INSTABILITY IN MULTICOLORING
AND MULTICLIQUE SEQUENCES

by

Sarah A. Sutorik

A.A., Colorado Mountain College, 1982

B.S., University of Colorado at Denver, 1993

A thesis submitted to the

University of Colorado at Denver

in partial fulfillment

of the requirements for the degree of

Master of Science

Applied Mathematics

1995
This thesis for the Master of Science
degree by
Sarah A. Sutorik
has been approved
by

________________________
David C. Fisher

________________________
Kathryn Fraughnaugh

________________________
J. Richard Lundgren

________________________
Date
Instability in Multicoloring and Multiclique Sequences

ABSTRACT

The $m$-chromatic number $\chi_m(G)$ of a graph $G$ is the fewest colors needed so each node has $m$ colors and no color appears on adjacent nodes. Stahl has shown that for any graph $G$, there is a $p$ and $k_0$ such that for the multicoloring sequence $\chi_1(G), \chi_2(G), \chi_3(G), \ldots$, we have $\chi_{k+p}(G) = \chi_k(G) + \chi_p(G)$ for all $k > k_0$. Stahl defined as “chromatically unstable” all graphs for which $k_0 > 0$, and found the first known such graph, Grötzch’s graph, for which $k_0 = 1$. Here we give several other graphs which are chromatically unstable, including two for which $k_0 = 3$. Additionally, we examine the multiclique sequences of these graphs, finding that most are “cliquishly unstable” as well.

This abstract accurately represents the content of the candidate’s thesis. I recommend its publication.

Signed

David C. Fisher
DEDICATION

To John Marriott

and

To Joseph and Jeanne Sutorik
## CONTENTS

Figures ................................................................. vi

Tables ................................................................. vii

Chapter
1. Introduction ............................................................. 1

2. Instability in Multicoloring Sequences ................................. 7

2.1 Cycles ................................................................. 7

2.2 Mycielskians of Cycles ............................................... 8

2.2.1 Mycielskians of Even Cycles ................................... 10

2.2.2 Mycielskians of Odd Cycles ................................... 10

3. Instability in Multiclque Sequences ................................. 28

3.1 Cycles ................................................................. 28

3.2 Mycielskians of Cycles ............................................... 29

3.2.1 Mycielskians of Even Cycles ................................... 29

3.2.2 Mycielskians of Odd Cycles ................................... 30

References ................................................................. 47
FIGURES

1. The Mycielskian of $C_5$ ................................................................. 4
2. A 4-coloring of $C_{15}$ with nine colors ........................................... 9
3. 1-, 2-, and 3-colorings of $\mu(C_3)$ ................................................. 11
4. One of 11 Type Z independent sets of $\mu(C_{11})$ ............................. 18
5. Type Y independent set of $\mu(C_{11})$ ............................................... 19
6. One of 11 Type $X_5$ independent sets of $\mu(C_{11})$ .......................... 21
7. One of 11 Type $X_3$ independent sets of $\mu(C_{11})$ ........................... 22
8. One of 11 Type $X_4$ independent sets of $\mu(C_{11})$ ........................... 23
9. 1-, 2-, and 3-cliques of $\mu(C_3)$ ....................................................... 31
TABLES

Table

1. The weight on each independent set in a k-coloring of $\mu(C_5)$ 13
2. The weight on each independent set in a k-coloring of $\mu(C_7)$ 14
3. The weight on each independent set in a k-coloring of $\mu(C_9)$ 16
4. The weight on each independent set in a k-coloring of $\mu(C_{11})$ 26
5. The weight on each independent set in a k-coloring of $\mu(C_{13})$ 27
6. The weight on each node in a k-clique of $\mu(C_5)$ 33
7. The weight on each node in a k-clique of $\mu(C_7)$ 37
8. The weight on each node in a k-clique of $\mu(C_9)$ 42
9. The weight on each node in a k-clique of $\mu(C_{11})$ 46
1 Introduction

A coloring of a graph \( G \) is a labeling of the nodes with \( 1, 2, \ldots, m \), so that adjacent nodes have different labels. Alternatively, it is easy to see but tedious to explain that a coloring can be thought of as a collection of maximal independent sets such that each node is in at least one set. The coloring number \( \chi(G) \) is the fewest number of maximal independent sets in a coloring of \( G \). Similarly, the \( k \)-coloring number \( \chi_k(G) \) is the fewest maximal independent sets so that each node is in at least \( k \) of the sets. The \( k \)-coloring number may be found through the solution of an integer program. Let \( K \) be the node maximal independence matrix of \( G \), where the rows of \( K \) represent the nodes of \( G \) and the columns represent all maximal independent sets of \( G \). If node \( i \) is in independent set \( j \), then \( k_{ij} = 1 \); otherwise \( k_{ij} = 0 \). Then \( \chi_k(G) \) is the value of the integer program:

\[
\text{Minimize } 1^T x \text{ subject to } K x \geq k 1, x \geq 0 \text{ and } x \text{ integer,} \tag{1}
\]

where \( 1 \) and \( 0 \) are vectors of all ones and all zeros, respectively, and \( x \) is a vector such that \( x_j \) is the integer weight on independent set \( j \). Since \( \chi(G) = \chi_1(G) \), this method also finds the value of \( \chi(G) \) by setting \( k = 1 \).

Fractional coloring is a generalization of coloring in that the weights on the maximal independent sets need not be integer. The fractional coloring number
\( \chi_f(G) \) is the value of the linear program:

\[
\text{Minimize } 1^T x \text{ subject to } Kx \geq 1, \text{ and } x \geq 0. \tag{2}
\]

The problem of finding the chromatic number is dual to that of finding a maximum clique. A *clique* of \( G \) is a complete subgraph of \( G \). An alternative definition is that a clique is a set of nodes such that each maximal independent set of \( G \) contains at most one node of the set. The *clique number* \( \omega(G) \) is the maximum number of nodes in any clique of \( G \). Expanding the idea, the *\( k \)-clique number* \( \omega_k(G) \) is the largest sum of integer weights on the nodes so that the sum of weights in any maximal independent set is at most \( k \). The value of the following integer program, where \( y \) is a vector such that \( y_i \) is a weight on node \( i \), is \( \omega_k(G) \):

\[
\text{Maximize } 1^T y \text{ subject to } K^T y \leq k1, y \geq 0 \text{ and } y \text{ integer.} \tag{3}
\]

The dual to the fractional coloring linear program (2) is the following:

\[
\text{Maximize } 1^T y \text{ subject to } K^T y \leq 1, \text{ and } y \geq 0. \tag{4}
\]

The value of (4) is the *fractional clique number*, \( \omega_f(G) \), that is the largest sum of (not necessarily integer) weights which may be placed on the nodes of \( G \) so that for each maximal independent set, the sum of weights on that set is at most one. Since (4) is the linear programming dual of (2), we have:

\[
\omega(G) \leq \omega_f(G) = \chi_f(G) \leq \chi(G). \tag{5}
\]
The fractional clique number and the $k$-clique number are related in that a maximal $k$-clique may be transformed into a fractional clique by dividing the weight on each node by $k$. Since this fractional clique may not be maximal,

$$\omega_f(G) \geq \frac{\omega_k(G)}{k},$$

which implies

$$\omega_k(G) \leq \lfloor k \omega_f(G) \rfloor \quad (6)$$

By similar reasoning,

$$\chi_f(G) \leq \frac{\chi_k(G)}{k},$$

implying

$$\chi_k(G) \geq \lceil k \chi_f(G) \rceil \quad (7)$$

A graph construction which will be utilized heavily in this report is that of Mycielski. Let $G$ be a graph with nodes $v_1, v_2, \ldots, v_m$. Then the Mycielskian $\mu(G)$ of $G$ is the graph with nodes $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m, z$ with $x_i$ adjacent to $x_j$ and $y_j$ if and only if $v_i$ is adjacent to $v_j$ in $G$; with $y_i$ adjacent to $z$ for all $i$; and no other edges. For example, the Mycieskian of $C_5$ is illustrated in Figure 1.

Larsen, Propp, and Ullman [4] proved the following useful results for the Mycielskian of any graph $G$:

$$\omega(\mu(G)) = \omega(G) \quad (8)$$

$$\chi(\mu(G)) = \chi(G) + 1 \quad (9)$$
Figure 1: The Mycielskian of $C_5$. 
\[ \chi_f(\mu(G)) = \chi_f(G) + \frac{1}{\chi_f(G)} \]  

These results will be used extensively in later proofs, where we expand upon several results by Stahl.

Stahl [5] showed that if a subadditive integer sequence (subadditive meaning \( f_{l+m} \leq f_l + f_m \)) has the property that \( \frac{f_l}{l} \geq \frac{f_p}{p} \) for some \( p \) and all \( l \), then there exists a \( k_0 \) such that \( f_{k+p} = f_k + f_p \) for all \( k > k_0 \). Since an \((l+m)\)-coloring may be formed from a minimal \( l \)-coloring and a minimal \( m \)-coloring, \( \chi_{l+m}(G) \leq \chi_l(G) + \chi_m(G) \), and therefore the multicoloring sequence \( \chi_1(G), \chi_2(G), \chi_3(G), \ldots \) is subadditive. If \( p \) is the least common denominator of a minimal fractional coloring, then \( p \) times that fractional coloring is a \( p \)-coloring, so by (7), \( \frac{\chi_k(G)}{p} \geq \frac{\chi_{k+p}(G)}{p} \) for some \( p \) and all \( k \). Thus

\[ \chi_{k+p}(G) = \chi_k(G) + \chi_p(G) \text{ for all } k \geq k_0, \]  

where \( p \) is the least common denominator of a minimal fractional coloring. Therefore if \( k_0 \) is known, we can find \( \chi_k(G) \) for all \( k \) in finite time.

For most common graphs, \( k_0 = 0 \). Stahl [6] termed graphs for which \( k_0 > 0 \) chromatically unstable, and published the first known example of such a graph. The graph is known as Grötzch’s graph, or alternatively, as \( \mu(C_5) \). The chromatic stability number \( k_0 \) of Grötzch’s graph is 1.

