2}{2}\right) \\ &< s + \left(\frac{\left(\frac{j-2}{2}\right)m - s}{j} + 1 \right) j - m\left(\frac{j-2}{2}\right) \\ &= j. \end{aligned}$$

If $B = \emptyset$, then there were no remaining vertices after forming $m - r$ color classes of size j. Therefore, j divides n and $\chi(G, \neg P_j) \leq \left\lfloor \frac{n}{j} \right\rfloor = \left\lceil \frac{n}{j} \right\rceil$. If $B \neq \emptyset$, then j does not divide n and we get $\chi(G, \neg P_j) \leq \left\lfloor \frac{n}{j} \right\rfloor + 1 = \left\lceil \frac{n}{j} \right\rceil$. In either case, we have produced a $\neg P_j$-coloring with $\left\lceil \frac{n}{j} \right\rceil$ colors. □

Theorem 6.6 *Let G be a graph of order n. If $j \geq 3$ is odd and $\langle E(\bar{G})\rangle = K_{\frac{j+3}{2}} - e$, where e is an edge, then $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$.*

Proof. Observe that $\langle E(\bar{G})\rangle = K_{\frac{j+3}{2}} - e$ if and only if $G = R \vee (S + T)$, where $R = K_{n-\left(\frac{j+3}{2}\right)}$, $S = I_{\frac{j-1}{2}}$ and $T = K_2$. Let A be a color class in a $\neg P_j$-coloring of G. Suppose that $|A| \geq j+1$. Let $B \subseteq A$ such that $|B| = j + 1$. Now, B contains up to $\frac{j+3}{2}$ vertices from $S \cup T$ and at least $\frac{j-1}{2}$ vertices from R. For all combinations of vertices from these sets, we get $\langle C\rangle = I_{\frac{j+3}{2}} \vee K_{\frac{j-1}{2}} \subseteq \langle B\rangle$. Let $U = \{u_1, u_2, \ldots, u_{\frac{j+3}{2}}\}$ be the set of vertices of degree $\frac{j-1}{2}$ in $\langle C\rangle$ and $W = \{w_1, w_2, \ldots, w_{\frac{j-1}{2}}\}$ be the set of vertices of degree j in $\langle C\rangle$. Now, $u_1w_1u_2w_2 \ldots u_{\frac{j-1}{2}}w_{\frac{j-1}{2}}u_{\frac{j+1}{2}}$ forms $P_j \subseteq \langle C\rangle$, a contradiction. Therefore, $|A| \leq j$.

Now, each color class of size j must contain strictly more than $\frac{j+3}{4}$ vertices from $S \cup T$. Suppose by way of contradiction that a color class A of size j contains at most $\frac{j+3}{4}$ vertices from $S \cup T$. Since j is odd, either $j+3$ or $j+1$ is divisible by 4. If $j+3$ is divisible by 4, then A contains at most $\frac{j+3}{4}$ vertices from $S \cup T$ and at least $\frac{3j-3}{4}$ vertices from R. For all combinations of vertices from these sets, we get $\langle D\rangle = I_{\frac{j+3}{4}} \vee K_{\frac{3j-3}{4}} \subseteq \langle A\rangle$. Let $x_1, x_2, \ldots, x_{\frac{j+3}{4}}$ be the vertices of degree $\frac{3j-3}{4}$ in $\langle D\rangle$ and $y_1, y_2, \ldots, y_{\frac{3j-3}{4}}$ be the vertices of degree $j-1$ in $\langle D\rangle$. Now, $x_1y_1x_2y_2 \ldots x_{\frac{j+3}{4}}y_{\frac{j+3}{4}}y_{\frac{j+7}{4}}y_{\frac{j+11}{4}} \ldots y_{\frac{3j-3}{4}}$ forms $P_j \subseteq \langle D\rangle$, a contradiction. Therefore, $j+1$ must be divisible by 4, A contains at most $\frac{j+1}{4}$ vertices from $S \cup T$, and A must contain at least $\frac{3j-1}{4}$ vertices from R. For all combinations of vertices from these sets, we get $\langle D\rangle = I_{\frac{j+1}{4}} \vee K_{\frac{3j-1}{4}} \subseteq \langle A\rangle$. Let $x_1, x_2, \ldots, x_{\frac{j+1}{4}}$ be the vertices of degree $\frac{3j-1}{4}$ in $\langle D\rangle$ and $y_1, y_2, \ldots, y_{\frac{3j-1}{4}}$ be the vertices of degree $j-1$ in $\langle D\rangle$. Now, $x_1y_1x_2y_2 \ldots x_{\frac{j+1}{4}}y_{\frac{j+1}{4}}y_{\frac{j+5}{4}}y_{\frac{j+9}{4}} \ldots y_{\frac{3j-1}{4}}$ forms $P_j \subseteq \langle D\rangle$, a contradiction.

