Analysis Prelim—January 2015

Name:

• All seven answers will be graded, the problem with the lowest point score will be dropped.
• Be sure to show all your work.
• Only write on one side of each sheet.
• Start a new sheet of paper for every problem, and write your name and the problem number on every sheet.
• If you use a statement from Rudin or class, state it. If you are unsure if a statement must be proved or may merely be stated, ask your friendly proctor.
Problems

1. Let \((X, d)\) be a complete metric space. Prove that a subspace of \((X, d)\) is complete if and only if it is closed.

**Proof:** Let \(A \subset X\) be any metric subspace. Suppose \(A\) is complete. If \(x \in X\) is a limit point of \(A\), then there exists a sequence \((x_n) \subset A\) such that \(x_n \to x\) in \(X\). Since limit points are unique, convergent sequences are Cauchy, and \(A\) is complete, this implies \(x \in A\). Now suppose \(A\) is closed and consider an arbitrary Cauchy sequence \((x_n) \subset A\). Since \(X\) is complete, there exists \(x \in X\) such that \(x_n \to x\). If the sequence does not eventually become constant (which implies \(x_n = x\) for all sufficiently large \(n\) and \(x \in A\)), then \(x\) is a limit point of \(A\). Since \(A\) is closed, \(x\) belongs to \(A\). \(\square\)
2. Let $X$ be any nonempty set and $d_X$ the discrete metric on $X$. Let $(Y, d_Y)$ be a metric space. Let $\mathcal{F} = \{f_\alpha\}_{\alpha \in \mathcal{A}}$ denote a family of functions $f_\alpha : X \to Y$ for each $\alpha \in \mathcal{A}$. Prove that $\mathcal{F}$ is uniformly equicontinuous.

**Proof:** Consider any $\epsilon > 0$ and fix any $\delta \in (0, 1)$. For $x, y \in X$ with $d_X(x, y) < \delta$, $x = y$ since $d_X$ is the discrete metric. Thus, for any $\alpha \in \mathcal{A}$, and $x, y \in X$ with $d_X(x, y) < \delta$, $d_Y(f_\alpha(x), f_\alpha(y)) = d_Y(f_\alpha(x), f_\alpha(x)) = 0 < \epsilon$, which proves the result. □
3. Let \( x = (\xi_j), \ y = (\eta_j) \) denote points in \( \ell^2 \).

(a) Prove that the function \( d_1(x, y) \) defined by
\[
d_1(x, y) = \sum_{j=1}^{\infty} |\xi_j - \eta_j|
\]
is not a metric on \( \ell^2 \).

(b) Prove that the function \( d_2(x, y) \) defined by
\[
d_2(x, y) = \left( \sum_{j=1}^{\infty} |\xi_j - \eta_j|^2 \right)^{1/2}
\]
is a metric on \( \ell^2 \). Hint: Use Minkowski’s inequality.

**Proof (a):** Note that \( x = (1/j) \in \ell^2 \) and \( y = (0) \in \ell^2 \) (i.e., \( y \) is the sequence of all 0’s), yet \( d_1(x, y) = \sum_{j=1}^{\infty} 1/j = +\infty \), so \( d_1 \) does not map \( \ell^2 \) into \([0, \infty)\) and is not a metric. □

**Proof (b):** Consider any \( x = (\xi_j), \ y = (\eta_j) \in \ell^2 \). Clearly, \( d_2(x, y) \geq 0 \) by properties of the absolute value. Applying Minkowski’s inequality to \( x \) and \(-y\), we have
\[
(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^2)^{1/2} \leq (\sum_{k=1}^{\infty} |\xi_k|^2)^{1/2} + (\sum_{m=1}^{\infty} |\eta_m|^2)^{1/2} < \infty
\]
Thus, \( d_2(x, y) \in [0, \infty) \).

Suppose \( d_2(x, y) = 0 \iff d_2^2(x, y) = 0 \iff \sum_{j=1}^{\infty} |\xi_j - \eta_j|^2 = 0 \iff |\xi_j - \eta_j|^2 = 0 \) for all \( j \in \mathbb{N} \)

\( \iff |\xi_j - \eta_j| = 0 \) for all \( j \in \mathbb{N} \iff \xi_j = \eta_j \) for all \( j \iff x = y \).

For any \( a, b \in \mathbb{R}, |a - b| = |b - a| \). It follows that \( d_2(x, y) = (\sum_{j=1}^{\infty} |\xi_j - \eta_j|^2)^{1/2} = (\sum_{j=1}^{\infty} |\eta_j - \xi_j|^2)^{1/2} = d(y, x) \).

Now consider any \( z = (\zeta_k) \in \ell^2 \). The triangle inequality of the absolute value on \( \mathbb{R} \) implies that for each \( j \in \mathbb{N} \) we have \( |\xi_j - \eta_j|^2 \leq |\xi_j - \zeta_j| + |\zeta_j - \eta_j|^2 \). Note that Minkowski’s inequality implies that the term-wise addition or subtraction of any two sequences in \( \ell^2 \) is also in \( \ell^2 \). Thus, we apply Minkowski’s inequality below to get
\[
d_2(x, y) = (\sum_{j=1}^{\infty} |\xi_j - \eta_j|^2)^{1/2}
\leq (\sum_{j=1}^{\infty} |\xi_j - \zeta_j| + |\zeta_j - \eta_j|^2)^{1/2}
\leq (\sum_{k=1}^{\infty} |\zeta_k|^2)^{1/2} + (\sum_{m=1}^{\infty} |\eta_m|^2)^{1/2}
\]
\[
= d(x, z) + d(z, y)
\] □
4. Consider the metric space \((\ell^2, d_2)\) from problem 3 where \(x = (\xi_i) \in \ell^2\). For each \(n \in \mathbb{N}\), let \(f_n : \ell^2 \to \mathbb{R}\) be defined by

\[ f_n(x) = f_n((\xi_i)) = \sum_{i=n}^{\infty} |\xi_i|^2 \]

Prove that (a) \(f_n(x) \to 0\) for every \(x \in \ell^2\), but (b) the convergence to the function \(f(x) = 0\) is not uniform.

