Analysis Prelim—January 2014

Name:

- All seven answers will be graded, the problem with the lowest point score will be dropped.
- Be sure to show all your work.
- Start a new sheet of paper for every problem, and write your name and the problem number on every sheet.
- If you use a statement from Rudin or class, state it. If you are unsure if a statement must be proved or may merely be stated, ask your friendly proctor.

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Problems

1. Let $f : A \subset \mathbb{R} \to \mathbb{R}$ have the property that for every $x \in A$, there exists $\varepsilon > 0$ such that $f(t) > \varepsilon$ if $t \in (x - \varepsilon, x + \varepsilon) \cap A$. Prove that if the set $A$ is compact, there exists $c > 0$ such that $f(x) > c$ for all $x \in A$.

2. Suppose $\{x_n\}$ is a Cauchy sequence in a metric space $(X, d)$. Show that $\{x_n\}$ converges if and only if $\{x_n\}$ has a convergent subsequence.

3. For $f, g : [0, 1] \to \mathbb{R}$, let
   \[d(f, g) = \sup_{x \in [0, 1]} \{x^2 |f(x) - g(x)|\} \]

   (a) Prove that $d$ defines a metric on the linear space $C([0, 1])$ of continuous functions $f : [0, 1] \to \mathbb{R}$.
   
   (b) Prove that the resulting metric space $(C([0, 1]), d)$ is not complete.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose there exists $T > 0$ such that $f(x + T) = f(x)$ for all $x \in \mathbb{R}$, i.e., $f$ is periodic. Prove that $f$ is uniformly continuous on $\mathbb{R}$.

5. Suppose that $T : X \to X$ is a continuous mapping of a complete metric space, $(X, d)$, onto itself. Define the $n$-time composition of $T$
   \[T^n = T \circ T \circ \ldots \circ T.\]

   Recall that $f$ is a contraction mapping if $d(f(x), f(y)) \leq cd(x, y)$ for some $c$, $0 \leq c < 1$ for all $x, y$ in the metric space.

   (a) Proof the following statement from Rudin: If $T$ is contracting, then $T$ has a unique fixed point in $X$.
   
   (b) Use the statement in (5a) to show: If $T^n$ is contracting for some $n \geq 1$, then $T$ has a unique fixed point in $X$.

6. Give a proof of the following formulation of the fundamental theorem of calculus:

   **Theorem.** Let $F : [a, b] \to \mathbb{R}$ be continuously differentiable, and let $f = \frac{d}{dx} F$. Then
   \[\int_a^b f(x) \, dx = F(b) - F(a).\]
7. The following is a theorem in Rudin:

**Theorem.** Suppose that

i. $K$ is compact,

ii. $\{f_n\}$ is a sequence of continuous functions on $K$,

iii. $\{f_n\}$ converges pointwise to a continuous function $f$ on $K$,

iv. $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, \ldots$.

Then $f_n \to f$ uniformly.

Show counterexamples (with justifications!) for the following statements missing or weakening one of the conditions:

(a) Suppose that

i.

ii. $\{f_n\}$ is a sequence of continuous functions on $K$,

iii. $\{f_n\}$ converges pointwise to a continuous function $f$ on $K$,

iv. $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, \ldots$.

Then $f_n \to f$ uniformly.

(b) Suppose that

i. $K$ is compact,

ii. $\{f_n\}$ is a sequence of functions on $K$,

iii. $\{f_n\}$ converges pointwise to a continuous function $f$ on $K$,

iv. $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, \ldots$.

Then $f_n \to f$ uniformly.

(c) Suppose that

i. $K$ is compact,

ii. $\{f_n\}$ is a sequence of continuous functions on $K$,

iii. $\{f_n\}$ converges pointwise to a function $f$ on $K$,

iv. $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, \ldots$.

Then $f_n \to f$ uniformly.

(d) Suppose that

i. $K$ is compact,

ii. $\{f_n\}$ is a sequence of continuous functions on $K$,

iii. $\{f_n\}$ converges pointwise to a continuous function $f$ on $K$,

iv.

Then $f_n \to f$ uniformly.