Therefore, each color class of size j must contain strictly more than $\frac{j+3}{4}$ vertices from the $S \cup T$, which is strictly more than half the vertices of $S \cup T$. Therefore, there can be at most one color class of size j and $\chi(G, \neg P_j) \geq \left\lceil \frac{n-j}{j-1} \right\rceil + 1 = \left\lceil \frac{n-1}{j-1} \right\rceil$.

Next, we will show the inequality in the other direction. Color G as follows: form one color class A of size j by using all the vertices in S together with $\frac{j-3}{2}$ vertices from R and both vertices from T. Now, $\langle A \rangle = K_{\frac{j-3}{2}} \vee \left(I_{\frac{j-1}{2}} + K_2 \right)$ and since $\langle A \rangle$ contains $\frac{j-1}{2}$ vertices of degree $\frac{j-3}{2}$ with exactly $\frac{j-3}{2}$ common neighbors, $\langle A \rangle$ is P_j-free. With the remaining vertices, form as many color classes of size $j-1$ as possible. With this coloring, we get $\chi(G, \neg P_j) \leq \left\lceil \frac{n-j}{j-1} \right\rceil + 1 = \left\lceil \frac{n-1}{j-1} \right\rceil$. So, $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$. □

Theorem 6.7 *Let G be a graph of order n. If $j \geq 4$ and $\langle E(\bar{G}) \rangle = I_{j-3} \vee K_2$, then $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$.*

Proof. Let $j \geq 4$ and G be a graph of order n such that $\langle E(\bar{G}) \rangle = I_{j-3} \vee K_2$. Then $G = R \vee (S + T)$ where $R = K_{n-(j-1)}$, $S = K_{j-3}$, $T = I_2$. Let $\{r_1, r_2, \ldots, r_{n-j+1}\}$ be the vertices of R, $\{s_1, s_2, \ldots, s_{j-3}\}$ be the vertices of S, and $\{t_1, t_2\}$ be the vertices of T. Let A be a color class in a $\neg P_j$-coloring of G. Suppose that $|A| \geq j+1$. Let $B \subseteq A$ such that $|B| = j+1$. If $t_1 \notin B$ and $t_2 \notin B$, then $\langle B \rangle$ is complete, which implies that $P_j \subseteq \langle B \rangle$, a contradiction. If $t_1 \in B$ and $t_2 \notin B$, then $B = \{t_1, r_1, \ldots, r_p, s_1, s_2, \ldots, s_{j-1-p}\}$ for some $3 \leq p \leq j-1$. Now, $t_1 r_1 r_2 r_3 \ldots r_p s_1 s_2 \ldots s_{j-1-p}$ forms $P_j \subseteq \langle B \rangle$, a contradiction. Therefore, we must have $\{t_1, t_2\} \subseteq B$. Since B can contain at most $j-3$ vertices from S, the subgraph $\langle B \rangle$ must contain at least two vertices r_1 and r_2 from R. So $B = \{t_1, t_2, r_1, \ldots, r_t, s_1, s_2, \ldots, s_{j-1-p}\}$ for some $2 \leq p \leq j-1$. Now, $t_1 r_1 t_2 r_2 r_3 \ldots r_p s_1 s_2 \ldots s_{j-1-p}$ forms $P_j \subseteq \langle B \rangle$, which again is a contradiction. Therefore, we must have $|A| \leq j$. Suppose $|A| = j$. By Theorem 2.9, if $\langle A \rangle$ is missing fewer than $j-1$ edges, then $\langle A \rangle$ has a Hamiltonian path. Since G is missing at most $2j-5$ edges, there can be at most one color class of size j. Thus, $\chi(G, \neg P_j) \geq \left\lceil \frac{n-j}{j-1} \right\rceil + 1 = \left\lceil \frac{n-1}{j-1} \right\rceil$. Next we produce a minimum $\neg P_j$-coloring. Form one color class A of size j with $A = \{t_1, t_2, r_1, s_1, s_2, \ldots, s_{j-3}\}$. Now $\langle A \rangle = K_1 \vee (K_{j-3} + I_2)$, which by Lemma 6.2, is P_j-free. We partition the remaining $n-j$ vertices into as many color classes of size $j-1$ as possible. Thus, we have $\chi(G, \neg P_j) \leq \left\lceil \frac{n-j}{j-1} \right\rceil + 1 = \left\lceil \frac{n-1}{j-1} \right\rceil$. □

In earlier theorems we have derived a value of $\left\lceil \frac{n-m}{j-1} \right\rceil$ for the $\neg P_j$-chromatic number for graphs of order n whose complements consist of m pairwise vertex disjoint copies of a complete graph of a particular order. The following theorem derives a new value for the $\neg P_j$-chromatic number, namely $\left\lceil \frac{n-2m}{j-1} \right\rceil$, for graphs whose complements consist of m pairwise vertex disjoint copies of a complete graph of a new order.

