Name: _______________________________________

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

     Good luck!

1. __________  5. __________
2. __________  6. __________
3. __________  7. __________
4. __________  8. __________

    Total __________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
Steve Billups (Chair), Alexander Engau, Julien Langou.
1. Find an orthogonal basis for the space $P_2$ of quadratic polynomials with the inner product $\langle f, g \rangle = f(-1)g(-1) + f(0)g(0) + f(1)g(1)$.

**Solution**

Two ways.

First way. Take a first nonzero quadratic polynomial, $x(x + 1)$, whose value is 0 in -1 and 0, and nonzero in 1; a second polynomial, $(x - 1)(x + 1)$, whose value is 0 in -1 and 1, and nonzero in 0; and a third polynomial, $x(x - 1)$, whose value is 0 in 0 and 1, and nonzero in -1. Then it is easy to see that these three polynomials are orthogonal with respect to the given scalar product. We just need to normalize accordingly. We find:

$$\frac{\sqrt{2}}{2}x(x + 1), \quad (x - 1)(x + 1), \quad \frac{\sqrt{2}}{2}x(x - 1).$$

Second way. We can use the Gram-Schmidt process on three linearly independent vectors in $P_2$, for example: 1, $x$, and $x^2$. 
2. A real \( n \times n \) matrix \( A \) is an isometry if it preserves length: \( \|Ax\| = \|x\| \) for all vectors \( x \in \mathbb{R}^n \). Show that the following are equivalent.

(a) \( A \) is an isometry (preserves length).
(b) \( \langle Ax, Ay \rangle = \langle x, y \rangle \) for all vectors \( x, y \), so \( A \) preserves inner products.
(c) \( A^{-1} = A^* \).
(d) The columns of \( A \) are unit vectors that are mutually orthogonal.

Solution

(b)\( \Rightarrow \) (a). Trivial since \( \|x\| \) is defined as \( \sqrt{\langle x, x \rangle} \). So if an application preserves inner products, it preserves length.

(a)\( \Rightarrow \) (b). Assume that \( A \) preserves lengths. Let \( x \) and \( y \in \mathbb{R}^n \). We have \( \|A(x + y)\|^2 = \|(x + y)\|^2 \). Let us consider \( \|A(x + y)\|^2 - \|(x + y)\|^2 = \langle A(x + y), A(x + y) \rangle - \langle x + y, x + y \rangle = \langle Ax, Ax \rangle + \langle Ax, Ay \rangle + \langle Ay, Ax \rangle - \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle - \langle y, y \rangle \). We note that \( \langle Ax, Ay \rangle = \langle Ay, Ax \rangle \) (symmetry of the inner product) and that \( \langle Ay, Ay \rangle = \|y\|^2 \). All in all, we obtain that \( \|A(x + y)\|^2 - \|(x + y)\|^2 = 2\langle Ax, Ay \rangle - 2\langle x, y \rangle \). Setting this to zero implies: \( \langle Ax, Ay \rangle = \langle x, y \rangle \). Therefore \( A \) preserves inner products.

We proved that (a)\( \iff \) (b).

(c)\( \Rightarrow \) (b). Assume \( A^{-1} = A^* \). Let \( x \) and \( y \in \mathbb{R}^n \). \( \langle Ax, Ay \rangle = \langle A^*Ax, y \rangle = \langle A^{-1}Ax, y \rangle = \langle x, y \rangle \). So \( A \) preserves inner products.

(b)\( \Rightarrow \) (d). Assume \( A \) preserves inner products. Let \( a_j \) be the \( j \)th column of \( A \). Then \( \langle a_i, a_j \rangle = \langle Ae_i, Ae_j \rangle = \langle e_i, e_j \rangle \). This proves that the columns of \( A \) are unit vectors that are mutually orthogonal.

(d)\( \Rightarrow \) (c). Assume that the columns of \( A \) are unit vectors that are mutually orthogonal. Let \( a_j \) be the \( j \)th column of \( A \). This means that \( \langle a_j, a_j \rangle = 1 \) and for \( i \neq j \), \( \langle a_i, a_j \rangle = 0 \). We know that \( A^* = A^H \), we know that \( (A^H A)_{ij} = a_i^H a_j = \langle a_i, a_j \rangle \), so \( A^* A = A^H A = I \). So \( A^* = A^{-1} \).

We proved that (b)\( \iff \) (c)\( \iff \) (d).
3. Let \( p \geq q \). Let \( A \) be a real \( p \times q \) matrix with rank \( q \). Prove that the QR-decomposition \( A = QR \) is unique if \( R \) is forced to have positive entries on its main diagonal, \( Q \) is \( p \times q \) and \( R \) is \( q \times q \).

**Solution**

Assume that \( A = Q_1R_1 \) and \( A = Q_2R_2 \) with \( R_1, R_2 \) upper triangular with positive entries on the diagonal and \( Q_1^TQ_1 = I_q \) and \( Q_2^TQ_2 = I_q \).

