Unit-Consistency and Inequality Orderings

by

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Abstract: This paper examines the implications of the newly proposed unit-consistency axiom for partial inequality orderings. The unit-consistency axiom requires the ordinal rankings of income distributions to remain unaffected when incomes are expressed in a different measuring unit. The partial inequality ordering condition we consider is Lorenz dominance. While both relative and absolute Lorenz dominance conditions are unit-consistent, some intermediate Lorenz dominance conditions may violate the axiom. By introducing additional axioms on the notion of intermediate inequality, we consider a class of “single-parameter” intermediate Lorenz curves. We then show that within such a general framework of intermediate Lorenz dominance, the only unit-consistent dominance condition is the one related to Krtscha (1994)’s intermediate notion of inequality which has also been further studied and generalized by Zoli (1998, 1999) and Yoshida (2003).

Key Words: unit-consistency, unit-invariance, intermediateness, inequality orderings.

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I. Introduction

When comparing income distributions across countries or across time, researchers often need to choose a common measuring unit for all incomes. For example, to compare income inequality or welfare between the United Kingdom and the United States, one may need to decide whether to use sterling or dollar in expressing both countries’ incomes. Since the choice of any specific measuring unit is arbitrary, it is natural to expect that the inequality rankings will not be affected by what specific measuring unit is used. That is, if the U.S. income distribution is more unequal than the U.K. distribution when incomes are measured in U.S. dollars, then the same conclusion should hold when all incomes are converted to British pounds. This “unit-consistency” requirement is essential for inequality comparisons (and, in fact, for all distributional comparisons) to be meaningful — just imagine the possibility that, in the absence of this requirement, the U.S. may be more unequal than the U.K. when all incomes are expressed in U.S. dollars but the ranking can be reversed when all incomes are converted into British pounds.

Given the importance of unit-consistency in inequality comparisons, it is surprising to see that until recently (Zheng, 2002) the issue has not been directly and properly addressed in the inequality measurement literature. Rather, the issue has been sidestepped by maintaining a unit-invariance assumption — the numerical indices of inequality remain unchanged with respect to the measuring unit of income. While unit-invariance clearly implies unit-consistency, it is inappropriate to be regarded as an axiom in lieu of unit-consistency for inequality measurement. First, this unit-invariance argument imposes a cardinal requirement on an ordinal concept. It is generally accepted that an inequality index is an ordinal number — it is meaningful only when the number is compared with other inequality indices. In this sense, there is no reason to impose any invariance restriction on an inequality index when the measuring unit is changed, say, from dollars to thousands of dollars. Second, the argument focuses entirely on the use of nominal income in inequality measurement. This would be an appealing requirement if we deal with distributions having the same real mean income but different nominal mean incomes. However, in practice, situations like this rarely if ever occur. Thus, the unit-invariance requirement in this sense has little relevance to empirical applications. Third, and more importantly, while unit-invariance is justified on the ground of using nominal income, it has a far-fetched implication on inequality measurement with real income: the inequality level remains unaffected when all real incomes are doubled. This implication represents a distinct value judgement on the notion of income inequality.

In the literature, how a measure of inequality should react to a uniform change in all incomes has been (and will perhaps remain to be) an issue of debate. For a given income distribution, should inequality be the same if all (real) incomes are increased by 10 percent or if all (real) incomes are increased by $10? The answers to this
question have been sharply different among economists as well as the general public. Over time, these different answers have formed three distinct value judgements on inequality measurement which Kolm (1976) intuitively referred to as the “rightist” view, the “leftist” view and the “centralist” view, respectively. The rightist view believes that income inequality should be invariant to an equal-proportion change to all (real) incomes. The leftist view, in contrast, feels that income inequality should be invariant to an equal-dollar-amount change to all (real) incomes. The centralist view, however, believes that both rightist view and leftist view are too extreme and that an equal-proportion addition to all incomes should increase income inequality while an equal-dollar-amount addition to all incomes should reduce income inequality. To a centralist view holder, the inequality level may be held unchanged through a uniform addition of, say, 10 percent plus $10 to all incomes; neither part of the addition alone can keep inequality unchanged. When these views are implemented in measuring income inequality, three distinct classes of inequality measures can be characterized and they are, respectively, the relative, absolute and intermediate classes of inequality indices.

Clearly, the value judgement embedded in the unit-invariance assumption is the rightist view. Thus, the unit-invariance condition excludes all value judgements in inequality measurement other than rightist and excludes all inequality indices except relative indices. In contrast, as we will see below, unit-consistency accommodates all value judgments and accepts all notions of inequality measurement.

In a different paper, Zheng (2003) examined the implication of the axiom for decomposable class of inequality indices. By replacing the unit-invariance axiom with the unit-consistency axiom in Shorrocks (1980), Zheng (2003) derived a extended generalized entropy family of inequality indices which contain relative indices (the generalized entropy class), absolute indices and intermediate inequality indices. In this paper, we investigate the implications of the unit-consistency axiom for partial inequality orderings. The partial ordering conditions that we consider in this paper are of the Lorenz type. Corresponding to the three classes of inequality indices, three types of Lorenz partial ordering conditions have been established in the literature, namely, relative Lorenz dominance, absolute Lorenz dominance, and intermediate Lorenz dominance. The unit-consistency axiom in the case of Lorenz dominance states that the dominance relation between any two distributions should not be affected by the use of a different income unit. For both relative and absolute Lorenz orderings, unit-consistency is satisfied since each Lorenz ordinate is a homogenous function of incomes. For intermediate Lorenz orderings, however, things are quite different. This is because, compared with the concepts of relativity and absoluteness, the notion of intermediateness is naturally very vague and, consequently, there are many ways it can be perceived and defined. For example, following the different notions of intermediateness proposed by Pfingsten (1986) and Krtscha (1994), two types of intermediate Lorenz dominance can be constructed (Pfingsten, 1986a, and Yoshida, 2003). We observe that the Pfingsten-type intermediate Lorenz dominance
violates the unit-consistency axiom but the Krtscha-type intermediate Lorenz dominance satisfies the axiom. Following this observation, we attempt to characterize a general unit-consistent intermediate Lorenz dominance condition. To do so, we define a class of “single-parameter intermediate Lorenz dominance” and consider two additional axioms on the notion of intermediate inequality orderings. Viewing an intermediate Lorenz ordinate as some kind of average between the absolute Lorenz ordinate and the relative Lorenz ordinate, we consider a set of properties that such an intermediate Lorenz curve may possess. Within this general framework, we show that a single-parameter intermediate Lorenz ordinate must be a quasilinear weighted-mean between the absolute Lorenz ordinate and the relative Lorenz ordinate. When the unit-consistency axiom is considered for all quasilinear-mean intermediate Lorenz orderings, the Krtscha-type intermediate Lorenz dominance emerges as the only one that can be unit-consistent.