Theorem 6.8 *Let G be a graph of order $n \geq 1$. If $j \geq 4$ is even, $m \geq 0$, and $\langle E(\bar{G}) \rangle = mK_{\frac{j+4}{2}}$, then $\chi(G, \neg P_j) = \max\left(\left\lceil \frac{n-2m}{j-1} \right\rceil, \left\lceil \frac{n}{j+1} \right\rceil\right)$.*

Proof. Observe that $G = \bigvee_{i=1}^{m+n-\left(\frac{j+4}{2}\right)m} S_i$ where $S_i = I_{\frac{j+4}{2}}$ for $i = 1, 2, \ldots, m$ and $S_i = I_1$ for $i = m+1, m+2, \ldots, n - \left(\frac{j+4}{2}\right) m$. Let A be a color class in a $\neg P_j$-coloring of G. Suppose that $|A| > j+1$. Let $B \subseteq A$ such that $|B| = j+2$. Then either $\langle B \rangle = \bigvee_{i=1}^{k} I_{q_i}$ where $q_1 = (j+4)/2$, and $1 \leq q_i \leq j/2$ for all $2 \leq i \leq k$ **(1)**, or $\langle B \rangle = \bigvee_{i=1}^{k} I_{q_i}$ where $1 \leq q_1 \leq (j+2)/2$, $1 \leq q_2 \leq (j+2)/2$, and $1 \leq q_i \leq j/2$ for all $3 \leq i \leq k$ **(2)**. If **(1)**, then form $C_1 \subseteq B$ such that $|C_1| = j$ by removing two vertices from I_{q_1}. If **(2)**, then form $C_2 \subseteq B$ such that $|C_2| = j$ by removing one vertex from each of I_{q_1} and I_{q_2}. Now, $\langle C_1 \rangle = I_{q_1-2} \vee \left(\bigvee_{i=2}^{k} I_{q_i} \right)$, and $\max_i\{q_i\} \leq \frac{j}{2}$. Also, $\langle C_2 \rangle = I_{q_1-1} \vee I_{q_2-1} \vee \left(\bigvee_{i=3}^{k} I_{q_i} \right)$, and $\max_i\{q_i\} \leq \frac{j}{2}$. By Lemma 5.1, $\langle C_1 \rangle$ and $\langle C_2 \rangle$ are Hamiltonian. This contradicts the assumption that $\langle A \rangle$ is P_j-free. So $|A| \leq j+1$ and $\chi(G, \neg P_j) \geq \left\lceil \frac{n}{j+1} \right\rceil$.

Let A be a color class of size $j+1$. We will show that $I_{\frac{j+4}{2}} \subseteq \langle A \rangle$. Suppose by way of contradiction that $I_{\frac{j+4}{2}} \not\subseteq \langle A \rangle$. Then $\langle A \rangle = \bigvee_{i=1}^{k} I_{q_i}$ where $q_1 \leq (j+2)/2$, and $1 \leq q_i \leq j/2$ for all $2 \leq i \leq k$. Form $B \subseteq A$ such that $|B| = j$ by removing one vertex from I_{q_1}. Now, $\langle B \rangle = I_{q_1-1} \vee \left(\bigvee_{i=2}^{k} I_{q_i} \right)$, and $\max_i\{q_i\} \leq \frac{j}{2}$. By Lemma 5.1, $\langle B \rangle$ is Hamiltonian. This contradicts the assumption that $\langle A \rangle$ is P_j-free. So $I_{\frac{j+4}{2}} \subseteq \langle A \rangle$.

Let C be a color class of size j. By Lemma 5.1, either $\langle C \rangle$ is Hamiltonian or $I_{\frac{j+2}{2}} \subseteq \langle C \rangle$. Since $\langle C \rangle$ is not Hamiltonian, we must have that $I_{\frac{j+2}{2}} \subseteq \langle C \rangle$.

Let a be the number of color classes of size $j+1$ and b be the number of color classes of size j in a $\neg P_j$-coloring of G. Since each color class of size $j+1$ must contain all the vertices from S_i for some $1 \leq i \leq m$, and since color classes are vertex disjoint, we must have $a \leq m$. We have just shown that each color class of size j must contain $\frac{j+2}{2}$ independent vertices, which is more than half the vertices from some S_i $(1 \leq i \leq m)$ since $\frac{j+2}{2} > \frac{1}{2}\left(\frac{j+4}{2}\right)$ when $j \geq 2$. Thus we cannot use the vertices of S_i $(1 \leq i \leq m)$ to form two different color classes of size j. Since a color class of size at least j requires more than half the vertices from some S_i $(1 \leq i \leq m)$ and color classes are vertex disjoint, we get $a + b \leq m$. This inequality and $a \leq m$ imply that $2a + b \leq 2m$.