Here we prove that several in the series of Mycielskians of odd cycles are chromatically unstable, including two where the stability number is 3. Addition-
ally, we examine the multiclique sequences of this series of graphs, and find that several are *cliquishly unstable*, meaning that $k_0 > 0$ for the multiclique sequence $\omega_1(G), \omega_2(G), \omega_3(G), \ldots$. The highest clique stability number we have found computationally is 18.
2 Instability in Multicoloring Sequences

Stahl [6] has shown that the Mycielskian of $C_5$ has the property of chromatic instability. We will show that the Mycielskians of $C_7, C_9, C_{11}$, and $C_{13}$ have this property as well.

2.1 Cycles

For completeness, we first show that all cycles are chromatically stable.

**Theorem 2.1.** For $n$ even, $\chi_k(C_n) = 2k$.

**Proof:** For a $k$-coloring of an even cycle, any two adjacent vertices require $k$ distinct colors each. Therefore, $\chi_k(C_n) \geq 2k$. But since an even cycle can be $k$-colored by alternating between two sets of $k$ colors as the cycle is traversed, $\chi_k(C_n) \leq 2k$. Then for even cycles, $\chi_k(C_n) = 2k$ for all $k$, and therefore even cycles are chromatically stable. □

**Lemma 2.1.** $\chi_f(C_n) = \frac{2n}{n-1}$ for $n$ odd.

**Proof:** An odd cycle contains $n$ maximal independent sets of size $\frac{n-1}{2}$. So placing a weight of $\frac{2}{n-1}$ on each node gives a fractional clique of weight $\frac{2n}{n-1}$. Therefore, $\chi_f(C_n) \geq \frac{2n}{n-1}$. Additionally, since we can form a fractional coloring of an odd cycle by placing a weight of $\frac{2}{n-1}$ on each of the $n$ maximal independent sets,
Theorem 2.2. For n odd, \( \chi_k(C_n) = \left\lfloor \frac{2nk}{n-1} \right\rfloor \).

Proof: Since \( \chi_k(G) \geq \lfloor k\chi_j(G) \rfloor \), by Lemma 2.1, \( \chi_k(C_n) \geq \left\lfloor \frac{2nk}{n-1} \right\rfloor \). To show \( \chi_k(G) \leq \lfloor k\chi_j(G) \rfloor \), we first note that \( \left\lfloor \frac{2nk}{n-1} \right\rfloor = 2k + 1 \) when \( k < \frac{n}{2} \). Suppose \( k < \frac{n}{2} \). Then for \( n \) odd, we may \( k \)-color \( C_n \) with \( 2k + 1 \) colors as follows. Number the vertices sequentially around the cycle, beginning with 1. Use numbers to represent colors, and for each vertex \( i = 1, \ldots, n \), color vertex \( i \) with a set of colors as follows:

\[
\begin{cases}
1 + \{k(i-1), k(i-1) + 1, \ldots, ik - 1 \mod (2k + 1)\} & \text{if } i \leq 2k + 1 \\
1 \text{ to } k & \text{if } i > 2k + 1 \text{ and even} \\
k + 1 \text{ to } 2k & \text{if } i > 2k + 1 \text{ and odd}
\end{cases}
\]

For example, to 4-color \( C_{15} \) with nine colors, we would follow the above procedure to produce the result illustrated in Figure 2.

We have shown that \( \chi_k(G) = \left\lfloor \frac{2nk}{n-1} \right\rfloor \) for \( k < \frac{n}{2} \). Now if \( k > \frac{n}{2} \), then:

\[
\chi_k(C_n) \leq \chi_{\frac{n-1}{2}}(C_n) + \chi_{k-(\frac{n-1}{2})}(C_n) = \left\lfloor \frac{2n}{n-1} \right\rfloor + \left\lfloor \frac{2n(k - (\frac{n-1}{2})}{n-1} \right\rfloor = \left\lfloor \frac{2nk}{n-1} \right\rfloor.
\]

Therefore, \( \chi_k(C_n) = \left\lfloor \frac{2nk}{n-1} \right\rfloor \) for \( n \) odd. 

2.2 Mycielskians of Cycles

We now prove results regarding the chromatic stability of Mycieskians of cycles.
Figure 2: A 4-coloring of $C_{15}$ with nine colors.
2.2.1 Mycielskians of Even Cycles

As with even cycles, the Mycielskians of even cycles have the property of chromatic stability.

**Theorem 2.3.** For $n$ even, $\chi_k(\mu(G)) = \left\lceil \frac{5k}{2} \right\rceil$.

**Proof:** Each $\mu(C_n)$, $n$ even, contains an induced 5-cycle. By Theorem 2.2, a minimum $k$-coloring of a 5-cycle requires $\left\lceil \frac{5k}{2} \right\rceil$ colors, therefore $\chi_k(\mu(C_n)) \geq \left\lceil \frac{5k}{2} \right\rceil$.

To show $\chi_k(\mu(C_n)) \leq \left\lceil \frac{5k}{2} \right\rceil$, we use a proof by induction. For $n$ even, $\mu(C_n) - \{z\}$ is a bipartite graph, and hence, may be 1-colored with two colors. Coloring the center with a third color shows that $\chi_1(\mu(C_n)) \leq \left\lceil \frac{5}{2} \right\rceil$. Additionally, $\mu(C_n) - \{z\}$ may be 2-colored with four colors by coloring one partite set with the colors 1 and 2, and the other with colors 3 and 4. Now replace 1 and 3 on the inner nodes by 5. Then the center may be colored by 1 and 3. Therefore, $\chi_2(\mu(C_n)) \leq \left\lceil \frac{5(2)}{2} \right\rceil$.

For $k > 2$, assume $\chi_{k-2}(\mu(C_n)) \leq \left\lceil \frac{5(k-2)}{2} \right\rceil$. Then

$$\chi_k(\mu(C_n)) \leq \chi_{k-2}(\mu(C_n)) + \chi_2(\mu(C_n)) \leq \left\lceil \frac{5(k-2)}{2} \right\rceil + 5 = \left\lceil \frac{5k}{2} \right\rceil.$$

Therefore, the Mycielskian of an even cycle is chromatically stable. $\square$

2.2.2 Mycielskians of Odd Cycles

Stahl [6] showed that $\mu(C_n)$ is chromatically unstable when $n = 5$. We will show the same property holds when $n = 7, 9, 11$, and 13. Of special importance is the fact that the chromatic stability number rises to three for $\mu(C_{11})$. For
completeness, we begin by showing that $\mu(C_3)$ is chromatically stable.

**Theorem 2.4.** $\chi_k(\mu(C_3)) = \left\lceil \frac{10k}{3} \right\rceil$ for $k = 1, 2, 3, \ldots$

**Proof:** Since $\chi_f(C_3) = 3$ and $\chi_f(\mu(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}$, we have $\chi_f(\mu(C_3)) = \frac{10}{3}$. Then $\chi_k(\mu(C_3)) \geq \lceil k \chi_f(C_3) \rceil = \left\lceil \frac{10k}{3} \right\rceil$.

To show $\chi_k(\mu(C_3)) \leq \left\lceil \frac{10k}{3} \right\rceil$, we first illustrate that the inequality holds for $k = 1, 2, \text{ and } 3$ by showing the appropriate $k$-colorings in Figure 3.

![Figure 3: 1-, 2-, and 3-colorings of $\mu(C_3)$.](image)

Using induction, for $k > 3$, assume $\chi_{k-3}(\mu(C_3)) \leq \left\lceil \frac{10(k-3)}{3} \right\rceil$. Then

$$\chi_k(\mu(C_3)) \leq \chi_{k-3}(\mu(C_3)) + \chi_3(\mu(C_3)) \leq \left\lceil \frac{10(k-3)}{3} \right\rceil + 10 = \left\lceil \frac{10k}{3} \right\rceil.$$

Therefore, $\chi_k(\mu(C_3)) = \left\lceil \frac{10k}{3} \right\rceil$ for $k > 1$. □
We now present examples of graphs which are chromatically unstable, beginning with a proof of Stahl’s result for $\mu(C_5)$. We will use the following lemma.

**Lemma 2.2.** $\chi_1(\mu(C_n)) = 4$ for $n$ odd.

**Proof:** Since the chromatic number of any odd cycle is 3, the result follows from (9). □

**Theorem 2.5.**

$$\chi_k(\mu(C_5)) = \begin{cases} 4 \quad & \text{for } k = 1 \\ \left\lceil \frac{29k}{10} \right\rceil \quad & \text{otherwise} \end{cases}$$

**Proof:** Since $\chi_f(C_5) = \frac{5}{2}$ and $\chi_f(\mu(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}$, we have $\chi_f(\mu(C_5)) = \frac{29}{10}$. Then $\chi_k(\mu(C_5)) \geq [k \chi_f(C_5)] = \left\lceil \frac{29k}{10} \right\rceil$. This lower bound holds for all $k > 1$.

However, for $k = 1$, Lemma 2.2 shows $\chi_1(\mu(C_5)) = 4$.

We now have only to show that $\chi_k(\mu(C_5)) \leq \left\lceil \frac{29(k-10)}{10} \right\rceil$ for $k > 1$. This is done with a proof by induction. For $k = 2, 3, 10, \text{and } 11$, Table 1 illustrates that the inequality holds.

