Problem 1. Let $f : A \subset \mathbb{R} \rightarrow \mathbb{R}$ have the property that for every $x \in A$, there exists $\epsilon > 0$ such that $f(t) > \epsilon$ if $t \in (x - \epsilon, x + \epsilon) \cap A$. If the set $A$ is compact, prove there exists $c > 0$ such that $f(x) > c$ for all $x \in A$.

Proof:

For any $x \in A$, there exists $\epsilon_x > 0$ such that $f(t) > \epsilon_x$ for $t \in (x - \epsilon_x, x + \epsilon_x) \cap A$. The set \{$(x - \epsilon_x, x + \epsilon_x) : x \in A$\} forms an open cover of $A$. Since $A$ is compact, there exists a finite subcover that can be written as \{$(x_i - \epsilon_{x_i}, x_i + \epsilon_{x_i}) : 1 \leq i \leq n, n \in \mathbb{N}$\}. Choose such a finite subcover.

Let $c = \min \{\epsilon_{x_1}, \epsilon_{x_2}, \ldots, \epsilon_{x_n}\}$.

For all $x \in A, x \in (x_i - \epsilon_{x_i}, x_i + \epsilon_{x_i}) \cap A$ for some $i \in \{1, 2, \ldots, n\}$, so $f(x) > \epsilon_{x_i} \geq c$. \(\square\)
Problem 2. Suppose \( \{x_n\} \) is a Cauchy sequence in a metric space \((X,d)\). Show \( \{x_n\} \) converges if and only if \( \{x_n\} \) has a convergent subsequence.

Proof:

(\(\Rightarrow\)) Suppose \( \{x_n\} \) converges. Since \( \{x_n\} \) is trivially its own subsequence, \( \{x_n\} \) has a convergent subsequence.

(\(\Leftarrow\)) Suppose \( \{x_n\} \) has a convergent subsequence \( \{x_{n_k}\} \) with limit \( x \).

Consider any \( \epsilon > 0 \).

Since \( \{x_n\} \) is Cauchy, there exists \( N_1 \) such that for all \( n,m > N_1 \), we have that \( d(x_n,x_m) < \epsilon/2 \). Choose such an \( N_1 \).

Since \( \{x_{n_k}\} \) converges to \( x \), there exists \( N_2 \) such that for all \( k > N_2 \), we have that \( d(x_{n_k},x) < \epsilon/2 \). Choose such an \( N_2 \). Note that for \( k > N_2 \), we have that \( n_k > N_2 \) by the definition of a subsequence.

Set \( N = \max \{N_1, N_2\} \).

For any \( n > N \), we have

\[
\begin{align*}
d(x_n,x) &\leq d(x_n,x_{n_k}) + d(x_{n_k},x) \text{ where we chose any fixed } k > N \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon \quad \Box
\end{align*}
\]
Problem 3. For \( f, g : [0, 1] \to \mathbb{R} \), let
\[
d(f, g) = \sup_{x \in [0, 1]} \{ x^2 |f(x) - g(x)| \}.
\]

(a) Prove that \( d \) defines a metric on the linear space \( C([0, 1]) \) of continuous functions \( f : [0, 1] \to \mathbb{R} \).

(b) Prove that the resulting metric space \( (C([0, 1]), d) \) is not complete.

Proof of (a):

• \( d(f, g) \geq 0 \): a square times an absolute value is non-negative, taking the sup does not change this

• \( d(f, g) = 0 \implies f = g \): \( d(f, g) = 0 \) implies that \( f(x) = g(x) \) for all \( x \in (0, 1] \). As \( f \) and \( g \) are continuous in 0, this implies that \( f(0) = g(0) \).

• \( d(f, g) = d(g, f) \): trivial (the definition is symmetric in \( f \) and \( g \)).

• \( d(f, h) \leq d(f, g) + d(g, h) \): By the triangle inequality of \( | \cdot | \) in the reals, \( x^2|f(x) - h(x)| - x^2|f(x) - g(x)| - x^2|g(x) - h(x)| \leq 0 \) for all \( x \in [0, 1] \). This implies the inequality for the supremum.

Proof of (b): Consider the sequence of functions \( f_n(x) = (1 - x)^n \). Then
\[
\lim_{n \to \infty} f_n(x) = \begin{cases} 
1, & x = 0 \\
0, & x \neq 0,
\end{cases}
\]
a function not in \( C([0, 1]) \). On the other hand,
\[
\lim_{n \to \infty} d(f_n, 0) = \lim_{n \to \infty} \sup_{x \in [0, 1]} \{ x^2 |f_n(x)| \} = \lim_{n \to \infty} \sup_{x \in [0, 1]} \{ x^2(1 - x)^n \} \leq \lim_{n \to \infty} \sup_{x \in (0, 1)} \{ x^2(1 - x)^n \} \text{ (as } x^2(1 - x)^n = 0 \text{ for } x \in \{0, 1\}) \leq \lim_{n \to \infty} \sup_{x \in (0, 1)} \{ (1 - x)^n \} = 0,
\]
so \( f_n \) is Cauchy.
Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose there exists $T > 0$ such that $f(x+T) = f(x)$ for all $x \in \mathbb{R}$, i.e., $f$ is periodic. Prove $f$ is uniformly continuous on $\mathbb{R}$.