Now,

$$\chi(G, \neg P_j) \geq \left\lceil \frac{n - a(j+1) - bj}{j-1} \right\rceil + a + b = \left\lceil \frac{n - 2a - b}{j-1} \right\rceil \geq \left\lceil \frac{n - 2m}{j-1} \right\rceil$$

Hence, $\chi(G, \neg P_j) \geq \max\left(\left\lceil \frac{n}{j+1} \right\rceil, \left\lceil \frac{n-2m}{j-1} \right\rceil\right)$.

To show the inequality in the other direction, we produce a minimum $\neg P_j$-coloring. Recall that $G = \bigvee_{i=1}^{m+n-\left(\frac{j+4}{2}\right)m} S_i$ where $S_i = I_{\frac{j+4}{2}}$ for $i = 1, 2, \ldots, m$ and $S_i = I_1$ for $i = m+1, m+2, \ldots, n - \left(\frac{j+4}{2}\right) m$. Let $S = V(G) - \bigcup_{i=1}^{m} V(S_i)$. Now, $n = \left(\frac{j+4}{2}\right) m + s$. We consider two cases: $\frac{j-2}{2}m \leq s$ and $0 \leq s < \frac{j-2}{2}m$.

If $\frac{j-2}{2}m \leq s$, then $n \geq m(j+1)$ and $\left\lceil \frac{n-2m}{j-1} \right\rceil \geq \left\lceil \frac{n}{j+1} \right\rceil$. For each $i = 1, \ldots, m$, form a color class of size $j+1$ by using all the vertices from S_i together with $\frac{j-2}{2}$ vertices from S not yet assigned to a color class. Since $\frac{j-2}{2}m \leq s$, the set S contains enough vertices to form these m color classes. Now, all the subgraphs induced by these color classes are isomorphic to $C = I_{\frac{j+4}{2}} \vee K_{\frac{j-2}{2}}$ and since C contains $\frac{j}{2}$ vertices of degree $\frac{j-2}{2}$ with exactly $\frac{j-2}{2}$ common neighbors, C is P_j-free. We partition the remaining $s - \frac{j-2}{2}m$ vertices into as many color classes of size $j-1$ as possible. Thus, we have $\chi(G, \neg P_j) \leq m + \left\lceil \frac{s-\left(\frac{j-2}{2}\right)m}{j-1} \right\rceil = \left\lceil \frac{\left(\frac{j+4}{2}\right)m+s-2m}{j-1} \right\rceil = \left\lceil \frac{n-2m}{j-1} \right\rceil$.

If $0 \leq s < \frac{j-2}{2}m$, then $n < m(j+1)$ and $\left\lceil \frac{n}{j+1} \right\rceil \geq \left\lceil \frac{n-2m}{j-1} \right\rceil$. Let $r = \left\lceil \frac{\left(\frac{j-2}{2}\right)m-s}{j+1} \right\rceil$ and $T = S \cup \left(\bigcup_{i=m-r+1}^{m} V(S_i)\right)$. For each $i = 1, \ldots, m-r$, form a color class of size $j+1$ by using all the vertices from S_i together with $\frac{j-2}{2}$ vertices from T not yet assigned to a color class using the vertices from S first. Note that $m - r = m - \left\lceil \frac{\left(\frac{j-2}{2}\right)m-s}{j+1} \right\rceil = \left\lfloor \frac{m(j+1)-\left(\frac{j-2}{2}\right)m+s}{j+1} \right\rfloor = \left\lfloor \frac{\left(\frac{j+4}{2}\right)m+s}{j+1} \right\rfloor = \left\lfloor \frac{n}{j+1} \right\rfloor$ and there are sufficient elements in T to form $\left\lfloor \frac{n}{j+1} \right\rfloor$ color classes since $|T| = s + r\left(\frac{j+4}{2}\right) = s + \left\lceil \frac{\left(\frac{j-2}{2}\right)m-s}{j+1} \right\rceil \left(\frac{j+4}{2}\right) \geq s + \frac{\left(\frac{j-2}{2}\right)m-s}{j+1}\left(\frac{j+4}{2}\right) = \left(\frac{j-2}{2}\right)\left(\frac{n}{j+1}\right) \geq \left(\frac{j-2}{2}\right)\left\lfloor \frac{n}{j+1} \right\rfloor$. Again, all the subgraphs induced by these color classes are isomorphic to $C = I_{\frac{j+4}{2}} \vee K_{\frac{j-2}{2}}$ and we have already shown that $\langle C \rangle$ is P_j-free.

