University of Colorado Denver  
Department of Mathematical and Statistical Sciences  
Applied Linear Algebra Ph.D. Preliminary Exam  
January 13, 2014

Name: ___________________________________________

Exam Rules:

• This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.

• Each problem is worth 20 points.

• Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.

• If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.

• Begin each solution on a new page and use additional paper, if necessary.

• Write only on one side of paper.

• Write legibly using a dark pencil or pen.

• Ask the proctor if you have any questions.

Good luck!

1. ___________ 5. ___________
2. ___________ 6. ___________
3. ___________ 7. ___________
4. ___________ 8. ___________

Total ___________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
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Let $A$ be a full column rank $n$-by-$k$ matrix (so $k \leq n$) and $b$ to be a column vector of size $n$. We want to minimize the squared Euclidean norm $L(x) := ||Ax - b||_2^2$ with respect to $x$.

(a) Prove that, if rank($A$) = $k$, then $A^T A$ is invertible.

(b) Compute the gradient of $L(x)$.

(c) Directly derive the normal equations by minimizing $L(x)$, and then provide the closed-form expression for $x$ that minimizes $L(x)$.

(d) We consider a QR factorization of $A$ where $Q$ is $n$-by-$k$ and $R$ is $k$-by-$k$. Show that an equivalent solution for $x$ is $x = R^{-1}Q^T b$.

Solution

(a) Let $x$ such that $A^T Ax = 0$, then $x^T A^T Ax = 0$ so that $||Ax||^2 = 0$ so that $Ax = 0$. But, since $A$ is full column rank, Null($A$) = $\{0\}$, so that $Ax = 0 \Rightarrow x = 0$. We proved that $A^T Ax = 0 \Rightarrow x = 0$. Since $A^T A$ is square, this means that $A^T A$ is invertible.

(b) The gradient of $L(x) = (Ax - b)^T (Ax - b) = x^T A^T Ax - 2x^T bA^T b + b^T b$ is $\nabla L(x) = 2A^T Ax - 2A^T b$.

(c) Setting the gradient to zero, we get the normal equations $A^T Ax = A^T b$, by question (a), we know that $A^T A$ is invertible, the unique solution of the normal equations is obtained as $x = (A^T A)^{-1} A^T b$.

(d) The QR factorization of $A$ has the property $A = QR$, with $Q^T Q = I$. (We note that $R$ is upper triangular but this does not matter here.) Starting from the normal equations in (a), we have $R^T Q^T Q Rx = R^T Q^T b$, which simplifies to $R^T Rx = R^T Q^T b$ since $Q^T Q = I$. We note that, since $A$ has full column rank, this means that $R$ is invertible. (Proof. By contrapositive. Assume $R$ is not invertible, then there exists $x$ nonzero such that $Rx = 0$, so that $Q Rx = 0$ so that $Ax = 0$ (with $x$ nonzero) so dim(Null($A$)) > 0 so Rank($A$) < $k$ so $A$ is not full column rank.) Since $R$ is invertible, (so is $R^T$,) from $R^T Rx = R^T Q^T b$, we get $x = R^{-1}Q^T b$. 

2. Let $V$ be a real vector space.

(a) Give the definition of a real inner product $\langle \cdot, \cdot \rangle$ over the vector space $V$. (That is the set of properties from the definition of a real inner product.)

We define $\|x\|$ as $\|x\| = \sqrt{\langle x, x \rangle}$.

(b) From these two definitions, state and prove the Cauchy-Schwarz inequality.

(c) Now, state and prove the triangular inequality.

(d) Now, prove that $\|x\|$ is a norm.

Solution

(a) A real inner product on $V$ is a function from $V^2$ to $\mathbb{R}$ with the following properties:

i. for all $x$ in $V$, $\langle x, x \rangle \geq 0$,
ii. $\langle x, x \rangle = 0$ if and only if $x = 0$,
iii. for all $x$ in $V$, for all $y$ in $V$, $\langle x, y \rangle = \langle y, x \rangle$,
iv. for all $x$ in $V$, for all $y$ in $V$, for all $z$ in $V$, $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$,
v. for all $\alpha$ in $\mathbb{R}$, for all $x$ in $V$, for all $y$ in $V$, $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$.