Proof:
Let $A = [0, T] \subset \mathbb{R}$, $B = [T, 2T] \subset \mathbb{R}$ and $C = A \cup B$. Since $A$, $B$, and $C$ are compact, $f$ is uniformly continuous on $A$, $B$, and $C$ (standard theorem: continuous functions on compact sets are uniformly continuous).

Thus, for any $\varepsilon > 0$, there exists $\delta_C > 0$ such that for all $a, b \in C$ with $|a - b| < \delta_C$ implies $|f(a) - f(b)| < \varepsilon$.

Set $\delta = \min\{\delta_C, T/2\}$. (Note: You can use $T$ instead of $T/2$.)

Consider any $x, y \in \mathbb{R}$ with $|x - y| < \delta$. Without loss of generality assume that $x \leq y$.

Observe that there exists $k \in \mathbb{Z}$ such that $x = a + kT$ and $y = b + kT$ for some $a, b \in C$. (Note: It looks like there are two cases to consider, (I) $a, b \in A$ or (II) $a \in A$ and $b \in B$, but the proof for each is exactly the same by construction. Also, if we had not made sure that $\delta$ was bounded by $T$ then we cannot necessarily state that the same $k \in \mathbb{Z}$ can be used to represent $x = a + kT$ and $y = b + kT$, which means we cannot necessarily get to $|a - b| < \delta$ below.)

Since $f$ is periodic, $f(x) = f(a)$ and $f(y) = f(b)$. Also, $|x - y| = |a - b| < \delta \leq \delta_C$ so we have that $|f(x) - f(y)| = |f(a) - f(b)| < \varepsilon$. □
Problem 5. [...] 

Proof of (a):
Pick any arbitrary but fixed $x_0 \in X$, and define $\{x_n\}$ recursively as $x_{n+1} = T(x_n)$ for $n \in \mathbb{N}$. For $n \geq 1$, we have 
\[ d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq cd(x_n, x_{n-1}). \]
By induction, for any $n \in \{0, 1, 2, \ldots\}$,
\[ d(x_{n+1}, x_n) \leq c^n d(x_1, x_0). \]
If $n < m$, then
\[ d(x_n, x_m) \leq \sum_{i=n+1}^{m} d(x_i, x_{i-1}) \leq d(x_1, x_0) \sum_{i=n+1}^{m} c^i \leq d(x_1, x_0) \frac{c^n}{1 - c}. \]
Since $c < 1$, $c^n \to 0$ as $n \to \infty$, so $\{x_n\}$ is a Cauchy sequence. Since $X$ is a complete metric space, there exists some $x \in X$ that is the limit of $\{x_n\}$. Since $T$ is a contraction, it is a continuous function, so 
\[ T(x) = \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} x_{n+1} = x. \]
If $y \in X$ is any other fixed point, then $d(x, y) \leq cd(x, y)$, which implies $d(x, y) = 0$. Thus, the fixed point is unique. □.

Proof of (b):
Since $T^n$ is a contraction, by part (a), there exists unique fixed point $x$ of $T^n$. We show this is the unique fixed point of $T$. Observe that 
\[ T^n(Tx) = T^{n+1}(x) = T(T^{n+1}x) = Tx. \]
Thus, $Tx$ is a fixed point of $T^n$, and by uniqueness, $Tx = x$. Thus, $x$ is a fixed point of $T$. Since any fixed point of $T$ is also a fixed point of $T^n$, which is unique, $x$ is the unique fixed point of $T$. □
Problem 6. Give a proof of the following formulation of the fundamental theorem of calculus:

Let $F : [a, b] \to \mathbb{R}$ be continuously differentiable, and let $f = \frac{d}{dx} F$. Then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

Proof: Rudin 6.21
Problem 7. [..]

7(a): \( f_n(x) = \frac{1}{nx + 1} \) for \( x \in (0, 1) \). Then \( f_n \) are clearly continuous, \( f_n \to 0 \) monotonically, but convergence is not uniform (the problem occurs near \( x = 0 \) - students should show).

7(b): \( f_n(x) = x^n \) for \( x \in [0, 1) \) and \( f_n(1) = 0 \) for all \( n \in \mathbb{N} \). Then \( f_n \to 0 \) monotonically on \([0, 1], \) but convergence is not uniform (the problem occurs near \( x = 1 \) - students should show).

7(c): \( f_n(x) = x^n \) for \( x \in [0, 1) \). Same problem as in (b).

7(d): \( f_n(x) = n^2 x(1 - x^2)^n \) for \( x \in [0, 1] \). Then \( f_n \to 0 \). Convergence cannot be uniform since the limit of the integral is not the integral of the limit on \([0, 1].\)