Furthermore, since $\chi_{k+l}(G) \leq \chi_k(G) + \chi_l(G)$:

\[
\begin{align*}
\chi_4(\mu(C_5)) & \leq \chi_2(\mu(C_5)) + \chi_2(\mu(C_5)) = 12 \\
\chi_5(\mu(C_5)) & \leq \chi_2(\mu(C_5)) + \chi_3(\mu(C_5)) = 15 \\
\chi_6(\mu(C_5)) & \leq \chi_3(\mu(C_7)) + \chi_3(\mu(C_5)) = 18 \\
\chi_7(\mu(C_5)) & \leq \chi_3(\mu(C_5)) + \chi_4(\mu(C_5)) = 21 \\
\chi_8(\mu(C_5)) & \leq \chi_4(\mu(C_5)) + \chi_4(\mu(C_5)) = 24 \\
\chi_9(\mu(C_5)) & \leq \chi_4(\mu(C_5)) + \chi_5(\mu(C_5)) = 27 
\end{align*}
\]

Thus, $\chi_k(\mu(C_5)) \leq \left\lceil \frac{29(k-10)}{10} \right\rceil$ for $2 \leq k \leq 11$. For $k > 11$, assume $\chi_{k-10}(\mu(C_5)) \leq \left\lceil \frac{29(k-10)}{10} \right\rceil$. Then

$$\chi_k(\mu(C_5)) \leq \chi_{k-10}(\mu(C_5)) + \chi_{10}(\mu(C_5)) \leq \left\lceil \frac{29(k-10)}{10} \right\rceil + 29 = \left\lceil \frac{29k}{10} \right\rceil.$$
maximal independent sets

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{maximal} & k \\
\text{independent sets} & 2 & 3 & 10 & 11 \\
\hline
\{x_1, x_3, y_1, y_3\} & 1 & 1 & 3 & 3 \\
\{x_2, x_4, y_2, y_4\} & 0 & 1 & 3 & 3 \\
\{x_3, x_5, y_3, y_5\} & 0 & 1 & 3 & 4 \\
\{x_4, x_1, y_4, y_1\} & 1 & 1 & 3 & 3 \\
\{x_5, x_2, y_5, y_2\} & 1 & 1 & 3 & 3 \\
\{x_1, x_3, z\} & 0 & 1 & 2 & 3 \\
\{x_2, x_4, z\} & 1 & 0 & 2 & 2 \\
\{x_3, x_5, z\} & 1 & 0 & 2 & 1 \\
\{x_4, x_1, z\} & 0 & 1 & 2 & 2 \\
\{x_5, x_2, z\} & 0 & 1 & 2 & 3 \\
\{x_4, y_1, y_2, y_4\} & 0 & 0 & 0 & 1 \\
\{y_1, y_2, y_3, y_4, y_5\} & 1 & 1 & 4 & 4 \\
\hline
\chi_k(\mu(C_5)) & 6 & 9 & 29 & 32 \\
\hline
\end{array}
\]

Table 1: The weight on each independent set in a k-coloring of \(\mu(C_5)\).

Therefore, \(\chi_k(\mu(C_5)) = \left\lceil \frac{29k}{10} \right\rceil \) for \(k > 1\). \(\square\)

**Theorem 2.6.**

\[
\chi_k(\mu(C_7)) = \begin{cases} 
4 & \text{for } k = 1 \\
\left\lceil \frac{58k}{21} \right\rceil & \text{otherwise}
\end{cases}
\]

**Proof:** Since \(\chi_f(C_7) = \frac{7}{3}\), and \(\chi_f(\mu(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}\), we have \(\chi_f(\mu(C_7)) = \frac{58}{21}\). Then \(\chi_k(\mu(C_7)) \geq \lceil k\chi_f(\mu(C_7)) \rceil = \lceil \frac{58k}{21} \rceil\). This lower bound holds for all \(k > 1\). However, for \(k = 1\), Lemma 2.2 shows that \(\chi_1(\mu(C_7)) = 4\).

We now show that \(\chi_k(\mu(C_7)) \leq \left\lceil \frac{58k}{21} \right\rceil\) for \(k > 1\). We will perform a proof by induction. For \(k = 2, 3, 5, 6, 9, 13, 17,\) and \(21\), Table 2 illustrates that the inequality holds.

Furthermore, since \(\chi_{k+1}(G) \leq \chi_k(G) + \chi_l(G)\):
Table 2: The weight on each independent set in a k-coloring of $\mu(C_7)$. 

<table>
<thead>
<tr>
<th>Maximal Independent Sets</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>6</th>
<th>9</th>
<th>13</th>
<th>17</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>${x_1, x_3, x_5, y_1, y_3, y_5}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>${x_2, x_4, x_6, y_2, y_4, y_6}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>${x_3, x_5, x_7, y_3, y_5, y_7}$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>${x_4, x_6, x_1, y_4, y_6, y_1}$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>${x_5, x_7, x_2, y_5, y_7, y_2}$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>${x_6, x_1, x_3, y_6, y_1, y_3}$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>${x_7, x_2, x_4, y_7, y_2, y_4}$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>${x_1, x_3, x_5, z}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>${x_2, x_4, x_6, z}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>${x_3, x_5, x_7, z}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>${x_4, x_6, x_1, z}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>${x_5, x_7, x_2, z}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>${x_6, x_1, x_3, z}$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>${x_7, x_2, x_4, z}$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>${x_5, x_7, y_2, y_3, y_5, y_7}$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>${y_1, y_2, y_3, y_4, y_5, y_6, y_7}$</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>$\chi_k(\mu(C_7))$</td>
<td>6</td>
<td>9</td>
<td>14</td>
<td>17</td>
<td>25</td>
<td>36</td>
<td>47</td>
<td>58</td>
</tr>
</tbody>
</table>
Thus far, we have $\chi_k(\mu(C_7)) \leq \left[\frac{58k}{21}\right]$ for $2 \leq k \leq 22$. For $k > 22$, assume $\chi_{k-21}(\mu(C_7)) \leq \left[\frac{58(k-21)}{21}\right]$. Then

$$\chi_k(\mu(C_7)) \leq \chi_{k-21}(\mu(C_7)) + \chi_{21}(\mu(C_7)) \leq \left[\frac{58(k-21)}{21}\right] + 58 = \left[\frac{58k}{21}\right].$$

Therefore, $\chi_k(\mu(C_7)) = \left[\frac{58k}{21}\right]$ for $k > 1$. □

**Theorem 2.7.**

$$\chi_k(\mu(C_9)) = \begin{cases} 4 & \text{for } k = 1 \\ \left[\frac{97k}{36}\right] & \text{otherwise} \end{cases}$$

**Proof:** In a fashion similar to the previous theorem, we find that $\chi_k(\mu(C_9)) \geq \left[\frac{97k}{36}\right]$. Since $\chi_f(C_9) = \frac{9}{4}$, $\chi_f(\mu(C_9)) = \frac{97}{36}$. Then $\chi_k(\mu(C_9)) \geq \left[k\chi_f(\mu(C_9))\right] = \left[\frac{97k}{36}\right]$. But $\chi_1(\mu(C_9))$ does not meet its lower bound of 3, since Lemma 2.2 shows $\chi_1(\mu(C_9)) = 4$.

To show $\chi_k(\mu(C_9)) \leq \left[\frac{97k}{36}\right]$ for $k > 1$, we perform a proof by induction. For $k = 2, 3, 4, 5, 7, 10, 23, \text{and } 36$, Table 3 illustrates that the inequality holds.

Furthermore, since $\chi_{k+l}(G) \leq \chi_k(G) + \chi_l(G)$, we have:
Table 3: The weight on each independent set in a k-coloring of $\mu(C_9)$. 

<table>
<thead>
<tr>
<th>maximal independent sets</th>
<th>$k$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>10</th>
<th>23</th>
<th>36</th>
</tr>
</thead>
<tbody>
<tr>
<td>${x_1, x_3, x_5, x_7, y_1, y_3, y_5, y_7}$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>${x_2, x_4, x_6, x_8, y_2, y_4, y_6, y_8}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>${x_3, x_5, x_7, x_9, y_3, y_5, y_7, y_9}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>${x_4, x_6, x_8, x_1, y_4, y_6, y_8, y_1}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>${x_5, x_7, x_9, x_2, y_5, y_7, y_9, y_2}$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>${x_6, x_8, x_1, x_3, y_6, y_8, y_1, y_3}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>${x_7, x_9, x_2, x_4, y_7, y_9, y_2, y_4}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>${x_8, x_1, x_3, x_5, y_8, y_1, y_3, y_5}$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>${x_9, x_2, x_4, x_6, y_9, y_2, y_4, y_6}$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>${x_1, x_3, x_5, x_7, z}$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>${x_2, x_4, x_6, x_8, z}$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>${x_3, x_5, x_7, x_9, z}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>${x_4, x_6, x_8, x_1, z}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>${x_5, x_7, x_9, x_2, z}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>${x_6, x_8, x_1, x_3, z}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>${x_7, x_9, x_2, x_4, z}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>${x_8, x_1, x_3, x_5, z}$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>${x_9, x_2, x_4, x_6, z}$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>${x_3, x_5, x_7, y_1, y_3, y_5, y_7, y_9}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>${x_2, x_4, y_2, y_4, y_6, y_7, y_8, y_9}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>${y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9}$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>10</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>$\chi_k(\mu(C_9))$</td>
<td>6</td>
<td>9</td>
<td>11</td>
<td>14</td>
<td>19</td>
<td>27</td>
<td>62</td>
<td>97</td>
<td></td>
</tr>
</tbody>
</table>
For $k = 6, 8, 9, \text{and } 11$:

$$\chi_k(\mu(C_9)) \leq \chi_4(\mu(C_9)) + \chi_{k-4}(\mu(C_9)) \leq 11 + \left\lceil \frac{97(k - 4)}{36} \right\rceil = \left\lceil \frac{97k}{36} \right\rceil.$$ 

For $k = 12, 13, \ldots, 22, 24, 25, \ldots, 35, \text{and } 37$:

$$\chi_k(\mu(C_9)) \leq \chi_{10}(\mu(C_9)) + \chi_{k-10}(\mu(C_9)) \leq 27 + \left\lceil \frac{97(k - 10)}{36} \right\rceil = \left\lceil \frac{97k}{36} \right\rceil.$$ 

Thus far, we have $\chi_k(\mu(C_9)) \leq \left\lceil \frac{97}{36} \right\rceil$ for $2 \leq k \leq 37$. For $k > 37$, assume $\chi_{k-36}(\mu(C_9)) \leq \left\lceil \frac{97(k-36)}{36} \right\rceil$. Then

$$\chi_k(\mu(C_9)) \leq \chi_{k-36}(\mu(C_9)) + \chi_{36}(\mu(C_9)) \leq \left\lceil \frac{97(k - 36)}{36} \right\rceil + 97 = \left\lceil \frac{97k}{36} \right\rceil.$$ 

Therefore, $\chi_k(\mu(C_9)) \leq \left\lceil \frac{97}{36} \right\rceil$ for all $k > 37$, and the proof is complete. $\square$

We continue to proofs of the chromatic instability of $\mu(C_{11})$ and $\mu(C_{13})$. Note that the chromatic stability number has increased from 1 to 3.

**Lemma 2.3.** $\chi_3(\mu(C_{11})) = 9$.

**Proof:** We first show that $\chi_3(\mu(C_{11})) \geq 9$. Suppose not. Then we are able to find eight independent sets of $\mu(C_{11})$ such that each node is included in at least three sets. To color all twenty-three nodes with three colors each, we need at least $23 \cdot 3 = 69$ node-colors total.