(b) We note that by property (i) above, for all $x$ in $V$, $\langle x, x \rangle \geq 0$, and so $\|x\| = \sqrt{\langle x, x \rangle}$ is well defined for $x$ in $V$.

The Cauchy-Schwarz inequality states that, for all $u$ and all $v$, we have

$$|\langle u, v \rangle| \leq \|u\| \|v\|.$$ 

Now we write that

$$0 \leq \langle \|u\|v - \|v\|u, \|u\|v - \|v\|u \rangle$$

$$= \|u\|^2 \langle v, v \rangle - 2 \|u\| \|v\| \langle u, v \rangle + \|v\|^2 \langle u, u \rangle$$

$$= 2 \|u\|^2 \|v\|^2 - 2 \|u\| \|v\| \langle u, v \rangle.$$

Rearranging yields

$$2 \|u\| \|v\| \langle u, v \rangle \leq 2 \|u\|^2 \|v\|^2$$

$$\langle u, v \rangle \leq \|u\| \|v\|.$$

We can apply the same reasoning to $-u$ instead of $u$ and we obtain the Cauchy-Schwarz inequality.

(c) The triangle inequality states that, for all $u$ and all $v$, we have

$$\|u + v\| \leq \|u\| + \|v\|.$$
Note that
\[ \|u + v\|^2 = \langle u + v, u + v \rangle = \|u\|^2 + 2\langle u, v \rangle + \|v\|^2 \leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \]
by Cauchy-Schwarz inequality
\[ = (\|u\| + \|v\|)^2 \]
and the inequality follows by taking the square root of both sides.

(d) A norm is a function from \( V \) to \( \mathbb{R} \) with the following properties:

i. for all \( x \) in \( V \), \( \|x\| = 0 \Rightarrow x = 0 \),
ii. for all \( x \) in \( V \), for all \( \alpha \) in \( \mathbb{R} \), \( \|\alpha x\| = |\alpha|\|x\| \),
iii. for all \( x \) in \( V \), for all \( y \) in \( V \), \( \|x + y\| \leq \|x\| + \|y\| \).

Property (2.d.i) comes from property (2.a.ii). Property (2.d.ii) comes from property (2.a.iii) and property (2.a.v). Property (2.d.iii) is the triangular inequality which we prove in (2.c).
3. Suppose $A$ is a positive definite symmetric real $n$–by–$n$ matrix and $B$ is a real $m$–by–$n$ matrix such that $BB^T$ is positive definite. Prove that the matrix $B^T(BA^{-1}B^T)^{-1}B$ is symmetric positive definite.

Solution
Since $A$ is positive definite, $A^{-1}$ is positive definite. For $x \in \mathbb{R}^m$, $B^T x = 0 \in \mathbb{R}^n$ if and only if $x = 0$. (If $B^T x = 0$ for $x \neq 0$, then $BB^T x = 0$ which is impossible by $BB^T$ being positive definite.) Hence, $x^TBA^{-1}B^T x = 0$ if and only if $x = 0$, so $BA^{-1}B^T$ is positive definite. Therefore, $(BA^{-1}B^T)^{-1}$ is positive definite which implies, as before that $B^T(BA^{-1}B^T)^{-1}B$ is positive definite.
4. Suppose $A$ is a positive definite symmetric square real matrix and $B$ is a symmetric square real matrix. Show that there exists a square real matrix $C$ such that $C^TAC$ is the identity matrix and $C^TBC$ is a diagonal matrix.

**Solution**

Let $C_1 = A^{1/2}$. Then $C_1^{-1}AC_1^{-1}$ is the identity matrix and $C_1^{-1}BC_1^{-1}$ is symmetric. We can write $C_1^{-1}BC_1^{-1} = PD^P^T$, where $D$ is diagonal and $P$ is orthogonal. Then $D = (P^T C_1^{-1}) B (C_1^{-1} P)$ and $(P^T C_1^{-1}) A (C_1^{-1} P) = P^T (C_1^{-1} AC_1^{-1}) P$ is the identity matrix. Thus, one can take $C = C_1^{-1} P$. 

5. Let $\mathcal{P}_n$ represent the real vector space of polynomials in $x$ of degree less than or equal to $n$ defined on $[0, 1]$. Given a real number $a$, we define $Q_n(a)$ the subset of $\mathcal{P}_n$ of polynomials that have the real number $a$ as a root.