**Proof (a):** Consider any \(x = (\xi_i) \in \ell^2\) and \(\epsilon > 0\). Since \(x \in \ell^2\), \(\sum_{i=1}^{\infty} |\xi_i|^2 < \infty\), which implies that there exists \(N\) such that \(\sum_{i=n}^{\infty} |\xi_i|^2 < \epsilon\) for \(n > N\). Thus, for \(n > N\),

\[ |f_n(x) - 0| = \sum_{i=n}^{\infty} |\xi_i|^2 < \epsilon \]

□

**Proof (b):** Let \(\epsilon = 1/2\) and consider any \(m \in \mathbb{N}\). Consider the sequence \((x_n) = ((\xi_{i,n})) \subset \ell^2\) defined by \(\xi_{i,n} = 1\) if \(i = n\) and \(\xi_{i,n} = 0\) if \(i \neq n\). Then, \(f_m(x_n) = 1\) for any \(n > m\), so

\[ |f_m(x_n) - f(x_n)| = 1 > \epsilon \]

for any \(n > m\) and the convergence cannot be uniform. □
5. Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces. For each \(n \in \mathbb{N}\), suppose \(f_n : X \to Y\) is a continuous function. Prove if \(Y\) is complete and \((f_n)\) is uniformly Cauchy, then there exists continuous \(f : X \to Y\) such that \(f_n \to f\) uniformly.

**Proof:** Consider any \(x \in X\), then \((f_n(x)) \subset Y\) is Cauchy since \((f_n)\) is uniformly Cauchy. Since \(Y\) is complete, there exists \(y \in Y\) such that \(f_n(x) \to y\). Define \(f : X \to Y\) by this point-wise limit, i.e. \(f(x) = \lim_{n \to \infty} f_n(x) = y\).

We now prove that the convergence is uniform. Let \(\epsilon > 0\) be given. Since \((f_n)\) is uniformly Cauchy, there exists \(N\) such that for any \(n, m > N\), \(d_Y(f_n(x), f_m(x)) < \epsilon/2\) for every \(x \in X\). Thus, for fixed \(n > N\), since \(f_m(x) \to f(x)\), by the (sequential) continuity of the metric \(d_Y(f_n(x), f(x)) \leq \epsilon/2 < \epsilon\). This proves the convergence is uniform.

We now prove that the function \(f\) is continuous. Since \(f_n \to f\) uniformly, there exists \(N\) such that for every \(x \in X\), \(d_Y(f_n(x), f(x)) < \epsilon/3\) for \(n > N\). Choose such an \(N\) and let \(n = N + 1\). Consider any \(x \in X\). Since \(f_n\) is continuous at \(x\), there exists \(\delta > 0\) such that if \(y \in X\) with \(d_X(x, y) < \delta\), then \(d_Y(f_n(x), f_n(y)) < \epsilon/3\). Choose such a \(\delta\). Then, for any \(y \in X\) with \(d_X(x, y) < \delta\),

\[
d_Y(f(x), f(y)) \leq d_Y(f(x), f_n(x)) + d_Y(f_n(x), f_n(y)) + d_Y(f_n(y), f(y)) < \epsilon \square.
\]
6. Let \((g_n)\) and \((f_n)\) denote sequences of Riemann integrable functions, \([a, b] \to \mathbb{R}\). Suppose

(i) \(\forall m \in \mathbb{N}, \lim_{n \to \infty} \int_a^b |f_m g_n| dx = 0\),
(ii) \(f_n \to f\) uniformly on \([a, b]\), and
(iii) the sequence \(\{\int_a^b g_n dx\}\) is bounded.

Prove that

\[ \lim_{n \to \infty} \int_a^b f g_n dx = 0. \]

**Proof:** Choose \(\epsilon > 0\). Let \(K\) be a bound on \(\{\int_a^b |g_n| dx\}\), and choose \(m\) large enough so that \(|f_m(x) - f(x)| < \epsilon / 2K(b - a)\) for \(x \in [a, b]\). Then

\[
\int_a^b |f g_n| dx = \int_a^b |f_m g_n - f_m g_n + f g_n| dx \leq \int_a^b |f_m g_n| dx + \int_a^b |f_m - f| g_n dx
\]

By (i) we can choose \(N\) so that \(n > N\) implies \(\int_a^b |f_m g_n| dx < \epsilon / 2\). By our choice of \(m\) and \(K\), \(\int_a^b |f_m - f||g_n| dx < \epsilon / 2\). Thus, for \(n > N\), \(\int_a^b |f g_n| dx < \epsilon\). Since \(\epsilon > 0\) was arbitrary, the result is proved.