Name: ________________________________________________

Exam Rules:

• This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.

• Each problem is worth 20 points.

• Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.

• If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.

• Begin each solution on a new page and use additional paper, if necessary.

• Write only on one side of paper.

• Write legibly using a dark pencil or pen.

• Ask the proctor if you have any questions.

Good luck!

1. __________  5. __________
2. __________  6. __________
3. __________  7. __________
4. __________

Total __________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
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1. Assume the following general definition for a real positive semidefinite matrix: an \( n \times n \) real matrix \( A \) is said to be **positive semidefinite** if and only if, for all vector \( x \) in \( \mathbb{R}^n \), \( x^T A x \geq 0 \). In particular, this definition allows real matrices which are **not symmetric** to be **positive semidefinite**.

(a) Prove that if \( A \) and \( B \) are real symmetric positive semidefinite matrices and matrix \( A \) is nonsingular, then \( AB \) has only real nonnegative eigenvalues. (10 pts)

(b) Provide a counterexample showing that the requirement that the matrices are symmetric cannot be dropped. (10 pts)

**Solution**

(a) Since \( A \) is symmetric positive definite, \( A^{1/2} \) and \( A^{-1/2} \) are well defined. The matrix \( AB \) has the same eigenvalues as the matrix \( A^{-1/2}ABA^{1/2} = A^{1/2}BA^{1/2} \). The latter matrix is selfadjoint and positive semidefinite, so it has real nonnegative eigenvalues.

Note: The result also holds if we remove the assumption of \( A \) to be nonsingular. In other words, \( A \) and \( B \) only need to be two \( n \)-by-\( n \) symmetric positive semidefinite matrices. The proof gets a little trickier though.

(b) One needs to provide positive semidefinite matrices \( A \) and \( B \), \( A \) nonsingular, such that \( AB \) has an eigenvalue which is not “real and nonnegative”. Given question (a) we understand that either \( A \) or \( B \) (or both) have to be nonsymmetric. To create a positive semidefinite matrix \( A \), one simply takes a symmetric positive definite matrix \( H \) and then add an antisymmetric matrix \( S \), then \( A = H + S \) is positive semidefinite matrix.

In our case, we can take \( A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

In this case \( A \) is positive semidefinite nonsingular, \( B \) is positive semidefinite, and \( AB \) does not have real nonnegative eigenvalues.
2. (a) Suppose $A$ and $B$ are real-valued symmetric $n \times n$ matrices. Show that $|\text{trace}(AB)| \leq \sqrt{\text{trace}(A^2)\text{trace}(B^2)}$. What are the conditions for equality to hold? (10 pts)

(b) Let $A$ be a real $m \times n$ matrix. Show that

$$\sqrt{\text{trace}(AA^T)} \leq \text{trace}\left(\sqrt{AA^T}\right).$$

When does equality hold? (10 pts)

Solution

(a) By the Cauchy-Schwarz Theorem,

$$|\text{trace}(AB)| = \sum_{i,j} a_{ij} b_{ij} \leq \sqrt{\sum_{i,j} a_{ij}^2} \sqrt{\sum_{i,j} b_{ij}^2} = \sqrt{\text{trace}(A^2)\text{trace}(B^2)}.$$

For equality to hold, one of the matrices has to be a scalar multiple of the other.

(b) Let $AA^T = P^TDP$, where $D$ represents a nonnegative diagonal matrix and $P$ represents an orthogonal matrix. Then

$$\text{trace}(AA^T) = \text{trace}(D) = \sum_{i} \lambda_i \leq \left(\sum_{i} \sqrt{\lambda_i}\right)^2 = (\text{trace}(D^{1/2}))^2 = (\text{trace}((AA^T)^{1/2}))^2.$$

The fact that $\sum_{i} \lambda_i \leq (\sum_{i} \sqrt{\lambda_i})^2$ comes from developing the square on the right side. Equality holds if and only if $D$ has at most one nonzero entry, so $AA^T$ has at most one nonzero eigenvalue, so $A$ has at most one nonzero singular value.
3. Let 
\[ f : \mathcal{M}_n(\mathbb{R}) \rightarrow \mathcal{M}_n(\mathbb{R}) \]
\[ A \mapsto A^T \]

(a) What are the eigenvalues of \( f \)? (10 pts)
(b) Is \( f \) diagonalizable? If yes, give a basis of eigenvectors. If no, give as many linearly independent eigenvectors as possible. (10 pts)

Solution

It is clear that \( f^2 = I \), therefore \( p(x) = (x - 1)(x + 1) \) is such that \( p(f) = 0 \). This implies that the eigenvalues of \( f \) are part of the set \( \{1, -1\} \). Also \( p(f) = 0 \) implies that \( f \) is diagonalizable since \( p \) only has single roots.