We first note that since \( A \) is full rank, \( R_1 \) and \( R_2 \) are invertible. We have \( Q_1R_1 = Q_2R_2 \), multiplying by \( Q_1^T \) and \( R_2^{-1} \), this gives

\[
R_1R_2^{-1} = Q_1^TQ_2.
\]

This means that \( Q_1^TQ_2 \) is upper triangular. Now multiplying by \( Q_2^T \) and \( R_1^{-1} \), this gives

\[
R_2R_1^{-1} = Q_2^TQ_1.
\]

This means that \( Q_2^TQ_1 \) is upper triangular. So \( Q_1^TQ_2 \) is lower triangular. \( Q_1^TQ_2 \) is upper and lower triangular. So it is diagonal (and invertible).

Let us call \( D = Q_1^TQ_2 \), (from \( R_1R_2^{-1} = Q_1^TQ_2 \),) we see that \( R_1 = DR_2 \). From \( Q_1R_1 = Q_2R_2 \), we see that \( Q_1 = Q_2D^{-1} \). So now \( Q_1^TQ_1 = I \) and \( Q_2^TQ_2 = I \) give \( D^2 = I \). \( D \) has therefore \( \pm 1 \) on the diagonal.

We come back to the relation \( R_1 = DR_2 \). Since the diagonal entry of \( R_1 \) are given by \( (R_1)_{ii} = D_{ii}(R_2)_{ii} \) and that \( (R_1)_{ii} \) and \( (R_2)_{ii} \) are both positive, and that \( D_{ii} = \pm 1 \), we see that this implies: \( D_{ii} = 1 \). Finally \( D = I \) and so:

\[
Q_1 = Q_2 \quad \text{and} \quad R_1 = R_2.
\]
4. Let \( A \) and \( B \) be \( n \times n \) complex matrices such that \( AB = BA \). Show that if \( A \) has \( n \) distinct eigenvalues, then \( A, B, \) and \( AB \) are all diagonalizable.

**Solution**

Let \( \lambda_1, \ldots, \lambda_n \) be the \( n \) distinct eigenvalues of \( A \) with corresponding (nonzero) eigenvectors \( v_1, \ldots, v_n \). We know that a list of eigenvectors belonging to distinct eigenvalues must be a linearly independent list. Hence \( \mathcal{B} = (v_1, \ldots, v_n) \) is a basis of \( \mathbb{C}^n \) consisting of eigenvectors of \( A \), so that \( A \) is similar to the diagonal matrix \( \text{diag}(\lambda_1, \ldots, \lambda_n) \). Then \( ABv_i = BA v_i = B(\lambda_i)v_i = \lambda_i(Bv_i) \). So \( Bv_i \) belongs to the 1-dimensional eigenspace of \( A \) associated with the eigenvalue \( \lambda_i \). This means that \( Bv_i = \mu_i v_i \). Hence the basis \( \mathcal{B} \) is also a basis of eigenvectors of \( B \) so that \( v_i \) is associated with the eigenvalue \( \mu_i \) (which might be equal to 0). Then clearly \( AB \) is similar to the matrix \( \text{diag}(\mu_1 \lambda_1, \ldots, \mu_n \lambda_n) \).
5. In this problem, \( \mathbb{R} \) is the field of real numbers. Let \((u_1, u_2, \ldots, u_m)\) be an orthonormal basis for subspace \( W \neq \{0\} \) of the vector space \( V = \mathbb{R}^{n \times 1} \) (under the standard inner product), let \( U \) be the \( n \times m \) matrix defined by \( U = [u_1, u_2, \ldots, u_m] \), and let \( P \) be the \( n \times n \) matrix defined by \( A = UU^T \).

(a) First it is clear that \( v - w \perp W \) since for all \( x \in W \),
\[
\langle v - w, x \rangle = \langle (v - \langle v, u_1 \rangle u_1 - \ldots - \langle v, u_m \rangle u_m), x \rangle \\
= \langle v, x \rangle - \langle v, u_1 \rangle \langle u_1, x \rangle - \ldots - \langle v, u_m \rangle \langle u_m, x \rangle = 0.
\]
The last equality comes from the fact that since \( x \in W \), \( x = \langle x u_1 \rangle u_1 + \ldots + \langle x, u_m \rangle u_m \).

Now consider \( x \in W \). We define
\[
\|v - x\|^2 = \|(v - w) + (w - x)\|^2 \\
= \|v - w\|^2 + 2(v - w) \bullet (w - x) + \|w - x\|^2
\]
Since \( v - w \perp W \) and \( w - x \in W \), we have that \((v - w) \bullet (w - x) = 0\), so that
\[
\|v - x\|^2 = \|v - w\|^2 + \|w - x\|^2
\]
We see that the minimum for \( \|v - x\| \) is \( \|v - w\|^2 \) and is realized when \( x = w \).