The rest of this paper is organized as follows. The next section introduces various Lorenz dominance conditions and formally states the unit-consistency axiom. In this section, we also demonstrate that the unit-consistency axiom may be violated by some intermediate Lorenz dominance conditions. Section III focuses on the characterization of unit-consistent intermediate Lorenz dominance. Section IV concludes.

II. Unit-Consistency and Inequality Orderings

All income distributions are discrete and are assumed to be of the same fixed size \( n \geq 2 \); all results hold for populations with variable and different sizes. Each distribution \( \mathbf{x} \) is drawn from the space \( \Psi^n := \{ \mathbf{x} = (x_1, x_2, ..., x_n)|x_i \in (0, \infty) \text{ and } x_1 \leq x_2 \leq \cdots \leq x_n \} \). The mean of each distribution \( \mathbf{x} \) is denoted by \( \bar{x} \).

2.1. Relative, absolute and intermediate Lorenz orderings

Relative Lorenz dominance is the best known partial inequality ordering condition. As stated in the Introduction, this dominance condition reflects the rightist view of inequality measurement. For any two distributions \( \mathbf{x} \in \Psi^n \) and \( \mathbf{y} \in \Psi^n \), \( \mathbf{x} \) relative Lorenz dominates \( \mathbf{y} \), denoted as \( \mathbf{x} \mathcal{L}_r \mathbf{y} \), if and only if

\[
L_r(\mathbf{x}; l) = \frac{1}{n} \sum_{i=1}^{l} \frac{x_i}{\bar{x}} \geq \frac{1}{n} \sum_{i=1}^{l} \frac{y_i}{\bar{y}} = L_r(\mathbf{y}; l)
\] (2.1)

for all \( l = 1, 2, ..., n-1 \) and the strict inequality holds at least once. Clearly, the relative Lorenz ordinate \( L_r(\mathbf{x}; l) = \frac{1}{n} \sum_{i=1}^{l} \frac{x_i}{\bar{x}} \) is unaffected by an equal-proportion change to all incomes. Graphically, a relative Lorenz curve is a convex curve connecting between two coordinates \((0, 0)\) and \((1, 1)\).

The leftist view of inequality measurement gives rise to absolute Lorenz dominance. For any two distributions \( \mathbf{x} \in \Psi^n \) and \( \mathbf{y} \in \Psi^n \), \( \mathbf{x} \) absolute Lorenz dominates \( \mathbf{y} \), denoted as \( \mathbf{x} \mathcal{L}_a \mathbf{y} \), if and only if

\[
L_a(\mathbf{x}; l) = \frac{1}{n} \sum_{i=1}^{l} (x_i - \bar{x}) \geq \frac{1}{n} \sum_{i=1}^{l} (y_i - \bar{y}) = L_a(\mathbf{y}; l)
\] (2.2)
for all $l = 1, 2, ..., n - 1$ and the strict inequality holds at least once. Clearly, the absolute Lorenz ordinate $L_a(x; l) = \frac{1}{n} \sum_{i=1}^{l} (x_i - \bar{x})$ is unaffected by an equal-dollar-amount change to all incomes. Graphically, an absolute Lorenz curve is a U-shape curve connecting between coordinates $(0, 0)$ and $(1, 0)$.

Based on the centralist notion of inequality measurement, Kolm (1976) defined the class of “centralist” measures of inequality. Based upon Kolm’s characterization, Pfingsten (1986) refined the centralist notion and introduced a class of “intermediate” measures of inequality. Pfingsten’s intermediate measures of inequality are invariant to neither an equal-proportion increase in all incomes nor an equal-dollar-amount increase in all incomes but to some combination of the two increases (see the second part of (2.4) below). In terms of inequality orderings, Pfingsten (1986a) proposed the following Lorenz dominance condition for his intermediate class of inequality measures: for any two distributions $x \in \Psi^n$ and $y \in \Psi^n$, $x$ intermediate Lorenz dominates $y$ by all inequality measures in the Pfingsten class, denoted as $xL_p y$, if and only if

$$L_p(x; \mu, l) = \frac{1}{n} \sum_{i=1}^{l} \left( \frac{x_i - \bar{x}}{\mu \bar{x} + 1 - \mu} + \mu \right) \geq \frac{1}{n} \sum_{i=1}^{l} \left( \frac{y_i - \bar{y}}{\mu \bar{y} + 1 - \mu} + \mu \right) = L_p(y; \mu, l)$$

(2.3)

for some constant $\mu$ ($0 < \mu < 1$) and for all $l = 1, 2, ..., n - 1$ with the strict inequality holding at least once. Following Pfingsten’s definition, the intermediate Lorenz ordinate $L_p(x; \mu, l) = \frac{1}{n} \sum_{i=1}^{l} \left( \frac{x_i - \bar{x}}{\mu \bar{x} + 1 - \mu} + \mu \right)$ is unaffected if $x$ is transformed to $z$ through

$$z_i = x_i + \lambda(\mu x_i + 1 - \mu)$$

(2.4)

for some constant $0 < \mu < 1$ and all real $\lambda$. Clearly, for $\lambda > 0$, this transformation can be viewed as an equal-proportion increase to all incomes (the $(1 + \mu \lambda)x_i$ part) followed by an equal-dollar-amount addition to all incomes (the $\lambda(1 - \mu)$ part). Graphically, the intermediate Lorenz curve defined via (2.3) is a convex curve connecting between coordinates $(0, 0)$ and $(1, 1)$.