(a) Let $a$ be a real number. Show that $Q_n(a)$ is a subspace of $\mathcal{P}_n$. Determine the dimension of that subspace and exhibit a basis.

(b) Let the inner product in $\mathcal{P}_n$ be defined by $\langle p, q \rangle = \int_0^1 p(x)q(x)dx$. Determine the orthogonal complement of the subspace $Q_2(1)$ of $\mathcal{P}_2$.

Solution

(a) Polynomials in $Q_n(a)$ can be written as $p(x) = (x - a)q(x)$ where $q(x)$ is a polynomial of degree less than or equal to $n - 1$. The definition of a subspace is verified routinely. Since $Q_n(a)$ is isomorphic with $\mathcal{P}_{n-1}$, its dimension is $n$, $\{(x - a), (x - a)^2, \ldots, (x - a)^n\}$ is a basis.

(b) We can write a polynomial in $\mathcal{P}_2$ as $a_0 + a_1(x - 1) + a_2(x - 1)^2$. We need a polynomial orthogonal to $x - 1$ and $(x - 1)^2$, so

\[
\int_0^1 (a_0 + a_1(x - 1) + a_2(x - 1)^2)(x - 1)dx = 0,
\]

\[
\int_0^1 (a_0 + a_1(x - 1) + a_2(x - 1)^2)(x - 1)^2dx = 0,
\]

which yields

\[
\begin{align*}
-\frac{a_0}{2} + \frac{a_1}{3} - \frac{a_2}{4} &= 0, \\
\frac{a_0}{3} - \frac{a_1}{4} + \frac{a_2}{5} &= 0,
\end{align*}
\]

so

\[
\begin{pmatrix}
a_0 \\
a_1 \\
a_2
\end{pmatrix}
= a_2
\begin{pmatrix}
3/10 \\
6/5 \\
1
\end{pmatrix}
\]

Thus, $Q_2(a) \perp = \{3a_2 + 12a_2(x - 1) + 10a_2(x - 1)^2, a_2 \in \mathbb{R}\}$. 
6. Let $F$ be a commutative field, let $(V, +, \cdot)$ be a vector space over $F$, let $A$ and $B$ be two subspaces of $V$, let $A'$ be a subspace such that $A' \oplus (A \cap B) = A$ and let $B'$ be a subspace such that $B' \oplus (A \cap B) = B$. Show that $A + B = (A \cap B) \oplus A' \oplus B'$.

**Solution**

One can write

$$A + B = (A' + (A \cap B)) + (B' + (A \cap B)) = A' + B' + (A \cap B).$$

So the real question is not about the sum but about the direct sum of $(A \cap B)$, $A'$, and $B'$.

Let $x \in (A \cap B)$, $a' \in A'$, $b' \in B'$ such that

$$x + a' + b' = 0.$$

Then, on the one hand, $b' \in B'$ but $B' \subset B$, so $b' \in B$, on the other hand, $b' = -x - a'$, but $x \in A$ (since $x \in (A \cap B)$), and $a' \in A$ (since $a' \in A'$ and $A' \subset A$), so $b' \in A$. We see that $b' \in (A \cap B))$. However, we also have that $b' \in B'$. Therefore $b' \in (A \cap B) \cap B'$. But $(A \cap B)$ and $B'$ are in direct sum so $(A \cap B) \cap B' = \{0\}$, so $b' = 0$.

Now we have

$$x + a' = 0.$$

$x \in (A \cap B)$, $a' \in A'$, but, since $(A \cap B)$ and $A'$ are in direct sum, $x = 0$ and $a' = 0$. We prove that $x = 0$, $a' = 0$, and $b' = 0$. Therefore $(A \cap B)$, $A'$, and $B'$ are in direct sum and

$$A + B = (A \cap B) \oplus A' \oplus B'.$$
7. Let $\mathbb{F}$ be a commutative field, let $(V, +, \cdot)$ be a vector space over $\mathbb{F}$, let $n$ be a natural number, let $(e_1, \ldots, e_n)$ be a linear independent list in $V$, let $\lambda_1, \ldots, \lambda_n$ be $n$ scalars in $\mathbb{F}$, let $u = \sum_{i=1}^{n} \lambda_i e_i$, and let, for all $i = 1, \ldots, n$, $v_i = u + e_i$. Show that $(v_1, \ldots, v_n)$ is linearly dependent if and only if $\sum_{i=1}^{n} \lambda_i = -1$.