If there are vertices remaining in T, form one more color class B containing those vertices. To show $\langle B \rangle$ is P_j-free, we will first show that $|B| \le j$. Now,

$$\begin{aligned} |B| &= |T| - \left\lfloor \frac{n}{j+1} \right\rfloor \left(\frac{j-2}{2} \right) \\ &= s + r\left(\frac{j+4}{2}\right) - (m-r)\left(\frac{j-2}{2}\right) \\ &= s + r(j+1) - m\left(\frac{j-2}{2}\right) \\ &< s + \left(\frac{\left(\frac{j-2}{2}\right) m - s}{j+1} + 1 \right)(j+1) - m\left(\frac{j-2}{2}\right) \\ &= j+1. \end{aligned}$$

If $|B| = j$, then $\langle B \rangle = I_{\frac{j+4}{4}} \vee I_{\frac{j-4}{4}}$ since we formed color classes by using vertices from S first. By Lemma 6.2, $I_{\frac{j+4}{4}} \vee I_{\frac{j-4}{4}}$ is P_j-free.

If $B = \emptyset$, then there were no remaining vertices after forming $m - r$ color classes of size $j+1$. Therefore, $j+1$ divides n and $\chi(G, \neg P_j) \le \left\lfloor \frac{n}{j+1} \right\rfloor = \left\lceil \frac{n}{j+1} \right\rceil$. If $B \ne \emptyset$, then $j+1$ does not divide n and we get $\chi(G, \neg P_j) \le \left\lfloor \frac{n}{j+1} \right\rfloor + 1 = \left\lceil \frac{n}{j+1} \right\rceil$. In either case, we have produced a $\neg P_j$-coloring with $\left\lceil \frac{n}{j+1} \right\rceil$ colors. □

The following theorem is needed for the proof of Theorem 6.10.

Theorem 6.9 *Let G be a graph of order n. If $j \ge 2$ is even and $\langle E(\bar{G}) \rangle = K_{\frac{j+4}{2}} - e$, where e is an edge, then $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$.*

Proof. Observe that $\langle E(\bar{G}) \rangle = K_{\frac{j+4}{2}} - e$ if and only if $G = R \vee (S + T)$, where $R = K_{n - \left(\frac{j+4}{2}\right)}$, $S = I_{\frac{j}{2}}$, and $T = K_2$. Let A be a color class in a $\neg P_j$-coloring of G. Suppose that $|A| > j$. Let $B \subseteq A$ such that $|B| = j+1$. Now, B contains up to $\frac{j+4}{2}$ vertices from $S \cup T$ and at least $\frac{j-2}{2}$ vertices from R. For all combinations of vertices from these sets, we get $\langle C \rangle = \left(I_{\frac{j}{2}} + K_2\right) \vee K_{\frac{j-2}{2}} \subseteq \langle B \rangle$. Let $u_1, u_2, \ldots, u_{\frac{j}{2}}$ be the vertices of degree $\frac{j-2}{2}$ in $\langle C \rangle$, v_1, v_2 be the vertices of degree $\frac{j}{2}$ in $\langle C \rangle$, and $w_1, w_2, \ldots, w_{\frac{j-2}{2}}$ be the vertices of degree j in $\langle C \rangle$. Now, $u_1 w_1 u_2 w_2 \ldots u_{\frac{j-2}{2}} w_{\frac{j-2}{2}} v_1 v_2$ forms $P_j \subseteq \langle C \rangle$, a contradiction. Therefore, $|A| \leq j$.

Now each color class of size j must contain strictly more than $\frac{j+4}{4}$ vertices from $S \cup T$. Suppose by way of contradiction that a color class A of size j contains at most $\frac{j+4}{4}$ vertices from $S \cup T$. Since j is even, either j or $j+2$ is divisible by 4. If j is divisible by 4, then A contains at most $\frac{j+4}{4}$ vertices from $S \cup T$ and therefore, at least $\frac{3j-4}{4}$ vertices from R. For all combinations of vertices from these sets, we get $\langle D \rangle = I_{\frac{j+4}{4}} \vee K_{\frac{3j-4}{4}} \subseteq \langle A \rangle$. Let $x_1, x_2, \ldots, x_{\frac{j+4}{4}}$ be the vertices of degree $\frac{3j-4}{4}$ in $\langle D \rangle$ and $y_1, y_2, \ldots, y_{\frac{3j-4}{4}}$ be the vertices of degree $j-1$ in $\langle D \rangle$. Now, $x_1 y_1 x_2 y_2 \ldots x_{\frac{j+4}{4}} y_{\frac{j+4}{4}} y_{\frac{j+8}{4}} y_{\frac{j+12}{4}} \ldots y_{\frac{3j-4}{4}}$ forms $P_j \subseteq \langle D \rangle$, a contradiction. So $j+2$ must be divisible by 4, A contains at most $\frac{j+2}{4}$ vertices from $S \cup T$, and A must contain at least $\frac{3j-2}{4}$ vertices from R. For all combinations of vertices from these sets, we get $\langle D \rangle = I_{\frac{j+2}{4}} \vee K_{\frac{3j-2}{4}} \subseteq \langle A \rangle$. Let $x_1, x_2, \ldots, x_{\frac{j+2}{4}}$ be the vertices of degree $\frac{3j-2}{4}$ in $\langle D \rangle$ and $y_1, y_2, \ldots, y_{\frac{3j-2}{4}}$ be the vertices of degree $j-1$ in $\langle D \rangle$. Now, $x_1 y_1 x_2 y_2 \ldots x_{\frac{j+2}{4}} y_{\frac{j+2}{4}} y_{\frac{j+6}{4}} y_{\frac{j+10}{4}} \ldots y_{\frac{3j-2}{4}}$ forms $P_j \subseteq \langle D \rangle$, a contradiction.