The Pfingsten intermediate Lorenz dominance condition would contain both absolute Lorenz dominance and relative Lorenz dominance as polar cases if the parameter $\mu$ were to take the boundary values of 0 and 1, respectively. In this sense, the parameter $\mu$ can be interpreted as the degree of intermediateness — indicating how an intermediate measure is positioned between the absolute bound and the relative bound.

Krtscha (1994) proposed a different notion of intermediate inequality. The approach Krtscha suggested is a continuous “fair compromise” procedure. To distribute a fixed sum of income, this “fair compromise” procedure first divides the total income into (infinitely) many small portions, and then allocate these slices of income

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1See also Kolm (1996) where the centralist notion is further elaborated.
sequentially to each recipient in the amount of the (weighted) average between the equal-dollar amount and the equal-proportion amount. In a recent paper, Yoshida (2003) characterizes Krtscha’s idea and defines the inequality ordering condition for this new notion of intermediate inequality. This new dominance condition can be stated as follows: For any two distributions \( x \in \Psi^n \) and \( y \in \Psi^n \), \( x \) intermediate Lorenz dominates \( y \) in the sense proposed by Krtscha, denoted as \( x L_k y \), if and only if
\[
L_k(x; \lambda, l) = \frac{1}{n} \sum_{i=1}^l x_i - \bar{x}_\lambda \geq \frac{1}{n} \sum_{i=1}^l y_i - \bar{y}_\lambda = L_k(y; \lambda, l)
\]  
(2.5)

for some constant \( \lambda \) \((0 < \lambda < 1)\) and for all \( l = 1, 2, ..., n-1 \) with the strict inequality holding at least once. Since both absolute and (transformed) relative Lorenz orderings are included as polar cases of (2.5) with \( \lambda = 0 \) and 1, respectively, the parameter \( \lambda \) once again can be regarded as the degree of intermediateness between the absolute bound and the relative bound. If distribution \( x \) is transformed to \( z \) with \( \bar{z} = \gamma \bar{x} \) for \( \gamma > 0 \), then the income transformation under which the Krtscha-type intermediate Lorenz ordinate \( L_k(x; \lambda, l) = \frac{1}{n} \sum_{i=1}^l x_i - \bar{x}_\lambda \) remains unchanged is:
\[
z_i = \gamma^\lambda x_i + (\gamma - \gamma^\lambda)\bar{x}.
\]  
(2.6)

Graphically, the intermediate Lorenz curve defined via (2.5) is a U-shape curve connecting between coordinates \((0, 0)\) and \((1, 0)\).

In a thorough investigation of non-linear inequality equivalence, Zoli (1998, 1999) generalized the intermediate notion of inequality of both Pfingsten (1986) and Krtscha (1994). The Lorenz dominance condition corresponding to Zoli’s generalized notion of intermediate inequality can be formulated as follows: For any two distributions \( x \in \Psi^n \) and \( y \in \Psi^n \), \( x \) intermediate Lorenz dominates \( y \) in the sense proposed by Zoli, denoted as \( x L_z y \), if and only if
\[
L_z(x; \mu, \lambda, l) = \frac{1}{n} \sum_{i=1}^l \frac{x_i - \bar{x}}{(\mu \bar{x} + 1 - \mu)\lambda} \geq \frac{1}{n} \sum_{i=1}^l \frac{y_i - \bar{y}}{(\mu \bar{y} + 1 - \mu)\lambda} = L_z(y; \mu, \lambda, l)
\]  
(2.7)

for some constant \( \mu \) and \( \lambda \) \((0 < \mu, \lambda < 1)\) and for all \( l = 1, 2, ..., n-1 \) with the strict inequality holding at least once. Clearly, this generalized intermediate Lorenz dominance condition includes a variant of Pfingsten’s condition (when \( \lambda = 1 \)) and Krtscha’s condition (when \( \mu = 1 \)) as polar cases. Graphically, the Lorenz curve defined in (2.7) is a U-shape curve connecting between coordinates \((0, 0)\) and \((1, 0)\).

\(^2\)Yoshida (2003) also investigated the welfare implications of the Krtscha-type inequality orderings.

\(^3\)This condition in different yet equivalent forms are provided in Yoshida (2003) and Zheng (2003). The version presented here is from Zheng (2003).

\(^4\)Yoshida (2003) discussed Zoli’s notion of intermediate inequality in some more details; he also described the type of income transformations under which inequality would be unaffected.
In the literature, other notions of intermediate inequality have also been proposed and characterized. For example, Seidl and Pfingsten (1997) proposed the notion of “ray invariant” intermediate inequality measurement. A special case of Seidl and Pfingsten’s approach has also been formally formulated and characterized by del Río and Ruiz-Castillo (2000, 2001). In terms of Lorenz dominance, Chakravarty (1988) provided a different characterization for the Pfingsten notion of intermediate Lorenz dominance. Zoli (1998, 1999) also investigated dominance conditions in a somewhat different direction. To preserve space, we will refrain from discussing these additional contributions in any greater details.