Label the nodes as before, with the outer nodes labeled $x_1$ through $x_{11}$, the inner nodes $y_1$ through $y_{11}$, and the center $z$. The largest independent set which includes $z$ is the type of set shown in Figure 4. We refer to this type of set as a
Figure 4: One of 11 Type Z independent sets of $\mu(C_{11})$. 
Type Z independent set and notice that through rotation, there are eleven such sets.

Three Type Z independent sets are required to 3-color \( z \), providing at most \( 3 \cdot 6 = 18 \) node-colors. We now require at most five additional independent sets which together provide at least 51 node-colors.

The unique largest independent set of \( \mu(C_{11}) \) is the one consisting of the eleven inner nodes. Call this set a Type Y set, illustrated in Figure 5.

![Type Y independent set of \( \mu(C_{11}) \)](image)

Figure 5: Type Y independent set of \( \mu(C_{11}) \).

At least one Type Y set must be used to provide 51 node-colors using five
independent sets. However, if more than one Type Y set is used, there are three or fewer independent sets with which to color the outer nodes. Since any independent set contains at most five outer nodes, these three independent sets combined with the three independent sets of Type Z provide at most $3 \cdot 5 + 3 \cdot 5 = 30$ node-colors on the outer nodes. But since the outer nodes require at least $3 \cdot 11 = 33$ node-colors, we must use exactly one of the Type Y independent sets.

We have used four independent sets, providing at most $18 + 11 = 29$ node-colors. So we must find four more independent sets which together provide at least 40 node-colors. Since we can no longer use a Type Y set, the only type of independent set containing eleven nodes, each of the four independent sets must contain ten nodes, for a total of exactly 69 node-colors. In other words, each node must be included in exactly three independent sets.

Each Type Z independent set provides five node-colors on the outer nodes. Therefore, we need exactly $33 - (3 \cdot 5) = 18$ more node-colors on the outer nodes. There are two cases which satisfy this criteria. See Figures 6 through 8 for illustrations of the following types of independent sets.

Case 1: Three Type $X_5$, one Type $X_3$. One independent set of Type $X_3$ colors seven inner nodes. These nodes have been previously colored by an independent set of Type Y, so each requires inclusion in exactly one more independent set, a set of Type $X_5$. Consider the four sequential inner nodes
of the Type $X_3$ set. We require two distinct independent sets of Type $X_5$ to complete the 3-coloring of these four nodes. But then the last independent set of Type $X_5$ cannot be used without adding another color to at least one of the four sequential inner nodes, an impossibility since each must have exactly three colors.

![Graph](image)

**Figure 6:** One of 11 Type $X_5$ independent sets of $\mu(C_{11})$.

**Case 2:** Two Type $X_5$, two Type $X_4$. Any choice of the two independent sets of Type $X_4$ forces at least one inner node to be colored by both of these
Figure 7: One of 11 Type $X_3$ independent sets of $\mu(C_{11})$. 
Figure 8: One of 11 Type $X_4$ independent sets of $\mu(C_{11})$. 
independent sets. Call this node $y_j$. Since $y_j$ was already 1-colored by the independent set of Type $Y$, it is now 3-colored. Therefore, we cannot include $y_j$ in either of the two Type $X_3$ sets to be used. Then $x_j$ will not be included in a Type $X_3$ set either, and therefore must have been colored by three independent sets of Type $Z$. Consider $x_{j+1}$, which is adjacent to both $x_j$ and $y_j$. Since $x_j$ and $y_j$ are colored by six distinct independent sets, we would require three more distinct independent sets to color $x_{j+1}$, for a total of nine, a contradiction of our hypothesis that $\mu(C_{11})$ can be 3-colored with eight colors.

We have shown that $\chi_3(\mu(C_{11})) \geq 9$. Table 4 illustrates that $\mu(C_{11})$ can be 3-colored with 9 colors; therefore, $\chi_3(\mu(C_{11})) \leq 9$. □

**Theorem 2.8.**

$$\chi_k(\mu(C_{11})) = \begin{cases} 4 & \text{for } k = 1 \\ 9 & \text{for } k = 3 \\ \left[\frac{146k}{85}\right] & \text{otherwise} \end{cases}$$

**Proof:** Similar to previous theorems, $\chi_j(C_{11}) = \frac{11}{3}$, which implies $\chi_j(\mu(C_{11})) = \frac{146}{85}$, and $\chi_k(\mu(C_{11})) \geq \left[\frac{146k}{85}\right]$. But for $k = 1$ and 3, this lower bound is not met, since Lemma 2.2 shows $\chi_1(\mu(C_{11})) = 4$ and Lemma 2.3 shows $\chi_3(\mu(C_{11})) = 9$.

To show $\chi_k(\mu(C_{11})) \leq \left[\frac{146k}{85}\right]$ for $k = 2, 4, 5, 6, \ldots$, we perform a proof by induction. For $k = 2, 4, 5, 6, 7, 9, 29, 32, \text{ and } 55$, Table 4 illustrates that the inequality holds. For $k = 8, 10, 11, \ldots, 28, 30, 31, 33, 34, \ldots, 54, 56, \text{ and } 57$, we have $\chi_k(\mu(C_{11})) \leq \chi_6(\mu(C_{11}))+\chi_{k-6}(\mu(C_{11})) = \left[\frac{146k}{85}\right]$. For $k = 58$, we have

$$\chi_{58}(\mu(C_{11})) \leq \chi_6(\mu(C_{11}))+\chi_{52}(\mu(C_{11})) = \left[\frac{146\times58}{85}\right].$$
Thus far, we have \( \chi_k(\mu(C_{11})) \leq \frac{146(k-58)}{55} \) for \( k = 2, 4, 5, \ldots, 58 \). For \( k > 58 \), assume \( \chi_{k-55}(\mu(C_{11})) \leq \frac{146(k-55)}{55} \). Then \( \chi_k(\mu(C_{11})) \leq \chi_{k-55}(\mu(C_{11})) + \chi_{55}(\mu(C_{11})) = \frac{146k}{55} \) for \( k > 58 \), and we are done. \( \square \)

**Theorem 2.9.**

\[
\chi_k(\mu(C_{13})) = \begin{cases} 
4 & \text{for } k = 1 \\
8 \text{ or } 9 & \text{for } k = 3 \\
\frac{205k}{78} & \text{otherwise}
\end{cases}
\]

**Proof:** Since \( \chi_f(C_{13}) = \frac{13}{6} \), we have \( \chi_f(\mu(C_{13})) = \frac{205}{78} \) and \( \chi_k(\mu(C_{13})) \geq \frac{205k}{78} \).

Lemma 2.2 shows \( \chi_1(\mu(C_{13})) = 4 \). Computational results have shown that \( \chi_3(\mu(C_{13})) = 9 \). However, at the time of this paper, a theoretical proof does not exist. Table 5 shows \( \chi_3(\mu(C_{13})) \leq 9 \).

We prove \( \chi_k(\mu(C_{13})) \leq \frac{205k}{78} \) for \( k = 2, 4, 5, 6, 7, 9, 11, 14, 19, 27, 35, \) and \( 78 \), Table 5 illustrates that the inequality holds.

For \( k = 8, 10, \) and \( 12 \), we have \( \chi_k(\mu(C_{13})) \leq \chi_6(\mu(C_{13})) + \chi_{k-6}(\mu(C_{13})) = \frac{205k}{78} \).

For \( k = 54, 62, \) and \( 70 \), we have \( \chi_k(\mu(C_{13})) \leq \chi_{35}(\mu(C_{13})) + \chi_{k-35}(\mu(C_{13})) = \frac{205k}{78} \).

For \( k = 13 \) through \( 81 \) except \( 14, 19, 27, 35, 54, 62, \) and \( 70 \), we have \( \chi_k(\mu(C_{13})) \leq \chi_{11}(\mu(C_{13})) + \chi_{k-11}(\mu(C_{13})) = \frac{205k}{78} \).

For \( k > 81 \), assume \( \chi_{k-78}(\mu(C_{13})) \leq \frac{205(k-78)}{78} \). Then \( \chi_k(\mu(C_{13})) \leq \chi_{k-78}(\mu(C_{13})) + \chi_78(\mu(C_{13})) = \frac{205k}{78} \) for \( k > 81 \), and we are done. \( \square \)
<table>
<thead>
<tr>
<th>maximal independent sets</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>9</th>
<th>29</th>
<th>32</th>
<th>55</th>
</tr>
</thead>
<tbody>
<tr>
<td>{x_1, x_3, x_5, x_7, x_9, y_1, y_3, y_5, y_7, y_9}</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>{x_2, x_4, x_6, x_8, x_{10}, y_2, y_4, y_6, y_8, y_{10}}</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>{x_3, x_5, x_7, x_9, x_{11}, y_3, y_5, y_7, y_9, y_{11}}</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>{x_4, x_6, x_8, x_{10}, x_1, y_4, y_6, y_8, y_{10}, y_1}</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>{x_5, x_7, x_9, x_{11}, x_2, y_5, y_7, y_9, y_{11}, y_2}</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>{x_6, x_8, x_{10}, x_1, x_3, y_6, y_8, y_{10}, y_1, y_3}</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>{x_7, x_9, x_{11}, x_2, x_4, y_7, y_9, y_{11}, y_2, y_4}</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>{x_8, x_{10}, x_1, x_3, x_5, y_8, y_{10}, y_1, y_3, y_5}</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>{x_9, x_{11}, x_2, x_4, x_6, y_9, y_{11}, y_2, y_4, y_6}</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>{x_{10}, x_1, x_3, x_5, x_7, y_{10}, y_1, y_3, y_5, y_7}</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>{x_{11}, x_2, x_4, x_6, x_8, y_{11}, y_2, y_4, y_6, y_8}</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>{x_1, x_3, x_5, x_7, x_9, z}</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>{x_2, x_4, x_6, x_8, x_{10}, z}</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>{x_3, x_5, x_7, x_9, x_{11}, z}</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>{x_4, x_6, x_8, x_{10}, x_1, z}</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>{x_5, x_7, x_9, x_{11}, x_2, z}</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>{x_6, x_8, x_{10}, x_1, x_3, z}</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>{x_7, x_9, x_{11}, x_2, x_4, z}</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>{x_8, x_{10}, x_1, x_3, x_5, z}</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>{x_9, x_{11}, x_2, x_4, x_6, z}</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>{x_{10}, x_1, x_3, x_5, x_7, z}</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>{x_{11}, x_2, x_4, x_6, x_8, z}</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 4: The weight on each independent set in a k-coloring of $\mu(C_{11})$. 