(b) 
\[
w = \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2 + \ldots + \langle v, u_m \rangle u_m \\
= u_1(u_1^Tv) + u_2(u_2^Tv) + \ldots + u_m(u_m^Tv) \\
= (u_1u_1^T + u_2u_2^T + \ldots + u_mu_m^T)v \\
= UU^Tv = Pv.
\]

(c) First, \( P^T = (UU^T)^T = UU^T = P \), second, \( P^2 = (UU^T)^2 = U(U^TU)U^T = UU^T = P \) where we have used the fact that \( U^TU = I \).
(d) An orthogonal basis for $W$ is for example

$$(u_1, u_2) = \left( \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix} \right).$$

We get


Finally

$$w = Px = \begin{pmatrix} 14/9 \\ 16/9 \\ 22/9 \end{pmatrix}.$$
6. Let $V = \mathbb{R}^5$ and let $T \in \mathcal{L}(V)$ be defined by $T(a, b, c, d, e) = (2a, 2b, 2c + d, a + 2d, b + 2e)$.

(a) (8 points) Find the characteristic and minimal polynomial of $T$.

(b) (8 points) Determine a basis of $\mathbb{R}^5$ consisting of eigenvectors and generalized eigenvectors of $T$.

(c) (4 points) Find the Jordan form of $T$ with respect to your basis.

Solution

The matrix of $T$ in the standard basis $(e_1, e_2, e_3, e_4, e_5)$ is

$$
\begin{pmatrix}
2 & 0 & 0 & 1 & 0 \\
0 & 2 & 0 & 0 & 1 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}.
$$

We can reorder the basis in $(e_3, e_4, e_1, e_5, e_2)$, the matrix of $T$ in this basis is:

$$
\begin{pmatrix}
2 & 1 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}.
$$

This answers questions (b) and (c). To answer (a), we readily see that the characteristic polynomial of $T$ is $(x - 2)^5$ and the minimal polynomial of $T$ is $(x - 2)^3$. 

7. Suppose that $W$ is finite dimensional and $T \in \mathcal{L}(V,W)$. Prove that $T$ is injective if and only if there exists $S \in \mathcal{L}(W,V)$ such that $ST$ is the identity map on $V$.

**Solution**

First suppose that there exists $S \in \mathcal{L}(W,V)$ such that $ST$ is the identity map on $V$. Let $x$ and $y$ in $V$ such that $Tx = Ty$. Multiplying by $S$, this means $STx = STy$, but $ST$ is the identity so $STx = x$ and $STy = y$, so we get $x = y$, which means $T$ is injective.

Now suppose that $T$ is injective. Consider $w_1, \ldots, w_m$ a basis of $\text{Range}(T)$. (We use the fact that $W$ is finite dimensional.) Since $w_1, \ldots, w_m$ belongs to $\text{Range}(T)$, there exists $v_1, \ldots, v_m$ in $V$ such that $w_1 = Tv_1$, $w_2 = Tv_2$, ... Moreover since $T$ is injective, $v_1, \ldots, v_m$ are linearly independent. Finally since $w_1, \ldots, w_m$ span $\text{Range}(T)$, we get that $v_1, \ldots, v_m$ span $V$. We conclude that $v_1, \ldots, v_m$ is a basis of $V$. (So $V$ is itself finite dimensional.)

Now we use the incomplete basis theorem to extend $w_1, \ldots, w_m$ with $w_{m+1}, \ldots, w_n$ so as $w_1, \ldots, w_n$ is a basis of $W$. Now we define $S : W \to V$ (on the basis $w_1, \ldots, w_n$) such that

$$Sw_1 = v_1, \quad Sw_2 = v_2, \quad \ldots, \quad Sw_m = v_m.$$ 

and

$$Sw_{m+1} = Sw_{m+2} = \ldots = Sw_n = 0.$$ 

It is clear that $S \in \mathcal{L}(W,V)$ and that $ST$ is the identity map on $V$. 

8. (a) Prove that a normal operator on a finite dimensional complex inner product space with real eigenvalues is self-adjoint.

(b) Let $V$ be a finite dimensional real inner product space and let $T : V \rightarrow V$ be a self-adjoint operator. Is it true that $T$ must have a cube root? Explain. (A cube root of $T$ is an operator $S : V \rightarrow V$ such that $S^3 = T$.)

Solution

(a) Let $V$ be a finite dimensional complex inner product space and $T : V \rightarrow V$ be a normal operator with real eigenvalues. Let $A$ be the matrix of $T$ in an orthonormal basis. Since $T$ is normal, $T$ is diagonalizable in an orthonormal basis. Therefore there exists a unitary matrix $U$ ($U^H U = I$) such that $A = UDU^H$ with $D$ diagonal. We also know that the eigenvalues of $T$ are real, so $D$ is a real matrix; in particular, this implies $D = D^H$. In this case: $A^H = (UDU^H)^H = U(D^HU^H) = UDU^H = A$.

(b) $T$ has a cube root. The proof of existence is by construction. Let $A$ be the matrix of $T$ in an orthonormal basis. Since $T$ is a self-adjoint operator, then $T$ is diagonalizable in an orthonormal basis with real eigenvalues. Therefore there exists a unitary matrix $U$ ($U^H U = I$) such that $A = UDU^H$ with $D$ real and diagonal. Define $S = U D^{1/3} U^H$, (the cube root of $D$ is simply the cube root of the diagonal entries,) then it is clear that $S^3 = T$. 