2.2. Inequality orderings and the unit-consistency axiom

Now we turn to investigate whether the above ordering conditions can be affected by the use of different income units.

The basic idea of the unit-consistency requirement is that inequality comparisons and rankings must be independent of the specific income measuring unit that is adopted. For inequality orderings in terms of Lorenz dominance, the axiom can be formally stated as follows:

The unit-consistency axiom: For any two distributions \( x \in \Psi^n \) and \( y \in \Psi^n \), if \( x \) Lorenz dominates \( y \) then \( x' = \theta x \) also Lorenz dominates \( y' = \theta y \) for any \( \theta > 0 \).

Note that the axiom does not require the position or location of each Lorenz curve to remain unchanged, i.e., \( L(\theta x; l) = \theta L(x; l) \) for all \( l \) – as unit-invariance would require – all it requires is that the dominance relation between the two Lorenz curves is preserved through any uniform income transformation. It is easy to deduce from this axiom that if the Lorenz curves of \( x \) and \( y \) cross then the Lorenz curves of \( x' = \theta x \) and \( y' = \theta y \) remain crossing; also a Lorenz dominance condition is unit-consistent if and only if each ordinate of the Lorenz curve is unit-consistent.

Clearly, the relative Lorenz dominance condition \( L_r \) satisfies the unit-consistency axiom. The absolute Lorenz dominance condition \( L_a \) also satisfies this condition since each ordinate \( \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x}) \) is homogenous of degree one in incomes, i.e., \( L_a(\theta x; l) = \theta L_a(x; l) \).

As for the intermediate Lorenz dominance conditions, the Krtscha-type intermediate Lorenz dominance satisfies the unit-consistency axiom. This is so because the Lorenz ordinate is also homogenous in all incomes: \( L_k(\theta x; \mu, l) = \theta^{1-\lambda} L_k(x; \mu, l) \).

The Pfingsten-type Lorenz dominance, however, violates the axiom. Consider two distributions \( x = (1, 2, 3, 4, 5) \) and \( y = (0.2, 0.4, 0.5, 1, 2) \) and \( \mu = 0.01 \). It is easy to calculate that the corresponding vectors of Lorenz ordinates are \((0.39, -0.58, -0.58, -0.38, 0.01)\) and \((-0.12, -0.20, -0.27, -0.23, 0.01)\), respectively. Thus \( y \mathcal{L}_p x \). Multiplying both \( x \) and \( y \) by 1000, we can also calculate the vectors of Lorenz ordinates for \( x' = 1000x \).

While all relative summary measures of inequality are unit-consistent, not all absolute measures of inequality are unit-consistent. An example of such measures is the well-known Kölm’s absolute measure of inequality. See Zheng (2003) for a more detailed discussion along this line and a general characterization of the unit-consistent absolute measures of inequality.
and $y' = 1000y$ as $(-12.91, -19.36, -19.36, -12.90, 0.01)$ and $(-13.49, -22.63, -29.59, -25.67, 0.01)$, respectively. Thus $x' L_p y'$. If we suppose that $x$ and $y$ are measured in U.S. dollars, then the above comparisons suggest that $x$ is more unequally distributed than $y$ when incomes are expressed in thousands of dollars but $y$ becomes more unequally distributed when incomes are expressed in dollars!

The Zoli-type intermediate Lorenz dominance in general is not unit-consistent. It can be unit-consistent only when it collapses to the Krtscha-type, i.e., when $\mu = 0, 1$. This can be briefly demonstrated as follows.

Consider two mean incomes $\bar{x} < \bar{y}$. Since $(\frac{\mu x + t}{\mu y + t})^\lambda$ is strictly increasing in $t$ for $\mu > 0$, it follows that $(\frac{\mu \bar{x}}{\mu \bar{y}})^\lambda < (\frac{\mu \bar{x} + 1 - \mu}{\mu \bar{y} + 1 - \mu})^\lambda$ for $0 < \mu < 1$. Now choose a distribution $x$ with mean income $\bar{x}$ and a distribution $y$ with mean income $\bar{y}$ such that

$$(\frac{\mu \bar{x}}{\mu \bar{y}})^\lambda < \frac{1}{n} \sum_{i=1}^{l} \frac{x_i - \bar{x}}{(\mu \bar{x})^\lambda} < \frac{1}{n} \sum_{i=1}^{l} \frac{y_i - \bar{y}}{(\mu \bar{y})^\lambda} < (\frac{\mu \bar{x} + 1 - \mu}{\mu \bar{y} + 1 - \mu})^\lambda$$

for some $l \leq n - 1$. It is easy to see that the distributions $x$ and $y$ satisfying (2.8) can easily be constructed. This inequality implies that we can always have two distributions $x$ and $y$ such that

$$\frac{1}{n} \sum_{i=1}^{l} \frac{y_i - \bar{y}}{(\mu \bar{y} + 1 - \mu)^\lambda} < \frac{1}{n} \sum_{i=1}^{l} \frac{x_i - \bar{x}}{(\mu \bar{x} + 1 - \mu)^\lambda}$$

and

$$\frac{1}{n} \sum_{i=1}^{l} \frac{x_i - \bar{x}}{(\mu \bar{x} + 1 - \mu)^\lambda} < \frac{1}{n} \sum_{i=1}^{l} \frac{x_i - \bar{x}}{(\mu \bar{x} + 1 - \mu)^\lambda}.$$

Multiply all incomes in (2.10) by $\theta > 0$, the unit-consistency axiom implies that

$$\frac{1}{n} \sum_{i=1}^{l} \frac{y_i - \bar{y}}{(\mu \bar{y} + 1 - \mu)^\lambda} < \frac{1}{n} \sum_{i=1}^{l} \frac{x_i - \bar{x}}{(\mu \bar{x} + 1 - \mu)^\lambda},$$

and letting $\theta \to \infty$, we have

$$\frac{1}{n} \sum_{i=1}^{l} \frac{y_i - \bar{y}}{(\mu \bar{y})^\lambda} < \frac{1}{n} \sum_{i=1}^{l} \frac{x_i - \bar{x}}{(\mu \bar{x})^\lambda},$$

which contradicts (2.9). Thus, for $0 < \mu < 1$, the Zoli-type Lorenz dominance cannot satisfy the unit-consistency axiom. In the case of $\mu = 1$, the Zoli-type Lorenz dominance becomes the Krtscha-type Lorenz dominance; in the case of $\mu = 0$, the Zoli-type Lorenz curve becomes absolute Lorenz curve which is a special case of the Krtscha-type Lorenz curve. In both cases, the unit-consistency axiom is satisfied.

