Name: ________________________________

Exam Rules:

• This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
• Each problem is worth 20 points.
• Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
• If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
• Begin each solution on a new page and use additional paper, if necessary.
• Write only on one side of paper.
• Write legibly using a dark pencil or pen.
• Ask the proctor if you have any questions.

Good luck!

1. ___________ 5. ___________
2. ___________ 6. ___________
3. ___________ 7. ___________
4. ___________ 8. ___________

Total ___________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
Alexander Engau, Andrew Knyazev, Julien Langou (Chair).
1. Let \( V \) be a finite-dimensional real vector space. Let \( W_1 \) and \( W_2 \) be subspaces of \( V \). We define the following operations:
\[
(w_1, w_2) + (w'_1, w'_2) := (w_1 + w'_1, w_2 + w'_2)
\]
and
\[
\alpha * (w_1, w_2) := (\alpha w_1, \alpha w_2)
\]
for all \((w_1, w_2) \in W_1 \times W_2\) and \((w'_1, w'_2) \in W_1 \times W_2\) and all \(\alpha \in \mathbb{R}\). The set \(W_1 \times W_2\) is a vector space with respect to these operations.

(a) Let \(U := \{(u, -u) : u \in W_1 \cap W_2\}\). Prove that \(U\) is a subspace of \(W_1 \times W_2\). Also prove that \(U\) is isomorphic to \(W_1 \cap W_2\).

(b) Define the map \(T : W_1 \times W_2 \rightarrow W_1 + W_2\) by \(T(w_1, w_2) = w_1 + w_2\). Prove that \(T\) is a linear transformation.

(c) Use the above to prove that \(\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim W_1 + \dim W_2\).
2. Let $E_{ij} \in \mathbb{R}^{n \times n}$ denote the matrix with 1 in entry $(i, j)$ and 0 everywhere else.

(a) Prove that $E_{ii}$ and $E_{jj}$ are similar for all $1 \leq i, j \leq n$.

(b) Given $A, B \in \mathbb{R}^{n \times n}$, define $[A, B] := AB - BA$. A matrix $C \in \mathbb{R}^{n \times n}$ is called a commutator in $\mathbb{R}^{n \times n}$ if and only if $C = [A, B]$ for some $A, B \in \mathbb{R}^{n \times n}$. Show that $E_{ii} - E_{jj}$ and $E_{ij}$ are commutators in $\mathbb{R}^{n \times n}$ for all $1 \leq i, j \leq n$ with $i \neq j$. 
3. We consider a real linear space $V$ of polynomials on $[a, b]$ of degree no larger than 2012 with the scalar product $\langle f, g \rangle := \int_a^b f(t)g(t)dt$. Let a real-valued function $k(s, t)$ be continuous for $s \in [a, b]$ and $t \in [a, b]$. Let us define the linear map $F : V \rightarrow V$ by

$$f \mapsto F(f) = g \text{ such that } g(t) := \int_a^b k(s, t)f(s)ds \text{ for all } t \in [a, b].$$

In other words, we have

$$F(f)(t) = \int_a^b k(s, t)f(s)ds, \text{ for all } t \in [a, b].$$

(a) Determine an explicit expression for $F^*$, the adjoint of $F$.

(b) Let $n$ be a positive integer. Show that $F$ is normal if $k(s, t) = (s - t)^n$ and determine for which $n$ the linear map $F$ is self-adjoint.
4. We consider two real valued \( n \)-by-\( n \) matrices \( A \) and \( B \) such that \( A \) is symmetric positive definite and \( B \) is anti-symmetric. Prove that \( A + B \) is invertible.
5. Let \( a \) and \( b \in \mathbb{R} \) such that \( a \neq b \). Let \( A \) a 6-by-6 real valued matrix such that the characteristic polynomial of \( A \) is \( \chi_A(X) = (X - a)^4(X - b)^2 \) and the minimal polynomial of \( A \) is \( \pi_A(X) = (X - a)^2(X - b) \). Describe all different possible Jordan forms for \( A \).
6. Let $A$ and $B$ be two square matrices such that

$$AB = A^2 + A + I.$$  

Show that $A$ and $B$ commute. (Hint: First show that $A$ is invertible.)
7. (a) Let $A$ be a complex Hermitian matrix. Prove that $A$ is positive definite if and only if all the eigenvalues of $A$ are positive.

(b) Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$. Let $V = \mathbb{R}^3$. We define the map $*: V \times V \to \mathbb{R}$ by $u*v = u^TAv$ for all $u, v \in V$. Prove that $*$ is an inner product on $V$.

(c) Use the inner product from above and the Gram-Schmidt orthogonalization process to find an orthonormal basis for $V$. 
8. For a complex vector $x = [x_1 x_2]$, we define the function $f(x) = |x_1| + 2|x_2|$.

(a) Is $f(x)$ a vector norm?

(b) Is there some scalar product $(x, y)$ such that $(x, x) = f^2(x)$? (Hint: Use the parallelogram identity.)