Therefore, each color class of size j must contain strictly more than $\frac{j+4}{4}$ vertices from $S \cup T$, which is strictly more than half the vertices in $S \cup T$. Therefore, there can be at most one color class of size j and $\chi(G, \neg P_j) \geq \left\lceil \frac{n-j}{j-1} \right\rceil + 1 = \left\lceil \frac{n-1}{j-1} \right\rceil$.

Next, we will show the inequality in the other direction. If $j = 2$, then $\langle E(\bar{G}) \rangle = P_3$. Form one color class of size 2 using the two nonadjacent vertices in G and put the remaining $n-2$ vertices into $n-2$ color classes. Therefore, $\chi(G, \neg P_j) \leq n-1$. If $j \geq 4$, then color G as follows: form one color class A of size j by using all the vertices in S together with $\frac{j-4}{2}$ vertices from R and both vertices from T. Now, $\langle A \rangle = K_{\frac{j-4}{2}} \vee \left(I_{\frac{j}{2}} + K_2 \right)$ and $\langle A \rangle$ is a spanning subgraph of

$I_{\frac{j+2}{2}} \vee K_{\frac{j-2}{2}}$. By Lemma 6.2, $\langle A \rangle$ is P_j-free. With the remaining vertices, form as many color classes of size $j-1$ as possible. We get $\chi(G, \neg P_j) \leq \left\lceil \frac{n-j}{j-1} \right\rceil + 1 = \left\lceil \frac{n-1}{j-1} \right\rceil$. So, $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$. □

Theorem 6.10 *Let G be a graph of order n. If $j \geq 6$ is even and $\langle E(\bar{G}) \rangle = K_1 \vee (K_{\frac{j}{2}} + K_1)$, then $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$.*

Proof. Assume $j \geq 6$ is even and G is a graph of order n such that $\langle E(\bar{G}) \rangle = K_1 \vee (K_{\frac{j}{2}} + K_1)$. Then $G = R \vee (S + T)$ where $R = K_{n-(\frac{j}{2}+2)}$, $S = K_{1,\frac{j}{2}}$ and $T = I_1$. Let $\{r_1, r_2, \ldots, r_{n-(\frac{j}{2}+2)}\}$ be the vertices of R, $\{s_1, s_2, \ldots, s_{\frac{j}{2}}\}$ be the vertices of degree 1 in S, s be the vertex of degree $\frac{j}{2}$ in S, and t be the vertex in T. Now $H = K_{n-(\frac{j}{2}+2)} \vee (K_2 + I_{\frac{j}{2}}) \subseteq G$ and by Theorems 6.9 and 2.4, $\chi(G, \neg P_j) \geq \chi(H, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$.

To show the inequality in the other direction, color G as follows: form one color class A of size j with $A = \{s_1, s_2, \ldots, s_{\frac{j}{2}}, s, r_1, r_2 \ldots, r_{\frac{j-4}{2}}, t\}$. Now $\langle A \rangle = K_{\frac{j-4}{2}} \vee (K_{1,\frac{j}{2}} + I_1)$, which is a spanning subgraph of $I_{\frac{j+2}{2}} \vee K_{\frac{j-2}{2}}$ and by Lemma 6.2, is P_j-free. With the remaining vertices form as many color classes of size $j-1$ as possible. With this coloring, we get $\chi(G, \neg P_j) \leq \left\lceil \frac{n-1}{j-1} \right\rceil$.

Theorem 6.11 *Let G be a graph of order n. If $j \geq 6$ is even and $\langle E(\bar{G}) \rangle = K_{\frac{j+2}{2}} + K_2$, then $\chi(G, \neg P_j) = \left\lceil \frac{n-1}{j-1} \right\rceil$.*