**Solution**

First, let us that assume $(v_1, \ldots, v_n)$ is linearly dependent, then there exists $n$ scalars $\alpha_1, \ldots, \alpha_n$, not all zeros such that,

$$\sum_{i=1}^{n} \alpha_i v_i = 0.$$ 

Since, for all $i = 1, \ldots, n$, $v_i = u + e_i$, we have

$$\sum_{i=1}^{n} \alpha_i (u + e_i) = 0.$$ 

We split the $i$ sum in two sums:

$$\left( \sum_{i=1}^{n} \alpha_i u \right) + \left( \sum_{i=1}^{n} \alpha_i e_i \right) = 0.$$ 

Now, we use the fact that $u = \sum_{j=1}^{n} \lambda_j e_j$:

$$\left( \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \lambda_j e_j \right) + \left( \sum_{i=1}^{n} \alpha_i e_i \right) = 0.$$ 

Now, we swap the $i$ and the $j$ sum on the left term and change the dummy index $i$ to a $j$ in the right term:

$$\left( \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_i \lambda_j e_j \right) + \left( \sum_{j=1}^{n} \alpha_j e_j \right) = 0.$$ 

We merge the two $j$ sums and factor the $e_j$ term:

$$\sum_{j=1}^{n} \left( \left( \sum_{i=1}^{n} \alpha_i \right) \lambda_j + \alpha_j \right) e_j = 0.$$ 

(1)

The latter expression reads now as a zero linear combination of the $e_j$. Since the $e_j$ are linear independent, each of the coefficients in the linear combination has to be 0, this writes:

$$\left( \sum_{i=1}^{n} \alpha_i \right) \lambda_j + \alpha_j = 0, \text{ for } j = 1, \ldots, n.$$
We can take the sum for \( j = 1 \) to \( n \) of these \( n \) expressions and we get:

\[
\sum_{j=1}^{n} \left[ (\sum_{i=1}^{n} \alpha_i) \lambda_j + \alpha_j \right] = 0.
\]

We break the sum in two:

\[
\sum_{j=1}^{n} \left[ (\sum_{i=1}^{n} \alpha_i) \lambda_j \right] + \sum_{j=1}^{n} \alpha_j = 0.
\]

We factor the \( \sum_{i=1}^{n} \alpha_i \) on the left term:

\[
(\sum_{i=1}^{n} \alpha_i)(\sum_{j=1}^{n} \lambda_j) + \sum_{j=1}^{n} \alpha_j = 0.
\]

We get

\[
(\sum_{i=1}^{n} \alpha_i) \left( 1 + \sum_{j=1}^{n} \lambda_j \right) = 0. \tag{2}
\]

Now we come back to Equation (1), it read

\[
\sum_{j=1}^{n} \left[ (\sum_{i=1}^{n} \alpha_i) \lambda_j + \alpha_j \right] e_j = 0.
\]

We see that, if \( \sum_{i=1}^{n} \alpha_i = 0 \), then \( \sum_{j=1}^{n} \alpha_j e_j = 0 \), which would imply that the \( e_j \) are linearly dependent. Therefore, since the \( e_j \) are linearly independent, we have that \( \sum_{i=1}^{n} \alpha_i \neq 0 \). Now we see that \( \sum_{i=1}^{n} \alpha_i \neq 0 \) and Equation (2) implies

\[
\sum_{j=1}^{n} \lambda_j = -1.
\]

This proves that, if \((v_1, \ldots, v_n)\) is linearly dependent, then \(\sum_{j=1}^{n} \lambda_j = -1\).