Now it is clear that any symmetric matrix is eigenvector associated with eigenvalue 1, and that an eigenvector associated with eigenvalue 1 is a symmetric matrix. If we call the subspace of symmetric matrices, \( S_n \), and \( E_1 \) the eigenspace of \( f \) associated with eigenvalue 1, we have \( S_n = E_1 \).

It is also clear that any antisymmetric matrix is eigenvector associated with eigenvalue -1, and that an eigenvector associated with eigenvalue -1 is an antisymmetric matrix. If we call the subspace of antisymmetric matrices, \( A_n \), and \( E_{-1} \) the eigenspace of \( f \) associated with eigenvalue -1, we have \( A_n = E_{-1} \).

We know that
\[ \mathcal{M}_n = S_n \oplus A_n. \]

Therefore we can diagonalize \( f \) by taking a basis of \( S_n \) and a basis of \( A_n \) to form a basis of \( \mathcal{M}_n \).
4. Define the $n \times n$ matrix

$$A_n = \begin{bmatrix}
a + b & b & b & \ldots & b & b \\
a & a + b & b & \ddots & b & b \\
a & a & a + b & \ddots & b & b \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
a & a & a & \ddots & a + b & b \\
a & a & a & \ldots & a & a + b
\end{bmatrix}$$

(a) Compute $D_n = \det(A_n)$. (10 pts)

(b) Give the value of $D_n$ for $n = 10$, $a = 2$, and $b = -1$. (10 pts)

Solution

We perform (in this order) $L_n \leftarrow L_n - L_{n-1}$, then $L_{n-1} \leftarrow L_{n-1} - L_{n-2}$, ... and finally $L_2 \leftarrow L_2 - L_1$. (These transformations do not change the value of the determinant.) We get

$$D_n = \begin{vmatrix}
a + b & b & b & \ldots & b & b \\
-b & a & 0 & \ddots & 0 & 0 \\
0 & -b & a & \ddots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ddots & \ddots & a & 0 \\
0 & 0 & 0 & \ldots & -b & a
\end{vmatrix}$$

We develop with respect to last column and get

$$D_n = (-1)^{n-1}b + aD_{n-1}.$$ 

And so, we get

$$D_n = b^n + aD_{n-1}.$$ 

We have

$$D_1 = a + b.$$ 

(Note: We could get $D_1$ from $D_1 = b + aD_0$ if we define $D_0$ to be 1.)
So we get
\[ D_2 = b^2 + aD_1 = b^2 + ab + a^2. \]

Quick check:
\[ D_2 = \begin{vmatrix} a + b & b \\ a & a + b \end{vmatrix} = (a + b)^2 - ab = b^2 + ab + a^2. \]

So we get
\[ D_3 = b^3 + aD_2 = b^3 + ab^2 + a^2b + a^3 \]

Pursuing in an identical manner, we get
\[ D_n = b^n + ab^{n-1} + \ldots + a^{n-1}b + a^n = \sum_{k=0}^{n} a^k b^{n-k}. \]

We can simplify by noticing that
\[(a - b)(b^n + ab^{n-1} + \ldots + a^{n-1}b + a^n) = a^{n+1} - b^{n+1}.\]

So, if \( a \neq b \), we have
\[ D_n = \frac{a^{n+1} - b^{n+1}}{a - b}. \]

And, if \( a = b \), we get
\[ D_n = (n + 1)a^n. \]

(And we check that the latter expression for \( a = b \) is the limit of the expression for \( a \neq b \) when \( b \) goes to \( a \).)