Proof. Assume $j \geq 6$ is even and G is a graph of order n such that $\langle E(\bar{G}) \rangle = K_{\frac{j+2}{2}} + K_2$. Then $G = R \vee (S \vee T)$ where $R = K_{n-(\frac{j+2}{2}+2)}$, $S = I_{\frac{j+2}{2}}$ and $T = I_2$. Let A be a color class in a $\neg P_j$-coloring of G. Suppose that $|A| \geq j+1$. Let $B \subseteq A$ with $|B| = j+1$. Now B contains up to $\frac{j+2}{2} + 2$ vertices from $S \cup T$ and at least $\frac{j-4}{2}$ vertices from R. For all combinations of vertices from these sets, we get $C = K_{\frac{j-4}{2}} \vee (I_{\frac{j+2}{2}} \vee I_2) \subseteq \langle B \rangle$. Let $x_1, x_2, \ldots, x_{\frac{j-4}{2}}$ be the vertices of degree j in C, $y_1, y_2, \ldots, y_{\frac{j+2}{2}}$ be the vertices of degree $\frac{j}{2}$ in C, and z_1, z_2 be the vertices of degree $j-1$ in C. Now, $y_1 x_1 y_2 x_2 \ldots y_{\frac{j-4}{2}} x_{\frac{j-4}{2}} y_{\frac{j-2}{2}} z_1 y_{\frac{j}{2}} z_2 y_{\frac{j+2}{2}}$ forms $P_j \subseteq C$. Hence $|A| \leq j$.

To form a color class of size j, we must use all the vertices in S. For if A contains at most $\frac{j}{2}$ vertices from S, then for $v \in A$, we have $d_{\langle A \rangle}(v) \geq \frac{j}{2}$ and, by Dirac's Theorem (Theorem 2.7), $\langle A \rangle$ is Hamiltonian, a contradiction. Thus, there can be at most one color class of size j. Therefore, we get $\chi(G, \neg P_j) \geq \left\lceil \frac{n-j}{j-1} \right\rceil + 1 = \left\lceil \frac{n-1}{j-1} \right\rceil$.

To show the inequality in the other direction, color G as follows: form one color class of size j by using all $\frac{j+2}{2}$ vertices from S, one vertex from T, and $\frac{j-4}{2}$ vertices from R. Now $\langle A \rangle = I_{\frac{j+2}{2}} \vee K_{\frac{j-2}{2}}$, which by Lemma 6.2, is P_j-free. With the remaining vertices form as many color classes of size $j-1$ as possible. With this coloring, we get $\chi(G, \neg P_j) \leq \left\lceil \frac{n-1}{j-1} \right\rceil$. $\square$

B References

1. J. Akiyama, H. Era, S. Grevacio and M. Watanabe, Path chromatic numbers of graphs, *J. Graph Theory* 13(5): 569–575 (1989).

2. M. O. Albertson, R. Jamison, S. T. Hedetniemi and S. C. Locke, The subchromatic number of a graph, *Discr. Math.* 74:33–49 (1989).

3. J. A. Andrews and M. S. Jacobson, On a generalization of chromatic number, *Congr. Numer.* 47:33–48 (1985).

4. J. A. Andrews and M. S. Jacobson, On a generalization of chromatic number and two kinds of Ramsey numbers, *Ars Combin.* 23:97–102 (1987).

5. K. I. Appel and W. Haken, *The solution of the four-color-map problem*, Freeman, California, 1977.

6. P. Baldi, On a generalized family of colorings, *Graphs and Combinatorics* 6:95–110 (1990).

7. L. W. Beineke and A. T. White, *Selected Topics in Graph Theory*, Academic Press, London, 1978.

8. L. D. Bodin and A. J. Friedman, Scheduling of committees for the New York State Assembly, Technical report, State University of New York, 1971.

9. J. A. Bondy, Variations on the Hamiltonian theme, *Can. Math. B.* 15:57–62 (1972).

10. J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Elsevier North Holland, Inc., New York, 1976.

11. O. Borodin, Cyclic coloring of plane graphs, *Discr. Math.* 100:281–289 (1992).

12. I. Broere and C. M. Mynhardt, Generalized colorings of outerplanar and planar graphs, in *Graph Theory with Applications to Algorithms and Computer Science*, Wiley Intersci. Publ., New York, 1985.

13. J. I. Brown and D. G. Corneil, On uniquely $-G$ k-colorable graphs, *Quaestiones Mathematicae* 15:477–487 (1992).

14. J. I. Brown and D. G. Corneil, On generalized graph colorings, *J. Graph Theory* 11:87–99 (1987).

15. G. Chartrand, D. P. Geller and S. T. Hedetniemi, A generalization of the chromatic number, *Proc. Camb. Philos. Soc.* 64:265–271 (1968).

16. G. Chartrand, H. V. Kronk and C. E. Wall, The point-arboricity of a graph, *Isr. J. Math.* 6:169–175 (1968).

17. G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Wadsworth, Inc., California, 1986.

18. R. Cimikowski, On heuristics for determining the thickness of a graph, *Inf. Sciences* 85(1):87 (1995).

19. R. J. Cook, Point partition numbers and girth, *Proc. Am. Math. Soc.* 49:510–514 (1975).

20. M. B. Cozzens and F. S. Roberts, T-colorings of graphs and the channel assignment problem, *Congr. Numer.* 35:191–208 (1982).

21. K. C. Dargen and K. Fraughnaugh, Conditional chromatic numbers with forbidden cycles, *Linear Algebra its Appl.* 217:53–66 (1995).