Now a natural question to ask at this point is whether there are other types of intermediate Lorenz dominance besides the Krtscha-type that can be unit-consistent.
In the following section, we introduce and consider a general framework of intermediate Lorenz orderings which include both the Pfingsten-type and Krtscha-type Lorenz dominance conditions as special cases. Within this general framework, we show that the requirement of unit-consistency leads inexorably to the Krtscha-type Lorenz dominance condition.

III. A General Intermediate Lorenz Dominance and Unit-Consistency

To characterize intermediate Lorenz dominance axiomatically, we need to first provide a general definition for Lorenz dominance. We can then investigate the properties such a dominance condition ought to possess in order for it to be labeled as “intermediate.”

Note that in constructing Lorenz curves for a given notion of inequality measurement, all income distributions of interest are normalized to have the same mean income so that they can be compared on the same basis. For example, in the above examples of Lorenz dominance, all distributions are normalized to have unity mean for relative Lorenz orderings and all distributions are normalized to have zero mean for absolute Lorenz orderings. Obviously, the choice for which value, say between zero and unity, to be the normalized mean is entirely arbitrary and, thus, should have no consequence on inequality orderings. What matters is the normalization function that transforms each income in the original distribution to that in the normalized distribution. This normalization function dictates whether the resulting Lorenz dominance is relative, absolute or intermediate. In both the Pfingsten-type Lorenz curve and the Krtscha-type Lorenz curve, the distinction between relative and absolute is done through a single parameter of intermediateness. In this sense, both types of intermediate Lorenz curve can be referred to as “single-parameter intermediate Lorenz curve.”

In this section, we confine our investigation of unit-consistency to this class of “single-parameter” Lorenz curves. Denoting the normalization function as \( m \), the corresponding “single-parameter” intermediate Lorenz dominance can be defined formally as follows.\(^6\)

**Definition 3.1.** For any distribution \( x \in \Psi^n \) and \( y \in \Psi^n \), \( x \) intermediate Lorenz dominates \( y \), denoted as \( xL_m y \), if and only if

\[
L_m(x; \mu, l) = \frac{1}{n} \sum_{i=1}^{l} m_i(x; \mu) \geq \frac{1}{n} \sum_{i=1}^{l} m_i(y; \mu) = L_m(y; \mu, l)
\]

for some given degree of intermediateness \( \mu \) (\( 0 < \mu < 1 \)) and for all \( l = 1, 2, ..., n - 1 \) with the strict inequality holding at least once. At \( l = n \),

\[
L_m(x; \mu, n) = L_m(y; \mu, n) = 0.
\]

\(^6\)Clearly, this definition would exclude the Zoli-type Lorenz dominance since it contains two parameters of intermediateness \( \lambda \) and \( \mu \).
Graphically, the intermediate Lorenz curve connects between coordinates \((0,0)\) and \((1,0)\).

In this definition, condition (3.2) requires that all distributions be normalized to have zero mean – as in the Krtscha definition of intermediate inequality where \(m_i(x;\mu) = \frac{x_i - \bar{x}}{\bar{x}}\). The Pfingsten-type intermediate Lorenz dominance would also conform to this definition if the original Lorenz ordinate \(L_p(x;\mu, l)\) in (2.3) is modified as follows:

\[
\tilde{L}_p(x;\mu, l) = \frac{1}{n} \sum_{i=1}^{l} \frac{x_i - \bar{x}}{\mu \bar{x} + 1 - \mu},
\]

that is, \(m_i(x;\mu) = \frac{x_i - \bar{x}}{\mu \bar{x} + 1 - \mu}\) and the stand-alone \(\mu\) term is dropped from \(L_p(x;\mu, l)\). Clearly, as we pointed out before, the omission of the \(\mu\) term has no consequence on inequality comparisons. Consequently, in the rest of the paper, we will switch to this modified Pfingsten-type Lorenz curve in order for (3.2) to accommodate different dominance conditions.

Definition 3.1 will also accommodate both relative and absolute Lorenz orderings as polar cases when \(\mu\) takes values of 0 and 1, respectively. But the relative Lorenz dominance (2.1) must be modified as follows:

\[
\tilde{L}_r(x;l) = \frac{1}{n} \sum_{i=1}^{l} \frac{x_i - \bar{x}}{\bar{x}} \geq \frac{1}{n} \sum_{i=1}^{l} \frac{y_i - \bar{y}}{\bar{y}} = \tilde{L}_r(y;l).
\]

Clearly, this modification also does not alter the nature of relative Lorenz dominance; it changes only the graphical presentation – the Lorenz curve now connects between \((0,0)\) and \((1,0)\) instead of between \((0,0)\) and \((1,1)\).

### 3.1. Two axioms on intermediate inequality orderings

To further characterize the class of “single-parameter intermediate Lorenz dominance,” we need to be more specific about the meaning of being “intermediate” in measuring inequality. In this paper, we follow the usual interpretation that “intermediate” means “lying or occurring between two extremes or in a middle position or state” (American Heritage Dictionary) and consider two axioms that we believe are particularly pertinent to the notion of intermediate inequality.\(^7\)

The first axiom clarifies the issue of “intermediate between what?” and specifies the relationship between the centralist view and its two extremes or bounds.