26
| maximal set | \(i\) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 |
| \(\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}\) | \(\emptyset\) | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 1 | 3 | 3 | 3 | \(\emptyset\) |
| \(\{x_2, x_4, x_6, x_8, x_{10}, x_{12}, x_9, x_{10}, x_{12}\}\) | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 3 | 3 | 3 | \(\emptyset\) |
| \(\{x_3, x_2, x_7, x_{10}, x_{12}, x_9, x_{10}, x_{12}\}\) | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 3 | 3 | 3 | \(\emptyset\) |
| \(\{x_4, x_6, x_8, x_{10}, x_{12}, x_{14}, x_9, x_{10}, x_{12}\}\) | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | \(\emptyset\) |
| \(\{x_5, x_7, x_9, x_{11}, x_{13}, x_{15}, x_9, x_{10}, x_{12}\}\) | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 4 | 3 | 3 | 3 | 3 | \(\emptyset\) |
| \(\{x_6, x_8, x_{10}, x_{12}, x_{14}, x_9, x_{10}, x_{12}\}\) | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 2 | 2 | 3 | 3 | 3 | 3 | \(\emptyset\) |
| \(\{x_7, x_9, x_{11}, x_{13}, x_{15}, x_9, x_{10}, x_{12}\}\) | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 2 | 2 | 3 | 3 | 3 | 3 | \(\emptyset\) |
| \(\{x_8, x_{10}, x_{12}, x_{14}, x_9, x_{10}, x_{12}\}\) | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 2 | 2 | 3 | 3 | 3 | 3 | \(\emptyset\) |
| \(\{x_9, x_{10}, x_{12}, x_{14}, x_9, x_{10}, x_{12}\}\) | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 2 | 2 | 3 | 3 | 3 | 3 | \(\emptyset\) |
| \(\{x_{10}, x_{12}, x_{14}, x_9, x_{10}, x_{12}\}\) | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 2 | 2 | 3 | 3 | 3 | 3 | \(\emptyset\) |
| \(\{x_{11}, x_{13}, x_{15}, x_9, x_{10}, x_{12}, x_{14}\\}\) | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 2 | 2 | 3 | 3 | 3 | 3 | \(\emptyset\) |
| \(\{x_{12}, x_{14}, x_9, x_{10}, x_{12}, x_{14}\\}\) | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 2 | 2 | 3 | 3 | 3 | 3 | \(\emptyset\) |
| \(\{x_{13}, x_{15}, x_{17}, x_9, x_{10}, x_{12}, x_{14}\\}\) | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 2 | 2 | 3 | 3 | 3 | 3 | \(\emptyset\) |

Table 5: The weight on each independent set in a k-coloring of \(\mu(C_{13})\).
3 Instability in Multiclique Sequences

Mycielskians of odd cycles have provided interesting results in chromatic instability. These graphs also prove to be useful in discovering instability in multiclique sequences.

3.1 Cycles

We first show that all cycles are cliquishly stable.

**Theorem 3.1.** For $n$ even, $\omega_k(C_n) = 2k$.

**Proof:** The $k$-clique number of a graph is the maximum total integer weight which can be placed on the nodes so that each independent set has weight at most $k$. An even cycle is a bipartite graph; therefore, since we cannot put a weight of more than $k$ on either partite set, $\omega_k \leq 2k$. But by placing a total weight of $k$ on the nodes of each partite set, we find a $k$-clique, so $\omega_k \geq 2k$. Therefore, every even cycle is cliquishly stable. □

**Theorem 3.2.** For $n$ odd, $\omega_k(C_n) = \left\lfloor \frac{2nk}{n-1} \right\rfloor$.

**Proof:** Since $\omega_f(C_n) = \chi_f(C_n) = \frac{2n}{n-1}$ for $n$ odd, and $\omega_k \leq \lfloor k\omega_f \rfloor$, we have $\omega_k \leq \left\lfloor \frac{2nk}{n-1} \right\rfloor$. To show the converse, note that $2k = \left\lfloor \frac{2nk}{n-1} \right\rfloor$ when $k < \frac{n}{2}$. If $k < \frac{n}{2}$, then a $k$-clique of value $2k$ may be formed by placing a weight of $k$ on each of two
adjacent nodes in the cycle. Therefore, for $k < \frac{n}{2}, \omega_k(G) \geq \left\lfloor \frac{2nk}{n-1} \right\rfloor$. So if $k > \frac{n}{2}$, then:

$$\omega_k(C_n) \geq \omega_{\frac{n-1}{2}}(C_n) + \omega_{k-\frac{n-1}{2}}(C_n) = \left\lfloor \frac{2n(n-1)}{n-1} \right\rfloor + \left\lfloor \frac{2n(k-(n-1))}{n-1} \right\rfloor = \left\lfloor \frac{2nk}{n-1} \right\rfloor.$$ 

Therefore, all odd cycles are cliquishly stable. □

### 3.2 Mycielskians of Cycles

We now proceed to results regarding the cliquish stability of Mycielskians of cycles.

#### 3.2.1 Mycielskians of Even Cycles.

We show that Mycielskians of even cycles are cliquishly unstable.

**Theorem 3.3.** For $n$ even, $\mu(C_n) = \left\lfloor \frac{5k}{2} \right\rfloor$.

**Proof:** For $n$ even, $\mu(C_n)$ contains an induced 5-cycle. Since $\omega_k(C_5) = \left\lfloor \frac{5k}{2} \right\rfloor$, we have $\omega_k(\mu(C_n)) \geq \left\lfloor \frac{5k}{2} \right\rfloor$. We show that $\omega_k(\mu(C_n)) \leq \left\lfloor \frac{5k}{2} \right\rfloor$ through a proof by induction. For $n$ even, we may find a 1-clique of $\mu(C_n)$ with value 2 by assigning a weight of 1 to each of two adjacent outer nodes. Additionally, we may find a 2-clique of value 5 by assigning a weight of 1 to each of the following: two adjacent outer nodes, say $x_i$ and $x_{i+1}$, the corresponding inner nodes, $y_i$ and $y_{i+1}$, and the center node $z$. Therefore, when $k = 1$ or 2, we have $\omega_k(\mu(C_n)) \leq \left\lfloor \frac{5k}{2} \right\rfloor$. For $k > 2$, assume $\omega_{k-2}(\mu(C_n)) \leq \left\lfloor \frac{5(k-2)}{2} \right\rfloor$. Then

$$\omega_k(\mu(C_n)) \leq \omega_{k-2}(\mu(C_n)) + \omega_2(\mu(C_n)) \leq \left\lfloor \frac{5(k-1)}{2} \right\rfloor + 5 = \left\lfloor \frac{5k}{2} \right\rfloor.$$
Therefore, Mycielskians of even cycles are cliquishly stable. □

3.2.2 Mycielskians of Odd Cycles

We begin by showing that for $n = 3$ and 5, we have that $\mu(C_n)$ is stable in its multiclique sequence. In the following proofs, we let $a_i, b_i$, and $c$ represent the weights on nodes $x_i, y_i$, and $z$ respectively. In addition, let $a = a_1 + a_2 + \cdots + a_n$, and $b = b_1 + b_2 + \cdots + b_n$.

**Theorem 3.4.** $\omega_k(\mu(C_3)) = \left\lfloor \frac{10k}{3} \right\rfloor$.

**Proof:** Since $\omega_f(\mu(C_3)) = \chi_f(\mu(C_3)) = \frac{10}{3}$, we have $\omega_k(\mu(C_3)) \leq \lfloor k \chi_f(\mu(C_3)) \rfloor = \left\lfloor \frac{10k}{3} \right\rfloor$. We show that $\omega_k(\mu(C_3)) \geq \left\lfloor \frac{10k}{3} \right\rfloor$ for $k = 1, 2, \text{ and } 3$ by presenting the appropriate $k$-cliques in Figure 9.

For $k > 3$ assume $\omega_{k-3}(\mu(C_3)) \geq \left\lfloor \frac{10(k-3)}{3} \right\rfloor$. Then by induction,

$$\omega_k(\mu(C_3)) \geq \omega_{k-3}(\mu(C_3)) + \omega_3(\mu(C_3)) \geq \left\lfloor \frac{10(k-3)}{3} \right\rfloor + 10 = \left\lfloor \frac{10k}{3} \right\rfloor,$$

which concludes our proof. □

**Lemma 3.1.** For $k = 9 + 10h$, where $h \geq 0$ and integer, $\omega_k(\mu(C_5)) \leq \left\lfloor \frac{29k}{10} \right\rfloor - 1$.

**Proof:** For $k = 9 + 10h$, we can algebraically show that $\left\lfloor \frac{29k}{10} \right\rfloor - 1 = 25 + 29h$.

There are two cases.

**Case 1:** $c \geq 4 + 4h$. We have five independent sets of $\mu(C_5)$ of the form $\{x_1, x_3, z\}$, $\{x_2, x_4, z\}$, \ldots, $\{x_5, x_2, z\}$. Since $a_1 + a_3 + c \leq 9 + 10h$, $a_2 + a_4 + c \leq 9 + 10h$, and
Figure 9: 1-, 2-, and 3-cliques of $\mu(C_3)$. 
\[ a = \frac{a_1 + a_3}{2} + \frac{a_2 + a_4}{2} + \frac{a_3 + a_5}{2} + \frac{a_4 + a_1}{2} + \frac{a_5 + a_2}{2} \leq 5 \left( \frac{9 + 10h - c}{2} \right). \]

Additionally, the independent set \( \{y_1, y_2, y_3, y_4, y_5\} \) has weight at most 9 + 10h. Therefore, summing the weights over all nodes:

\[ \omega_{9+10h}(\mu(C_5)) = a + b + c \leq \frac{45 + 50h - 5}{2} + (9 + 10h) + c \leq 25 + 29h. \]

**Case 2:** \( c \leq 3 + 4h \). Now we use independent sets of the form \( \{x_1, x_3, y_1, y_3\}, \{x_2, x_4, y_2, y_4\}, \ldots, \{x_5, x_3, y_5, y_2\} \). Since \( a_1 + a_3 + b_1 + b_3 \leq 9 + 10h, a_2 + a_4 + b_2 + b_4 \leq 9 + 10h, \ldots, a_5 + a_2 + b_5 + b_2 \leq 9 + 10h \), we have

\[ \omega_{9+10h}(\mu(C_5)) = a + b + c = \frac{(a_1 + b_1) + (a_3 + b_3)}{2} + \frac{(a_2 + b_2) + (a_4 + b_4)}{2} \]
\[ + \frac{(a_3 + b_3) + (a_5 + b_5)}{2} + \frac{(a_4 + b_4) + (a_1 + b_1)}{2} + \frac{(a_5 + b_5) + (a_2 + b_2)}{2} + c \]
\[ \leq 5 \left( \frac{9 + 10h}{2} \right) + c \leq 25 + 29h. \]