For \( n = 10, a = -1, \) and \( b = 2, \) we get
\[ \frac{(-1)^{11} - (2)^{11}}{(-1) - 2} = \frac{2049}{3} = 683. \]
5. Suppose that \( u \) and \( v \) are vectors in a real inner product space \( V \).

(a) Prove that
\[
||u|| + ||v|| \frac{\langle u, v \rangle}{||u|| ||v||} \leq ||u + v||. \quad (10 \text{ pts})
\]

(b) Prove or disprove the following identity:
\[
(||u|| + ||v||) \frac{||u\rangle \langle v||}{||u|| ||v||} \leq ||u + v||. \quad (10 \text{ pts})
\]

Solution

(a) Case 1: \( \langle u, v \rangle \leq 0 \). The inequality follows trivially since a norm is nonnegative. Thus, the leftside is no more than 0 while the right side is no less than 0.

Case 2: \( \langle u, v \rangle > 0 \). Squaring the left side we have
\[
(||u|| + ||v||)^2 \frac{\langle u, v \rangle}{||u||^2 ||v||^2} \leq (||u||^2 + ||v||^2 + 2||u|| ||v||) \frac{\langle u, v \rangle ||u|| ||v||}{||u||^2 ||v||^2} \]
\[
= \frac{||u||}{||v||} \langle u, v \rangle + \frac{||v||}{||u||} \langle u, v \rangle + 2\langle u, v \rangle \quad (1)
\]
\[
= \frac{||u||}{||v||} ||u|| ||v|| + \frac{||v||}{||u||} ||u|| ||v|| + 2\langle u, v \rangle \quad (2)
\]
\[
= ||u + v||^2. \quad (3)
\]

Both (1) and (3) are obtained by applying the Cauchy-Schwarz inequality to \( \langle u, v \rangle \), while (2) and (4) are obtained by simplifying.

(b) Let \( u = (1, 0) \), \( v = (-1, 0) \), and use a Euclidean inner product (dot product).
Then the left side of the inequality becomes \( (1 + 1) \frac{1}{1(1)} = 1 \) while the right side is 0. (Note: one can also use one-dimensional vector: \( u = (1), v = (-1) \).)
6. Let $V$ be a vector space. Let $f \in \mathcal{L}(V)$. Let $p$ be a projection (so $p \in \mathcal{L}(V)$ and is such that $p^2 = p$). Prove that

$$\text{Null}(f \circ p) = \text{Null}(p) \oplus (\text{Null}(f) \cap \text{Range}(p)).$$

(20 pts)

**Solution**

Firstly, we would like to prove that

$$\text{Null}(p) \oplus (\text{Null}(f) \cap \text{Range}(p)) \subset \text{Null}(f \circ p).$$

Note: We recall that if $A$, $B$ and $C$ are subspaces, to prove that $A + B \subset C$, we just need to prove that $A \subset C$ and $B \subset C$.

Let $x \in \text{Null}(p)$, then $p(x) = 0$, so $(f \circ p)(x) = 0$, so $x \in \text{Null}(f \circ p)$.

Let $x \in \text{Null}(f) \cap \text{Range}(p)$. Since $x \in \text{Range}(p)$, there exists $y$ such that $x = p(y)$. Since $x \in \text{Null}(f)$, we have $f(x) = 0$. Now let us look at $(f \circ p)(x)$. (Note: we want to prove that $(f \circ p)(x) = 0$.) We have $(f \circ p)(x) = (f \circ p)(p(y)) = f(p^2(y)) = f(p(y)) = f(x) = 0$, We have used the facts that $1 \rightarrow 2$: $x = p(y)$, $3 \rightarrow 4$: $p^2 = p$, $4 \rightarrow 5$: $p(y) = x$, $5 \rightarrow 6$: $f(x) = 0$. This proves that $x \in \text{Null}(f \circ p)$.

We proved that

$$(\text{Null}(p) + (\text{Null}(f) \cap \text{Range}(p))) \subset \text{Null}(f \circ p).$$

Secondly, we would like to prove that

$$\text{Null}(f \circ p) \subset \text{Null}(p) \oplus (\text{Null}(f) \cap \text{Range}(p)).$$

Let $x \in \text{Null}(f \circ p)$, we can write $x$ as

$$x = (x - p(x)) + p(x),$$

where

(a) $$(x - p(x)) \in \text{Null}(p).$$ Indeed, $p(x - p(x)) = p(x) - p^2(x)$, but $p = p^2$ so $p(x - p(x)) = 0$, so $(x - p(x)) \in \text{Null}(p)$.