**The intermediateness axiom:** For any two distributions \(x \in \Psi^n\) and \(y \in \Psi^n\), if \(xL_r y\) and \(xL_a y\) then \(xL_m y\).

Thus, intermediate Lorenz dominance is between the polar cases of relative Lorenz dominance and absolute Lorenz dominance. In addition, if both relative Lorenz curve and absolute Lorenz curve move in the same direction, then the intermediate Lorenz dominance.

\(^7\)Both axioms were introduced in Zheng (2002) and discussed in Zheng (2003) in the case of unit-consistent poverty indices.
curve must also follow in the same direction. This implies that in the case where both rightist and leftist views agree upon the direction of the change in income inequality, the centralist view must agree as well. This implication is clearly consistent with the notion that the centralist (and thus an intermediate measure) is a mitigation between the relative view (and thus a relative measure) and the absolute view (and thus an absolute measure). We believe that this axiom is very intuitive and is important in characterizing the notion of intermediate inequality. It is also easy to see that an intermediate Lorenz dominance condition satisfies the intermediateness axiom if and only if each Lorenz ordinate satisfies the axiom, i.e., for any \( l = 1, 2, ..., n - 1 \), if \( L_r(x; l) \geq L_r(y; l) \) and \( L_a(x; l) \geq L_a(y; l) \) then \( L_m(x; l) \geq L_m(y; l) \).

The intermediateness axiom is stronger than Bossert and Pfingsten’s (1990) compromise property. In terms of Lorenz dominance, their compromise property can be stated as follows:

*The compromise property:* For any distribution \( x \in \Psi^n \), if \( \hat{x}_i = ax_i \) with \( a > 1 \) and \( \tilde{x}_i = x_i + b \) with \( b > 0 \), then \( \tilde{x} = \langle \tilde{x}_i \rangle \) intermediate Lorenz dominates \( x \) which, in turn, intermediate Lorenz dominates \( \hat{x} = \langle \hat{x}_i \rangle \).

Following this definition, it is clear that the compromise property can be viewed as a special case of the intermediateness axiom with \( y = \hat{x} \) and \( y = \tilde{x} \). The additional content of the intermediateness axiom beyond the compromise property lies in the fact that the axiom also applies to the cases where \( y \) is different from \( \hat{x} \) and \( \tilde{x} \). For example, distribution \( x = (1, 2, 3, 4, 5) \) dominates \( y = (2, 3, 3, 4, 5) \) in both relative and absolute Lorenz dominance but neither distribution can be obtained from the other distribution through a uniform equal-proportion transformation or a uniform equal-dollar-amount transformation.

The second axiom ensures the smoothness of the defined intermediate inequality orderings when the parameter of intermediateness \( \mu \) varies.

*The \( \mu \)-continuity axiom:* Each intermediate Lorenz ordinate \( L_m(x; \mu, l) \) is a continuous function of the degree of intermediateness \( \mu \).

Both the Pfingsten-type and Krtscha-type intermediate Lorenz dominance conditions satisfy the intermediateness axiom and the \( \mu \)-continuity axiom. The fact that the \( \mu \)-continuity axiom is satisfied is obvious. For the modified Pfingsten-type Lorenz dominance, since its ordinate (3.3) can be written as

\[
\tilde{L}_p(x; \mu, l) = \left\{ \mu \left[ \frac{1}{n} \sum_{i=1}^{l} \frac{x_i - \bar{x}}{\bar{x}} \right]^{-1} + (1 - \mu) \left[ \frac{1}{n} \sum_{i=1}^{l} (x_i - \bar{x}) \right]^{-1} \right\}^{-1},
\]

i.e., \( \tilde{L}_p(x; \mu, l) \) is a weighted harmonic mean between the relative Lorenz ordinate \( -\frac{1}{n} \sum_{i=1}^{l} \left( \frac{x_i - \bar{x}}{\bar{x}} \right) \) and the absolute Lorenz ordinate \( -\frac{1}{n} \sum (x_i - \bar{x}) \), it follows that the intermediateness axiom is fulfilled. For the Krtscha-type intermediate Lorenz dominance, its ordinate is a weighted geometric mean between the relative Lorenz
For a given income distribution 

Proof.

Proposition 3.1. A single-parameter intermediate Lorenz dominance satisfies exclusively determined by

Thus, the intermediateness axiom is also satisfied by the Krtscha-type intermediate Lorenz dominance.

Both equations (3.5) and (3.6) indicate that the intermediate Lorenz ordinates are functions of relative Lorenz ordinates and absolute Lorenz ordinates. This relationship is in fact true for all intermediate Lorenz dominance as long as the intermediateness axiom is fulfilled. This result is summarized as a proposition below.

Proposition 3.1. A single-parameter intermediate Lorenz dominance satisfies both the intermediateness axiom and the μ-continuity axiom if and only if there exists a continuous function \( F \) which is increasing in its first two arguments such that

\[
L_m(x; \mu, l) = F[\bar{L}_r(x; l), \bar{L}_a(x; l), \mu]
\]

for \( l = 1, 2, \ldots, n - 1 \).