Therefore, for \( k = 9 + 10h \), we have \( \omega_{9+10h}(\mu(C_5)) \leq 25 + 29h = \left\lfloor \frac{29k}{10} \right\rfloor - 1. \square \)

**Theorem 3.5.**

\[ \omega_k(\mu(C_5)) = \begin{cases} 
\left\lfloor \frac{29k}{10} \right\rfloor - 1 & \text{for } k = 9 + 10h, h \geq 0 \text{ and integer} \\
\left\lfloor \frac{29k}{10} \right\rfloor & \text{otherwise} 
\end{cases} \]

**Proof:** Since \( \omega_f(\mu(C_5)) = \chi_f(\mu(C_5)) = \frac{29}{10} \), then \( \omega_k(\mu(C_5)) \leq \lceil k \omega_f(G) \rceil = \left\lceil \frac{29k}{10} \right\rceil \); for \( k = 9 + 10h \), Lemma 3.1 shows \( \omega_k(\mu(C_5)) \leq \left\lfloor \frac{29k}{10} \right\rfloor - 1 \). To prove equality, we use induction. For \( k \leq 10 \), Table 6 illustrates that equality holds.
Table 6: The weight on each node in a k-clique of $\mu(C_5)$.

For $k > 10$, assume that the result holds for $k - 10$. Then since $\omega_k(\mu(C_5)) \geq \omega_{k-10}(\mu(C_5)) + \omega_{10}(\mu(C_5))$, the result follows for $k > 10$. □

We now present examples of graphs which are cliquishly unstable.

**Lemma 3.2.** $\omega_3(\mu(C_7)) \leq 7$.

**Proof:** There are three cases.

**Case 1:** $c \geq 2$. There are seven independent sets of $\mu(C_7)$ of the form $\{x_1, x_3, x_5, z\}$,

$\{x_2, x_4, x_6, z\}, \ldots, \{x_7, x_2, x_4, z\}$. Since $a_1 + a_3 + a_5 + c \leq 3$, $a_2 + a_4 + a_6 + c \leq 3$, $a_7 + a_2 + a_4 + c \leq 3$, we have

$$a = \frac{a_1 + a_3 + a_5}{3} + \frac{a_2 + a_4 + a_6}{3} + \cdots + \frac{a_7 + a_2 + a_4}{3} \leq 7 \left(\frac{3 - c}{3}\right).$$

The independent set $\{y_1, y_2, \ldots, y_7\}$ has weight at most 3, therefore $\omega_3(\mu(C_7)) = a + b + c \leq \frac{21 - 7c}{3} + 3 + c \leq 7$. 

\[ \]
Case 2: $c = 0$. Now we use independent sets of the form \(\{x_1, x_3, x_5, y_1, y_3, y_5\}\), \(\{x_2, x_4, x_6, y_2, y_4, y_6\}\), \(\{x_7, x_2, x_4, y_7, y_2, y_4\}\). Since \(a_1 + a_3 + a_5 + b_1 + b_3 + b_5 \leq 3\), \(a_2 + a_4 + a_6 + b_2 + b_4 + b_6 \leq 3\), \(a_7 + a_2 + a_4 + b_7 + b_2 + b_4 \leq 3\), we have

\[
\omega_3(\mu(C_7)) = a + b + c = \frac{(a_1 + b_1) + (a_3 + b_3) + (a_5 + b_5)}{3} + \frac{(a_2 + b_2) + (a_4 + b_4) + (a_6 + b_6)}{3} + \cdots + \frac{(a_7 + b_7) + (a_2 + b_2) + (a_4 + b_4)}{3} + c
\]

\[
\leq 7.
\]

Case 3: $c = 1$. Each independent set of outer nodes \(\{x_1, x_3, x_5\}\), \(\{x_2, x_4, x_6\}\), \(\ldots, \{x_7, x_2, x_4\}\) can have a total weight of at most 2, since including \(z\) in each set forms a maximal independent set. So at least one node of each of the above sets has weight 0. Since each node is in three of the above sets, at least three of the outer nodes are weighted 0. We have two cases.

Case a: No two zero-weighted outer nodes are sequential. Then there must be exactly three zero-weighted outer nodes. Without loss of generality, suppose \(a_1 = a_3 = a_5 = 0\). Since \(a_1 > 0\) for \(i \neq 1, 3, 5\), it follows that \(a_7 + a_2 + a_4 + c \geq 4\). But since \(\{x_7, x_2, x_4, z\}\) is an independent set, we have \(a_7 + a_2 + a_4 + c \leq 3\), a contradiction.

Case b: Two of the zero-weighted outer nodes are sequential. Without loss of generality, suppose the two sequential nodes are \(x_1\) and \(x_2\).
Then all other nodes of $\mu(C_7)$ are contained in $S = \{x_3, x_5, x_7, y_3, y_5, y_7\} \cup \{x_4, x_6, y_1, y_2, y_4, y_6\} \cup \{z\}$. But since $S$ is the union of two maximal independent sets and $z$, the total weight on the nodes of $S$ is at most $3+3+1 = 7$.

Therefore, $\omega_3(\mu(C_7)) \leq 7$. □

**Lemma 3.3.** For $k = g + 21h$, where $g = 4, 8, 20$ and $h$ a nonnegative integer, $\omega_k(\mu(C_7)) \leq \left\lceil \frac{58k}{21} \right\rceil - 1$.

**Proof:** Let $q_g$ represent an integer associated with each $g$, where $q_4 = 2, q_8 = 4$, and $q_{20} = 9$. For $k = g + 21h$, we can show $\left\lfloor \frac{58k}{21} \right\rfloor - 1 = \left\lfloor \frac{58k}{21} \right\rfloor + 58h - 1$. There are two cases.

**Case 1:** $c \geq q_g + 9h$. Since $a_1 + a_3 + a_5 + c \leq g + 21h$, $a_2 + a_4 + a_6 + c \leq g + 21h$, $\ldots$, $a_7 + a_2 + a_4 + c \leq g + 21h$, we have

$$a = \frac{a_1 + a_3 + a_5}{3} + \frac{a_2 + a_4 + a_6}{3} + \cdots + \frac{a_7 + a_2 + a_4}{3} \leq 7 \left( \frac{g + 21h - c}{3} \right).$$

Additionally, the independent set $\{y_1, y_2, \ldots, y_7\}$ has weight at most $g+21h$, therefore:

$$\omega_{g+21h}(\mu(C_7)) = a + b + c \leq \frac{7g + 147h - 7c}{3} + (g + 21h) + c \leq \left\lfloor \frac{58g}{21} \right\rfloor + 58h - 1.$$

**Case 2:** $c \leq q_g - 1 + 9h$. Since $a_1 + a_3 + a_5 + b_1 + b_3 + b_5 \leq g + 21h$, $a_2 + a_4 + a_6 + b_2 + b_4 + b_6 \leq g + 21h$, $\ldots$, $a_7 + a_2 + a_4 + b_7 + b_2 + b_4 \leq g + 21h$, we have

$$\omega_{g+21h}(\mu(C_7)) = a + b + c = \frac{(a_1 + b_1) + (a_3 + b_3) + (a_5 + b_5)}{3}.$$
\[
+ \frac{(a_2 + b_2) + (a_4 + b_4) + (a_6 + b_6)}{3} + \cdots + \frac{(a_7 + b_7) + (a_2 + b_2) + (a_4 + b_4)}{3} + c \\
\leq 7 \left( \frac{g + 21h}{3} \right) + c \leq \left\lfloor \frac{58g}{21} \right\rfloor + 58h - 1.
\]

Therefore, for \( g = 4, 8, 20 \), we have \( \omega_k(\mu(C_7)) = \omega_{g + 21h}(\mu(C_7)) \leq \left\lfloor \frac{58g}{21} \right\rfloor + 58h - 1 = \left\lfloor \frac{58k}{21} \right\rfloor - 1. \text{ } \square

**Theorem 3.6.**

\[
\omega_k(\mu(C_7)) = \begin{cases} 
7 & \text{for } k = 3 \\
\left\lfloor \frac{58k}{21} \right\rfloor - 1 & \text{for } k = g + 21h, g = 4, 8, 20, h \geq 0 \text{ and integer} \\
\left\lfloor \frac{58k}{21} \right\rfloor & \text{otherwise}
\end{cases}
\]

**Proof:** Since \( \omega_f(\mu(C_7)) = \chi_f(\mu(C_7)) = \frac{58}{21} \), we have \( \omega_k(\mu(C_7)) \leq \lceil k\omega_f(\mu(C_7)) \rceil = \frac{58k}{21} \). Lemma 3.2 shows \( \omega_3(\mu(C_7)) \leq 7 \). By Lemma 3.3, \( \omega_k(\mu(C_7)) \leq \left\lfloor \frac{58k}{21} \right\rfloor - 1 \) for \( k = g + 21h \) and the given values of \( g \). We use induction to show equality. For \( k = 1, 2, 5, 6, 7, 10, 11, 12, 15, 16, \text{ and } 21 \), Table 7 shows that equality holds.