(b) $$p(x) \in \text{Null}(f) \cap \text{Range}(p).$$ It is a fact that $p(x) \in \text{Range}(p)$. Moreover, since $x \in \text{Null}(f \circ p)$, we have that $(f \circ p)(x) = 0$, which proves that $p(x) \in \text{Null}(f)$. So $p(x) \in \text{Null}(f) \cap \text{Range}(p)$. 

Therefore we have that

$$\text{Null}(f \circ p) \subset \text{Null}(p) + (\text{Null}(f) \cap \text{Range}(p)) .$$

At this point, we proved that

$$\text{Null}(f \circ p) = \text{Null}(p) + (\text{Null}(f) \cap \text{Range}(p)) .$$

It remains to prove that the sum is direct. Let $x \in \text{Null}(p) \cap (\text{Null}(f) \cap \text{Range}(p))$, then $x \in \text{Range}(p)$, so there exists $u \in V$ such that $x = p(u)$, but $x \in \text{Null}(p)$, so $p(x) = 0$, so $p^2(u) = 0$, but $p^2 = p$, so $p(u) = 0$, so $x = 0$. We proved that $\text{Null}(p) \cap (\text{Null}(f) \cap \text{Range}(p)) = \{0\}$ so the sum in the previous paragraph is direct.

We are done and we can conclude that

$$\text{Null}(f \circ p) = \text{Null}(p) \oplus (\text{Null}(f) \cap \text{Range}(p)) .$$
7. (a) Let $n \in \mathbb{N}\setminus\{0,1\}$ (so $n \geq 2$) and $A \in \mathcal{M}_n(\mathbb{C})$ such that $\text{rank}(A) = 1$. Prove that $A$ is diagonalizable if and only if $\text{trace}(A) \neq 0$. (10 pts)

(b) Let $a_1, \ldots, a_n \in \mathbb{C}\setminus\{0\}$, (so the $a_i$’s are nonzero complex numbers,) and $A$ such that $A = \left( \frac{a_i}{a_j} \right)_{1 \leq i,j \leq n}$. (This means that the entry $(i,j)$ of $A$ is $\frac{a_i}{a_j}$.) Show that $A$ is diagonalizable. Give a basis of eigenvectors (with the associated eigenvalues) for $A$. (10 pts)

Solution

(a) First we note that $\text{rank}(A) = 1 \iff \text{dim}(\text{Null}(A)) = n - 1$ (by the rank theorem). So, if $\text{rank}(A) = 1$ and $n \geq 2$, then $\text{dim}(\text{Null}(A)) \geq 1$ and so 0 is an eigenvalue of $A$. We call $\nu_0$ the geometric multiplicity of the eigenvalue 0, and $\mu_0$ the algebraic multiplicity of the eigenvalue 0. We call $E_0$ the eigenspace associated with the eigenvalue 0. Now, since $\text{dim}(\text{Null}(A)) = n - 1$, we have that $\text{dim}(E_0) = n - 1$, or in other words, the geometric multiplicity of the eigenvalue 0, $\nu_0$, is $n - 1$. We know that, for a given eigenvalue, the algebraic multiplicity is always greater than or equal to the geometric multiplicity. For the eigenvalue 0, this reads: $\nu_0 \leq \mu_0$. For a rank–1 matrix, there are therefore only two cases: either $\nu_0 = \mu_0 = n - 1$, or $\nu_0 = n - 1$, $\mu_0 = n$.

[case $\nu_0 = \mu_0 = n - 1$] In this case, since $\mu_0 = n - 1$, there has to exist another eigenvalue $\lambda$ different from zero. (Because the sum of the algebraic multiplicities of the eigenvalues has to sum to $n$.) For that eigenvalue $\lambda$, the geometric multiplicity, $\nu_\lambda$, is at least 1, but can be no more than 1 (because $\nu_0 = n - 1$ and the sum of the algebraic multiplicities of two distinct eigenvalues has to be less than $n$). So $\nu_\lambda = 1$. So we have $\nu_\lambda = 1$ and $\nu_0 = n - 1$, so $A$ is diagonalizable. We also note that, in this case, $\text{trace}(A) = \lambda$, (the trace is the sum of the eigenvalues counted with their multiplicities,) and so, in this case, $\text{trace}(A) \neq 0$.