Proof. For a given income distribution \( x \) and the intermediateness parameter \( \mu \), consider the two transformed distributions \( \bar{x} = (\bar{x}_i) \) and \( \hat{x} = (\hat{x}_i) \) as defined in Bossert and Pfingsten’s compromise property above. Since \( \bar{L}_r(\bar{x}; l) = \bar{L}_r(x; l) \) and \( \bar{L}_a(\bar{x}; l) = \bar{L}_a(x; l) \), the intermediateness axiom implies that \( L_m(\bar{x}; \mu, l) > L_m(x; \mu, l) \). That is, \( L_m(x; \mu, l) \) is an increasing function of \( L_a(x; l) \). Similarly, the intermediateness axiom also implies that \( L_m(x; \mu, l) \) is an increasing function of \( \hat{L}_r(x; l) \). It follows that there exists a function \( F \) such that

\[
L_m(x; \mu, l) = F[\bar{L}_r(x; l), \bar{L}_a(x; l), \mu \text{ and other terms}].
\]

The continuity of \( L_m(x; \mu, l) \), \( \bar{L}_r(x; l) \) and \( \bar{L}_a(x; l) \) further implies the continuity of \( F \) in the first two arguments; the \( \mu \)-continuity axiom establishes the continuity of \( F \) in \( \mu \). Since for a given \( \mu \), the intermediateness axiom states that \( L_m(x; \mu, l) \) is exclusively determined by \( \bar{L}_r(x; l) \) and \( \bar{L}_a(x; l) \), it follows that \( F \) does not depend on any other terms. Thus (3.8) is simplified to (3.7) which proves the necessity of the proposition. The sufficiency of the proposition is obvious. \( \square \)

3.2. A general intermediate Lorenz dominance: \( L_m(x; \mu, l) \) as a quasilinear weighted mean of \( \bar{L}_r(x; l) \) and \( \bar{L}_a(x; l) \)

Proposition 3.1 clearly applies to both the modified Pfingsten-type and Krtscha intermediate Lorenz curves. Both intermediate Lorenz curves, however, reveal more about the meaning of intermediateness than either the intermediateness axiom or Proposition 3.1 contain. Specifically, as shown above, both intermediate Lorenz curves achieve the intermediateness by averaging the two bounds (i.e., relative and absolute Lorenz curves). By considering a more general averaging processes, we can
define and characterize a general intermediate Lorenz curve and we can then examine the unit-consistency axiom within the broader framework.

By reading up on both the Pfingsten-type Lorenz orderings and the Krtscha-type Lorenz orderings, we can extract additional desirable properties for an intermediate Lorenz dominance condition. That is, we can impose further reasonable restrictions on the functional form of \( F \). For example, since \( \mu \) is interpreted as a measure of the degree of intermediateness in the sense that \( \mu \) is the weight attached to the relative Lorenz dominance and \( 1 - \mu \) is the weight attached to the absolute Lorenz dominance, we may assume

\[
F(u, v, 1) = u \quad \text{and} \quad F(u, v, 0) = v. \tag{3.9}
\]

Further, one may desire that graphically the intermediate Lorenz curve should lie between the relative Lorenz curve and the absolute Lorenz curve – this may be a good intuitive interpretation of being “intermediate.”\(^8\) In terms of \( F \), this means:

\[
\min(u, v) \leq F(u, v, \mu) \leq \max(u, v) \quad \text{for} \quad 0 < \mu < 1. \tag{3.10}
\]

Also as a direct corollary of the above inequality, if \( \bar{L}_r(x; l) \) and \( L_a(x; l) \) happen to be equal (i.e., when \( \bar{x} = 1 \)), then \( L_m(x; \mu, l) \) should also be the same regardless of the value of \( \mu \), i.e.,

\[
F(u, u, \mu) = u. \tag{3.11}
\]

Furthermore, if \( u < v \) and \( \mu_1 > \mu_2 \), then

\[
F(u, v, \mu_1) < F(u, v, \mu_2). \tag{3.12}
\]

That is, if the absolute Lorenz ordinate is greater than the relative Lorenz ordinate, and if the degree of intermediateness decreases (moving closer to the absolute bound), then the intermediate Lorenz ordinate should increase as well. Graphically, this means that if the absolute Lorenz curve lies above the relative Lorenz curve, then a smaller value of \( \mu \) implies that the intermediate Lorenz curve should be closer to the absolute Lorenz curve and, hence, each ordinate becomes larger.

All these intuitive requirements on the notion of intermediate inequality point to the direction that \( L_m(x; \mu, l) \) is a quasilinear weighted mean (e.g., Aczél, 1966, p. 240) between \( \bar{L}_r(x; l) \) and \( L_a(x; l) \). Formerly, this quasilinear-weighted-mean intermediate Lorenz curve can be defined as follow.

**Definition 3.** For any distribution \( x \in \Psi^n \), its quasilinear-weighted-mean intermediate Lorenz curve \( \{L_m(x; \mu, l)\} \) is defined by

\[
L_m(x; \mu, l) = f\left\{ \mu f^{-1}[\bar{L}_r(x; l)] + (1 - \mu) f^{-1}[L_a(x; l)] \right\} \tag{3.13}
\]

\(^8\)It is easy to see that both the modified Pfingsten-type Lorenz curve and the Krtscha-type Lorenz curve possess this property. For the original Pfingsten definition of intermediate Lorenz dominance (Pfingsten, 1986a), it can be shown that the intermediate Lorenz curve may lie beyond the range between the relative Lorenz curve and the absolute Lorenz curves. The same is true for the Zoli-type intermediate Lorenz curve.
or, in terms of \( F \),

\[
F(u, v; \mu) = f[\mu f^{-1}(u) + (1 - \mu)f^{-1}(v)]
\]

(3.14)

for some continuous, invertible and strictly monotonic function \( f \).