For \( k = 3, 4, 8, \text{ and } 9 \), we have

\[
\omega_k(\mu(C_7)) \geq \omega_2(\mu(C_7)) + \omega_{k-2}(\mu(C_7)) = \begin{cases} 
\left\lfloor \frac{58k}{21} \right\rfloor - 1 & \text{for } k = 4, 8 \\
\left\lfloor \frac{58k}{21} \right\rfloor & \text{for } k = 3, 9
\end{cases}
\]

For \( k = 13, 14, 17, 18, 19, 20, 22, 23, \text{ and } 24 \), we have

\[
\omega_k(\mu(C_7)) \geq \omega_2(\mu(C_7)) + \omega_{k-2}(\mu(C_7)) = \begin{cases} 
\left\lfloor \frac{58k}{21} \right\rfloor - 1 & \text{for } k = 20 \\
\left\lfloor \frac{58k}{21} \right\rfloor & \text{otherwise}
\end{cases}
\]

For \( k > 24 \), assume that the result holds for \( k = 21 \). Since \( \omega_k(\mu(C_7)) \geq \omega_{k-21}(\mu(C_7)) + \omega_{21}(\mu(C_7)) \), the result holds for \( k > 24 \). \text{ } \square

36
Table 7: The weight on each node in a k-clique of $\mu(C_7)$.
For the next two proofs, we will refer to types of maximal independent sets of \( \mu(C_9) \) as follows:

Type \( X_3 \): \( \{x_1, x_3, x_5, y_1, y_3, y_5, y_7, y_8\}, \{x_2, x_4, x_6, y_2, y_4, y_6, y_8, y_9\}, \ldots, \{x_9, x_2, x_4, y_3, y_2, y_4, y_6, y_7\} \)

Type \( X_4 \): \( \{x_1, x_3, x_5, x_7, y_1, y_3, y_5, y_7\}, \{x_2, x_4, x_6, y_2, y_4, y_6, y_8\}, \ldots, \{x_3, x_2, x_4, x_6, y_3, y_2, y_4, y_5\} \)

Type \( Y \): \( \{y_1, y_2, \ldots, y_9\} \)

Type \( Z \): \( \{x_1, x_3, x_5, x_7, z\}, \{x_2, x_4, x_6, x_8, z\}, \ldots, \{x_9, x_2, x_4, x_6, z\} \)

Lemma 3.4. \( \omega_5(\mu(C_9)) \leq 12 \).

Proof: There are three cases.

Case 1: \( c \geq 3 \). There are nine Type \( Z \) independent sets of \( \mu(C_9) \). Since \( a_1 + a_3 + a_5 + a_7 + c \leq 5 \), \( a_2 + a_4 + a_6 + a_8 + c \leq 5 \), \ldots, \( a_9 + a_2 + a_4 + a_6 + c \leq 5 \), we have

\[
a = \frac{a_1 + a_3 + a_5 + a_7}{4} + \frac{a_2 + a_4 + a_6 + a_8}{4} + \cdots + \frac{a_9 + a_2 + a_4 + a_6}{4}
\]

\[
\leq 9 \left( \frac{5 - c}{4} \right).
\]

The Type \( Y \) independent set has weight at most 5, therefore \( \omega_5(\mu(C_9)) = a + b + c \leq 45 - 9c \leq 5 + c \leq 12 \).

Case 2: \( c \leq 1 \). Using the Type \( X_4 \) independent sets, since \( a_1 + a_3 + a_5 + a_7 + b_1 + b_3 + b_5 + b_7 \leq 5 \), \( a_2 + a_4 + a_6 + a_8 + b_2 + b_4 + b_5 + b_8 \leq 5 \), \ldots, \( a_9 + a_2 + a_4 + a_6 + b_3 + b_2 + b_4 + b_5 \leq 5 \), we have \( \omega_5(\mu(C_9)) = a + b + c \leq 9 \left( \frac{5}{4} \right) + c \leq 12 \).
Case 3: $c = 2$. Each Type $Z$ independent set has weight at most 5, and therefore contains at least one outer node of weight 0. Since each $x_i$ is an element of four Type $Z$ sets, there are at least three zero-weighted outer nodes.

Case a: Two zero-weighted outer nodes are sequential. For any two sequential outer nodes, there is one Type $X_4$ and one Type $X_3$ independent set that together contain all nodes except $z$ and the two sequential zero-weighted outer nodes. Since each independent set is weighted at most 5, for this case, $\omega_0(\mu(C_9)) \leq 12$.

Case b: No two zero-weighted outer nodes are sequential. It is easy to show that there exists but one arrangement of three nonsequential zero-weighted outer nodes which does not violate a maximum weight of 5 on all Type $Z$ independent sets; that is, the three zero-weighted outer nodes must be equally spaced around the cycle. All other outer nodes must be at least 1, to avoid falling into Case a. But the non-zero-weighted outer nodes cannot exceed 1, else we could find Type $Z$ independent sets of weight greater than 5.

Each zero-weighted outer node is an element of a Type $X_4$ independent set which contains three one-weighted outer nodes; therefore, its corresponding inner node has weight at most 2. If any inner node corresponding to a zero-weighted outer node has weight 0, then all other nodes except the center
are contained in two Type $X_4$ independent sets, and then $\omega_5(\mu(C_9)) \leq 5 + 5 + 2 = 12$. If one inner node corresponding to a zero-weighted outer node has weight 2, then investigation of the Type $X_4$ independent sets shows that the other two must have weight 1. Then to exceed a total weight of 12 and to avoid exceeding a maximum weight of 5 on the Type $Y$ independent set, exactly one other inner node must have weight 1; but all such placements result in a set of Type $X_3$ with weight 6. Therefore, all inner nodes corresponding to zero-weighted outer nodes must have weight 1.

The maximum weight on the Type $Y$ independent set then dictates that the remaining inner nodes have total weight 2, but any placement of the additional weight results in a Type $X_4$ independent set with weight 6.

Therefore, $\omega_5(\mu(C_9)) \leq 12$. □

**Lemma 3.5.** For $k = g + 36h$, where $g = 3, 6, 13, 19, 26, 35,$ and $h$ a nonnegative integer, $\omega_k(\mu(C_9)) \leq \left\lfloor \frac{97k}{36} \right\rfloor - 1$.

**Proof:** Let $q_g$ represent an integer associated with each $g$, such that $q_3 = 2, q_6 = 3, q_{13} = 6, q_{19} = 9, q_{26} = 12,$ and $q_{35} = 16$. For $k = g + 36h$, it can be shown that $\left\lfloor \frac{97k}{36} \right\rfloor - 1 = \left\lfloor \frac{97g}{36} \right\rfloor + 97h - 1$. There are two cases.

**Case 1:** $c \geq q_9 + 16h$. Using the nine Type $Z$ independent sets, since $a_1 + a_3 + a_5 + a_7 + c \leq g + 36h$, $a_2 + a_4 + a_6 + a_8 + c \leq g + 36h$, ... $a_9 + a_2 + a_4 + a_6 + c \leq g + 36h$, 

40
we have

\[
a = \frac{a_1 + a_3 + a_5 + a_7}{4} + \frac{a_2 + a_4 + a_6 + a_8}{4} + \cdots + \frac{a_9 + a_2 + a_4 + a_6}{4}
\]

\[
\leq 9 \left( \frac{g + 36h - c}{4} \right).
\]

Since the Type Y independent set has weight at most \(g + 36h\), we have

\[
\omega_{g+36h}(\mu(C_9)) = a + b + c \leq \frac{9g + 324h - 9c}{4} + (g + 36h) + c \leq \left\lfloor \frac{9g}{36} \right\rfloor + 97h - 1.
\]

**Case 2:** \(c \leq q_g - 1 + 16h\). Using the nine Type X independent sets, since

\[
a_1 + a_3 + a_5 + a_7 + b_1 + b_3 + b_5 + b_7 \leq g + 36h, \quad a_2 + a_4 + a_6 + a_8 + b_2 + b_4 + b_6 + b_8 \leq g + 36h, \quad \ldots, \quad a_9 + a_2 + a_4 + a_6 + b_2 + b_4 + b_6 \leq g + 36h,
\]

we have

\[
\omega_{g+36h}(\mu(C_9)) = a + b + c \leq \left\lfloor \frac{9g}{36} \right\rfloor + 97h - 1.
\]

Therefore, for \(g = 3, 6, 13, 19, 26, 35\), we have \(\omega_k(\mu(C_9)) \leq \left\lfloor \frac{9g}{36} \right\rfloor + 97h - 1 = \left\lfloor \frac{97k}{24} \right\rfloor - 1\). □

**Theorem 3.7.**

\[
\omega_k(\mu(C_9)) = \begin{cases} 
12 & \text{for } k = 5 \\
\left\lfloor \frac{97k}{36} \right\rfloor - 1 & \text{for } k = g + 36h, g = 3, 6, 13, 19, 26, 35, h \geq 0 \text{ and integer} \\
\left\lfloor \frac{97k}{36} \right\rfloor & \text{otherwise}
\end{cases}
\]

**Proof:** Since \(\omega_f(\mu(C_9)) = \chi_f(\mu(C_9)) = \frac{97}{36}\), then \(\omega_k(\mu(C_9)) \leq \left\lfloor k \omega_f(\mu(C_9)) \right\rfloor = \left\lfloor \frac{97k}{36} \right\rfloor\). By Lemma 3.4, we have \(\omega_5(\mu(C_9)) \leq 12\), and Lemma 3.5 shows \(\omega_k(\mu(C_9)) \leq \left\lfloor \frac{97k}{36} \right\rfloor - 1\) for \(k = g + 36h\) and the given values of \(g\). Induction will show equality.

For \(k = 1, 2, 7, 8, 9, 12, 15, 16, 22, 25, 29\) and 36, Table 8 shows that equality
holds. For \( k = 3, 4, 5, 6, 10, 11, 13, 14, 17, 18, 19, 20, 23, 24, 26, 27, 30, 31, 33, 34, 35, 38, 39, \) and 40, we have

\[
\omega_k(\mu(C_9)) \geq \omega_2(\mu(C_9)) + \omega_{k-2}(\mu(C_9)) = \begin{cases} 
\left\lfloor \frac{97k}{36} \right\rfloor - 1 & \text{for } k = 3, 6, 13, 19, 26, 35, 39 \\
\left\lfloor \frac{97}{36} \right\rfloor & \text{otherwise}
\end{cases}
\]