[case $\nu_0 = n - 1, \mu_0 = n$] In this case, since $\mu_0 = n$, $A$ only has the eigenvalue 0. We also have that $A$ is not diagonalizable and that $\text{trace}(A) = 0$.

Starting from a rank–1 matrix, we found two possibilities. Either $\nu_0 = \mu_0 = n - 1$, in which case, $A$ is diagonalizable and $\text{trace}(A) \neq 0$. Or $\nu_0 = n - 1, \mu_0 = n$, in which case, $A$ is not diagonalizable and $\text{trace}(A) = 0$.

This enables us to conclude that for a rank–1 matrix

$A$ is diagonalizable $\iff$ $\text{trace}(A) \neq 0$. 


(b) We observe that the matrix is of rank 1. Indeed

\[
A = \begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_{n-1} \\
a_n
\end{pmatrix}
\begin{pmatrix}
\frac{1}{a_1} & \frac{1}{a_2} & \cdots & \frac{1}{a_{n-1}} & \frac{1}{a_n} \\
\end{pmatrix}.
\]

We also have \(\text{trace}(A) = n\). So by the previous question, we see that \(A\) is diagonalizable (since \(\text{trace}(A) \neq 0\)). We also see that \(A\) has eigenvalue 0 with geometric multiplicity \(n-1\) and eigenvalue \(n\) with geometric multiplicity 1.

**eigenvalue 0** To find \(n-1\) linearly independent eigenvectors associated with eigenvalue 0, we want to find a basis for the null space of \(A\), which is same as null space of

\[
\begin{pmatrix}
\frac{1}{a_1} & \frac{1}{a_2} & \cdots & \frac{1}{a_{n-1}} & \frac{1}{a_n} \\
\end{pmatrix}.
\]

We have (for example) that \(x_1\) is a leading variable, and that \(x_2, x_3, \ldots, x_n\) are free variables. This gives for a general solution:

\[
\begin{pmatrix}
-a_1 x_2 - \frac{a_1}{a_3} x_3 - \cdots - \frac{a_1}{a_{n-1}} x_{n-1} - \frac{a_1}{a_n} x_n \\
x_2 \\
x_3 \\
\vdots \\
x_{n-1} \\
x_n
\end{pmatrix} =
\begin{pmatrix}
-a_1 \\
1 \\
0 \\
\vdots \\
0 \\
0
\end{pmatrix} x_2 +
\begin{pmatrix}
-a_1 \\
0 \\
0 \\
\vdots \\
0 \\
0
\end{pmatrix} x_3 + \cdots +
\begin{pmatrix}
-a_1 \\
0 \\
0 \\
\vdots \\
0 \\
1
\end{pmatrix} x_{n-1} +
\begin{pmatrix}
-a_1 \\
0 \\
0 \\
\vdots \\
0 \\
0
\end{pmatrix} x_n.
\]

So a basis for \(E_0\) is for example

\[
v_1 = \begin{pmatrix}
-a_1 \\
a_2 \\
0 \\
\vdots \\
0 \\
0
\end{pmatrix}, \ v_2 = \begin{pmatrix}
-a_1 \\
0 \\
0 \\
\vdots \\
0 \\
0
\end{pmatrix}, \ \ldots \ v_{n-2} = \begin{pmatrix}
-a_1 \\
0 \\
0 \\
\vdots \\
0 \\
1
\end{pmatrix}, \ v_{n-1} = \begin{pmatrix}
-a_1 \\
0 \\
0 \\
\vdots \\
0 \\
0
\end{pmatrix}.
\]

**eigenvalue n** We see that an eigenvector for eigenvalue \(n\) is for example

\[
v_n = \begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_{n-1} \\
a_n
\end{pmatrix}.
\]

**Answer:** The above given \((v_1, \ldots, v_n)\) is a basis of \(\mathbb{C}^n\) made of eigenvectors of \(A\).