22. G. A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.* 2:69–81 (1952).

23. G. S. Domke, S. T. Laskar, S. T. Hedetniemi and K. Peters, The partite-chromatic number of a graph, *Congr. Numer.* 53:235–246 (1986).

24. P. Erdös and T. Gallai, On maximal paths and circuits of graphs, *Acta Math. Acad. Sci. Hungar* 10:337–356 (1959).

25. R. J. Faudree and R. H. Schelp, Path Ramsey numbers in multicolorings, *J. Comb. Th. B* 19:150–160 (1975).

26. M. R. Garey and D. S. Johnson, An application of graph coloring to printed circuit testing, *IEEE Trans. CAS* 23:591–599 (1976).

27. M. R. Garey and D. S. Johnson, *Computers and Intractability, A Guide to the Theory of NP-Completeness*, W. H. Freeman, New York, 1979.

28. W. K. Hale, Frequency assignment: theory and applications, *Proc. IEEE* 68:1497–1514 (1980).

29. F. Harary, *Graph Theory*, Addison-Wesley, 1969.

30. F. Harary, Conditional colorability in graphs, in *Graphs and Applications, Proc. First Colo. Symp. Graph Theory* (F. Harary and J. Maybee, eds.), Wiley Intersci. Publ., New York, 1985.

31. F. Harary and K. Fraughnaugh (Jones), Conditional colorability II: bipartite variations, *Congr. Numer.* 50:205–218 (1985).

32. F. Harary and K. Fraughnaugh (Jones), Degree conditional bipartition numbers in graphs, *Congr. Numer.* 55:39–50 (1986).

33. F. Harary and L. Hsu, Conditional chromatic number and graph operations, *Bull. Inst. Math. Acad. Sinica* 19(2):125–134 (1991).

34. F. Harary and P. C. Kainen, On triangular colorings of a planar graph, *Bull. Calcutta Math Soc.* 69:393–395 (1977).

35. S. T. Hedetniemi, On partitioning planar graphs, *Can. Math. B.* 11(2):203-211 (1968).

36. S. T. Hedetniemi, Disconnected colorings of graphs, in *Combinatorial Structures and Their Applications, Proc. Calgary Internat. Conf.* (R. Guy, ed.), Gordon and Breach, New York, 1970.

37. J. P. Hutchinson, Coloring ordinary maps, maps of empires, and maps of the moon, *Math. Mag.* 66(4):211–225 (1993).

38. G. Johns and F. Saba, On the path-chromatic number of a graph, in *Graph Theory and its Applications: East and West* (M. F. Capobianco, ed.), Jinan, 1986.

39. F. Kramer and H. Kramer, Ein färbungsproblem der knotenpunkte eines graphen bezüglich der distanz p, *Rev. Roumaine Math. Pures Appl.* 14(2):1031–1038 (1969).

40. L. Lesniak-Foster and J. Straight, The cochromatic number of a graph, *Ars Combin.* 3:39–45 (1977).

41. D. R. Lick and A. R. White, k-degenerate graphs, *Can. J. Math.* 22(5):1082–1096 (1970).

42. C. Mynhardt and I. Broere, Generalized colorings of graphs, in *Graph Theory with Applications to Algorithms and Computer Science* (Y. Alavi, ed.), Wiley Intersci. Publ., New York, 1985.

43. R. J. Opsut and F. S. Roberts, On the fleet maintenance, mobile radio frequency, task assignment, and traffic phasing problems, in *The Theory and Applications of Graphs* (G. Chartrand, ed.), Wiley Intersci. Publ., New York, 1981.

44. O. Ore, Note on Hamiltonian graphs, *Am. Math. Mo.* 67:55 (1960).

45. O. Ore, Arc coverings of graphs, *Ann. Mat. Pura. Appl.* 55:315–321 (1961).

46. R. J. Pennotti, *Channel Assignment in Mobile Communication Systems*, PhD Thesis, Polytechnic Institute of New York, 1976.

47. F. S. Roberts, *Applied Combinatorics*, Prentice Hall, 1984.

48. H. Sachs and M. Schäuble, Konstruction von Graphen mit gewissen färbungseigenschafte, in *Beiträge zur graphentheorie, (Kolloquium, Maneback, 1967)* (H. Sachs, H. Voss and H. Walter, eds.) Teubner, Leipzig, 1968.

49. E. Sampathkumar, Generalizations of independence and chromatic numbers of a graph, *Discr. Math.* 115:245–251 (1993).

50. E. Sampathkumar, S. Prabha, S. Neeralagi and V. Venkatachalam, A generalization of the chromatic and line chromatic numbers of a graph, *J. Karnatak Univ. Sci.* 22:44–49 (1977).

51. D. B. West, *Introduction to Graph Theory*, Prentice Hall, New Jersey, 1996.