Now, let us assume that \(\sum_{j=1}^{n} \lambda_j = -1\). We want to prove that \((v_1, \ldots, v_n)\) is linearly dependent. That is, we want to find \(\alpha_i, i = 1, \ldots, n\), not all zeros, such that

\[
\sum_{i=1}^{n} \alpha_i v_i = 0.
\]

We will prove that a correct choice for the \(\alpha_i\) is \(\alpha_i = \lambda_i\). First note that the \(\lambda_i\) are
not all zeros since $\sum_{i=1}^{n} \lambda_i = -1$. Second:

$$\sum_{i=1}^{n} \lambda_i v_i = \sum_{i=1}^{n} \lambda_i (u + e_i),$$

$$= \sum_{i=1}^{n} (\lambda_i u) + \sum_{i=1}^{n} (\lambda_i e_i),$$

$$= \sum_{i=1}^{n} (\lambda_i (\sum_{j=1}^{n} \lambda_j e_j)) + \sum_{i=1}^{n} (\lambda_i e_i),$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda_i \lambda_j e_j) + \sum_{i=1}^{n} (\lambda_i e_i),$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda_i \lambda_j e_j) + \sum_{i=1}^{n} (\lambda_i e_i),$$

$$= \sum_{j=1}^{n} \left( (\sum_{i=1}^{n} \lambda_i) \lambda_j e_j \right) + \sum_{i=1}^{n} (\lambda_i e_i),$$

$$= (\sum_{i=1}^{n} \lambda_i) \sum_{j=1}^{n} (\lambda_j e_j) + \sum_{i=1}^{n} (\lambda_i e_i),$$

$$= (-1) \sum_{j=1}^{n} (\lambda_j e_j) + \sum_{i=1}^{n} (\lambda_i e_i),$$

$$= 0.$$

This proves that $(v_1, \ldots, v_n)$ is linearly dependent.
8. What is the rank of 
\[
\begin{pmatrix}
1 & a & 1 & b \\
1 & b & 1 & a \\
1 & b & 1 & a \\
1 & a & 1 & 1
\end{pmatrix}
\]?

The rank is a function of \(a\) and \(b\). You need to give the values of the rank for all values of \((a, b) \in \mathbb{R}^2\).

**Solution**

We perform some Gaussian elimination steps.

First, \(L_2 \leftarrow L_2 - aL_1, L_3 \leftarrow L_3 - L_1, L_4 \leftarrow L_4 - bL_1\) gives
\[
\begin{pmatrix}
1 & a & 1 & b \\
0 & 1 & 0 & -1 \\
0 & 1 - a^2 & b - a & 1 - ab \\
0 & 1 - ab & a - b & 1 - b^2
\end{pmatrix}
\]

We assume \(a \neq b\) so that we can simplify the third row with \(L_3 \leftarrow L_3/(b - a)\), after this we swap second and third row \(L_2 \leftrightarrow L_3\). This gives:
\[
\begin{pmatrix}
1 & a & 1 & b \\
0 & 1 & 0 & -1 \\
0 & 1 & 0 & -1 \\
0 & 1 - ab & a - b & 1 - b^2
\end{pmatrix}
\]

Now, \(L_3 \leftarrow L_3 - (1 - a^2)L_1, L_4 \leftarrow L_4 - (1 - ab)L_1\), gives
\[
\begin{pmatrix}
1 & a & 1 & b \\
0 & 1 & 0 & -1 \\
0 & 0 & b - a & 2 - a^2 - ab \\
0 & 0 & a - b & 2 - b^2 - ab
\end{pmatrix}
\]

Finally \(L_4 \leftarrow L_4 + L_3\), gives
\[
\begin{pmatrix}
1 & a & 1 & b \\
0 & 1 & 0 & -1 \\
0 & 0 & b - a & 2 - a^2 - ab \\
0 & 0 & 4 - (a + b)^2
\end{pmatrix}
\]

So we see that (1) if \(a \neq b\) and \(a + b \neq \pm 2\), then the rank is 4. (2) if \(a \neq b\), and \(a + b = \pm 2\), then the rank is 3.

Now let us see to the case when \(a = b\). In this case, the matrix is:
\[
\begin{pmatrix}
1 & a & 1 & a \\
1 & a & 1 & a \\
1 & a & 1 & a \\
1 & a & 1 & a
\end{pmatrix}
\].
It is clear that if $a = 1$ then the rank is 1, if $a \neq 1$, the rank is 2.

Let us repeat:

(a) If $a = b = 1$, then the rank is 1,
(b) If $a = b$ and $a \neq 1$, then the rank is 2,
(c) If $a \neq b$ and $a + b = \pm 2$, then the rank is 3,
(d) If $a \neq b$ and $a + b \neq \pm 2$, then the rank is 4.