For \( k = 21, \omega_k(\mu(C_9)) \geq \omega_9(\mu(C_9)) + \omega_{12}(\mu(C_9)) = \left\lfloor \frac{97(21)}{36} \right\rfloor. \) For \( k = 28, 32, 37, \) and 41, \( \omega_k(\mu(C_9)) \geq \omega_{16}(\mu(C_9)) + \omega_{k-16}(\mu(C_9)) = \left\lfloor \frac{97k}{36} \right\rfloor. \)

\[\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
k & 1 & 2 & 7 & 8 & 9 & 12 & 15 & 16 & 25 & 29 & 36 \\
\hline
x_1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 3 & 3 & 4 & 5 \\
x_2 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 5 \\
x_3 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 5 \\
x_4 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 5 \\
x_5 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 5 \\
x_6 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 5 \\
x_7 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 5 \\
x_8 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 5 \\
x_9 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 3 & 5 & 4 & 5 \\
y_1 & 0 & 1 & 0 & 1 & 0 & 2 & 2 & 2 & 1 & 4 & 4 \\
y_2 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 3 & 3 & 4 \\
y_3 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & 4 \\
y_4 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 2 & 3 & 3 & 4 \\
y_5 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 & 3 & 3 & 4 \\
y_6 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \\
y_7 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \\
y_8 & 0 & 0 & 0 & 1 & 2 & 1 & 2 & 2 & 1 & 3 & 4 \\
y_9 & 0 & 1 & 1 & 2 & 1 & 1 & 3 & 2 & 2 & 4 & 4 \\
z & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\omega_k(\mu(C_9)) & 2 & 5 & 18 & 22 & 24 & 32 & 40 & 43 & 67 & 78 & 97 \\
\hline
\end{array}\]

Table 8: The weight on each node in a k-clique of \( \mu(C_9). \)

For \( k > 41, \) assume the result holds for \( k = 36. \) Then since \( \omega_k(\mu(C_9)) \geq \omega_{k-36}(\mu(C_9)) + \omega_{36}(\mu(C_9)), \) the result holds for \( k > 41. \) □
In the next two proofs, we refer to the following types of maximal independent sets of $\mu(C_{11})$:

Type $X_4$: \{x_1, x_3, x_5, x_7, y_1, y_3, y_5, y_7, y_9, y_{10}\}, \{x_2, x_4, x_6, x_8, y_2, y_4, y_6, y_8, y_{10}, y_{11}\},
\ldots, \{x_{11}, x_2, x_4, x_6, y_{11}, y_2, y_4, y_6, y_8, y_9\}

Type $X_5$: \{x_1, x_3, x_5, x_7, x_9, y_1, y_3, y_5, y_7, y_9\}, \{x_2, x_4, x_6, x_8, x_{10}, y_2, y_4, y_6, y_8, y_{10}\},
\ldots, \{x_9, x_2, x_4, x_6, y_9, y_2, y_4, y_6, y_8\}

Type $Y$: \{y_1, y_2, \ldots, y_{11}\}

Type $Z$: \{x_1, x_3, x_5, x_7, x_9, z\}, \{x_2, x_4, x_6, x_{10}, z\}, \ldots, \{x_9, x_2, x_4, x_6, x_8, z\}

**Lemma 3.6.** $\omega_5(\mu(C_{11})) \leq 12$.

**Proof:** There are three cases.

**Case 1:** $c \geq 3$. There are eleven Type $Z$ independent sets of $\mu(C_{11})$. Since 
\[ a_1 + a_3 + a_5 + a_7 + a_9 + c \leq 5, \quad a_2 + a_4 + a_6 + a_8 + a_{10} + c \leq 5, \ldots, \]
\[ a_9 + a_2 + a_4 + a_6 + a_8 + c \leq 5, \]
we have $a \leq 11 \left(\frac{5+c}{5}\right)$. The Type $Y$ independent set has weight at most 5, therefore 
$\omega_5(\mu(C_{11})) = a + b + c \leq \frac{55-11c}{5} + 5 + c \leq 12$.

**Case 2:** $c \leq 1$. Using the Type $X_5$ independent sets, since 
\[ a_1 + a_3 + a_5 + a_7 + a_9 + b_1 + b_3 + b_5 + b_7 + b_9 \leq 5, \quad a_2 + a_4 + a_6 + a_8 + a_{10} + b_2 + b_4 + b_6 + b_8 + b_{10} \leq 5, \]
\[ \ldots, a_9 + a_2 + a_4 + a_6 + a_8 + b_3 + b_4 + b_6 + b_8 \leq 5, \]
we have $\omega_5(\mu(C_{11})) = a + b + c \leq 11 + c \leq 12$.

**Case 3:** $c = 2$. Each Type $Z$ independent set must contain at least two outer nodes with weight 0. Since each pair of outer nodes are contained in at most four Type $Z$ independent sets, and since some duplication may occur, there are at least five zero-weighted outer nodes.
**Case a:** Two zero-weighted outer nodes are sequential. Then all other nodes excluding the center are contained in the union of a Type $X_4$ and a Type $X_5$ independent set. Then $\omega_5(\mu(C_{11})) \leq 5 + 5 + 2 = 12$ in this case.

**Case b:** No two zero-weighted outer nodes are sequential. Then the five zero-weighted outer nodes must be spaced at least one apart. But since all other outer nodes have positive weight, there exists a Type $Z$ independent set with weight larger than 5, a contradiction.

Therefore, $\omega_5(\mu(C_{11})) \leq 12$. □

**Lemma 3.7.** For $k = g + 55h$, where $g = 8, 14, 17, 23, 26, 34, 43, 49, 52, 54,$ and $h$ a nonnegative integer, $\omega_k(\mu(C_{11})) \leq \left\lfloor \frac{146k}{55} \right\rfloor - 1$.

**Proof:** Let $q_g$ represent an integer associated with each $g$, such that $q_8 = 4, q_{14} = 7, q_{17} = 8, q_{23} = 11, q_{26} = 12, q_{34} = 16, q_{43} = 20, q_{49} = 23, q_{52} = 24$ and $q_{54} = 25$.

For $k = g + 55h$, we have $\left\lfloor \frac{146k}{55} \right\rfloor - 1 = \left\lfloor \frac{146q_g}{55} \right\rfloor + 146h - 1$.

**Case 1:** $c \geq q_g + 25h$. Since each Type $Z$ independent set has weight at most $g + 55h$, we have $a \leq 11 \left( \frac{g + 55h - c}{5} \right)$. Since the Type $Y$ independent set has weight at most $g + 55h$, we have $\omega_{g+55h}(\mu(C_{11})) = a + b + c \leq \frac{11g + 605h - 11c}{5} + (g + 55h) + c \leq \left\lfloor \frac{146q_g}{55} \right\rfloor + 146h - 1$.

**Case 2:** $c \leq q_g - 1 + 25h$. Since each Type $X_5$ independent set has weight at most $g + 55h$, and each node is an element of five such sets, we have $\omega_{g+55h}(\mu(C_{11})) = a + b + c \leq 11 \left( \frac{g + 55h}{5} \right) + c \leq \left\lfloor \frac{58q_g}{21} \right\rfloor + 58h - 1$. 44
Therefore, $\omega_k(\mu(C_{11})) \leq \left\lfloor \frac{58h}{21} \right\rfloor + 58h - 1 = \left\lfloor \frac{58k}{21} \right\rfloor - 1$. □

**Theorem 3.8.**

\[
\omega_k(\mu(C_{11})) = \begin{cases} 
12 & \text{for } k = 5 \\
17 \text{ or } 18 & \text{for } k = 7 \\
\left\lfloor \frac{146k}{55} \right\rfloor - 1 & \text{for } k = g + 55h, g = 8, 14, 17, 23, 26, 34, 43, 49, 52, 54, h \geq 0 \text{ and integer} \\
\left\lfloor \frac{146k}{55} \right\rfloor & \text{otherwise}
\end{cases}
\]

**Proof:** We have $\omega_k(\mu(C_{11})) \leq \left\lfloor k\omega_j(\mu(C_{11})) \right\rfloor = \left\lfloor \frac{146k}{55} \right\rfloor$. Lemma 3.6 shows $\omega_5(\mu(C_{11})) \leq 12$. Computational results show that $\omega_7(\mu(C_{11})) = 17$, however, a theoretical proof does not yet exist. Lemma 3.7 proves $\omega_{g+55h}(\mu(C_{11})) \leq \left\lfloor \frac{146k}{55} \right\rfloor - 1$ for the values of $g$ given above. By induction, we will show equality. For $k = 1, 2, 9, 10, 11, 16, 19, 20, 25, 28, 37, 46$ and 55, Table 9 shows that equality holds.

For $k = 3, 4, 5, 6, 7, 8, 12, 13, 14, 15, 17, 18, 21, 22, 23, 24, 26, 27, 29$ and 30, we have

\[
\omega_k(\mu(C_{11})) \geq \omega_2(\mu(C_{11}))+\omega_{k-2}(\mu(C_{11})) = \begin{cases} 
\left\lfloor \frac{146k}{55} \right\rfloor - 1 & \text{for } k = 8, 14, 17, 23, 26 \\
\left\lfloor \frac{146k}{55} \right\rfloor & \text{otherwise}
\end{cases}
\]

For $k = 31, 32, \ldots, 36, 38, 39, \ldots, 45, 47, 48, \ldots, 54, 56, 57, \ldots, 62$, we have

\[
\omega_k(\mu(C_{11})) \geq \omega_{20}(\mu(C_{11}))+\omega_{k-20}(\mu(C_{11})) = \begin{cases} 
\left\lfloor \frac{146k}{55} \right\rfloor - 1 & \text{for } k = 34, 43, 49, 52, 54 \\
\left\lfloor \frac{146k}{55} \right\rfloor & \text{otherwise}
\end{cases}
\]

For $k > 62$, assume that the result holds for $k = 55$. Since $\omega_k(\mu(C_{11})) \geq \omega_{k-55}(\mu(C_{11}))+\omega_{55}(\mu(C_{11}))$, the result holds for $k > 62$. □

Additionally, computational results have shown that $\mu(C_{13})$ is cliquishly unstable at $k = 7, 9$, and 18.
Table 9: The weight on each node in a $k$-clique of $\mu(C_{11})$. 

<table>
<thead>
<tr>
<th>nodes</th>
<th>1</th>
<th>2</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>16</th>
<th>19</th>
<th>20</th>
<th>25</th>
<th>28</th>
<th>37</th>
<th>46</th>
<th>55</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>$x_5$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>$x_6$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>$x_7$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>$x_8$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>$x_9$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>$x_{10}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>$x_{11}$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>$y_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>$y_2$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>$y_3$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>$y_4$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>$y_5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>$y_6$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>$y_7$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>$y_8$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$y_9$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$y_{10}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>$y_{11}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>$z$</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>9</td>
<td>11</td>
<td>13</td>
<td>17</td>
<td>21</td>
<td>25</td>
</tr>
</tbody>
</table>

$\omega_k(\mu(C_{11}))$ | 2 | 5 | 24 | 26 | 29 | 42 | 50 | 53 | 66 | 74 | 98 | 122 | 146 |
REFERENCES