The quasilinear-weighted-mean definition clearly includes both the modified Pfingsten and Krtscha Lorenz curves: for the Pfingsten definition, \( f(u) = \frac{1}{u} \); for the Krtscha definition, \( f(u) = -e^u \). With additional functions \( f(u) \), more intermediate Lorenz curves can be generated. For examples, with the root-mean-power function \( f(u) = -|u|^c \) where \( c > 0 \), the intermediate Lorenz curve ordinate is:

\[
L_m(x; \mu, l) = -\left[ \mu|\tilde{L}_r(x; l)|^{1/c} + (1 - \mu)|L_a(x; l)|^{1/c} \right]^c
\]

(3.15)

To characterize (3.14) as a necessary realization of \( F(u, v; \mu) \), Aczél (1966, p. 241) listed six requirements that \( F \) needs to fulfill. From what we have discussed above, five of these requirements are readily satisfied. Briefly and in Aczél’s terminologies, conditions (3.9) and (3.10) represent the internality requirement; condition (3.11) is the reflexivity requirement; condition (3.12) is the same as the requirement of increasing in the (2nd) weight — the second weight would be \( 1 - \mu \) and in (3.12) we would have \( 1 - \mu_1 < 1 - \mu_2 \); the intermediateness axiom fulfills the requirement of increasing in the (2nd) variable; the requirement of homogeneity (0th degree) in the weights is trivially satisfied by the way \( \mu \) is defined.

The only requirement that we need to additionally consider for \( F(u, v, \mu) \) is the bisymmetry condition which can be replaced by the symmetry and associativity requirements — they are somewhat easier to state and interpret than bisymmetry. In our case, symmetry means

\[
F(u, v; \mu) = F(v, u, 1 - \mu)
\]

(3.16)

which is a very natural requirement in the context of intermediate Lorenz curve (3.7). To see the meaning of the associativity requirement for intermediate inequality orderings, we need to express \( \mu \) as \( \mu = \frac{r}{r+s} \) with \( r, s \geq 0 \) and \( r + s > 0 \) and interpret \( r \) and \( s \) as the weights attached to the relative and absolute Lorenz ordinates, respectively. That is, we can write \( F(u, v; \mu) \) as \( F(u, v; r, s) \) and (3.14) as

\[
F(u, v; r, s) = f[\frac{rf^{-1}(u) + sf^{-1}(v)}{r + s}].
\]

(3.17)

The associativity condition then requires

\[
F[F(u, v; r, s), w; r + s, t] = F[u, F(v, w; s, t); r, s + t] \text{ for all } u, v, w, r, s, t.
\]

(3.18)

This seemingly complex condition is actually quite intuitive and reasonable and can be explained as follows.
Suppose \( \mu_1, \mu_2, \) and \( \mu_3 \) are three distinct values for the intermediateness parameter \( \mu \). For a given distribution \( x \in \Psi^n \), equation (3.13) generates three different intermediate Lorenz ordinates. Call these ordinates \( u, v, \) and \( w \), respectively. Clearly, these intermediate Lorenz ordinates can be further “averaged” to produce a new Lorenz ordinate. If we can average only two ordinates at a time, then there are three approaches one can take: average \( u \) and \( v \) first and then with \( w \); average \( u \) and \( w \) first and then with \( v \); average \( v \) and \( w \) first and then with \( u \). Should the final averaged Lorenz ordinate depend on a specific approach one takes? What the association condition states is that all three approaches should lead to the same result. The condition interpreted in this way is clearly consistent with the notion of intermediate Lorenz ordinate as an average of the absolute and relative Lorenz bounds, it thus can and should be viewed as a reasonable property.

With these conditions on the intermediate Lorenz ordinate function (3.8) and following Aczél (1966, Theorem 5.3.2, p. 241), we have:

**Proposition 3.2.** Conditions (3.9) - (3.12), (3.15) and (3.17) on the function \( F \) are necessary and sufficient for the intermediate Lorenz ordinate \( L_m(x; \mu, l) \) to be a quasi-linear weighted mean of \( \tilde{L}_r(x;l) \) and \( L_a(x;l) \) as shown in (3.13).

When the unit-consistency axiom is considered, of all conceivable quasilinear weighted-mean intermediate Lorenz conditions, it seems that the Krtscha-type Lorenz dominance is the only one that can satisfy the axiom. In the following proposition, we confirm this conjecture. To prove the proposition, however, we need to use a simple lemma below.

**Lemma 3.1.** An intermediate Lorenz curve \( \{L_m\} \) is unit-consistent if and only if there exists a continuous function \( G \) such that for any \( x \in \Psi^n \) and \( \theta > 0 \)

\[
L_m(\theta x; \mu, l) = G[\theta, L_m(x; \mu, l)], \quad l = 1, 2, ..., n - 1 \tag{3.19}
\]

and \( G \) is increasing in its second argument.

What this lemma states is something one may have realized through the examples of unit-consistent Lorenz dominance: in order for the measuring unit \( \theta \) not to have any influence on inequality rankings, there must be no cross-effect between the unit \( \theta \) and the pre-transformed Lorenz ordinate \( L_m(x; \mu, l) \) on the after-transformed Lorenz ordinate \( L_m(\theta x; \mu, l) \). Since a similar result has been established and proved elsewhere (Zheng, 2003), we will not furnish a proof here.

With Lemma 3.1, we can state and prove the following proposition.

**Proposition 3.3.** The quasilinear-mean intermediate Lorenz ordinate (3.13) is unit-consistent if and only if \( f(u) = -\frac{1}{\beta} e^{u/\alpha} \) for some constants \( \alpha \neq 0 \) and \( \beta > 0 \).

**Proof.** Since \( \tilde{L}_r(x;l) = \frac{1}{n} \sum_{i=1}^{l} \frac{x_i - \bar{x}}{\bar{x}} \) and \( L_a(x;l) = \frac{1}{n} \sum_{i=1}^{l} (x_i - \bar{x}) \), it follows that if all incomes are multiplied by \( \theta > 0 \), then

\[
L_m(\theta x; \mu, l) = f\{\mu f^{-1}[\tilde{L}_r(x;l)] + (1 - \mu)f^{-1}[\theta L_a(x;l)]\}. \tag{3.20}
\]
By lemma 3.1, there exists a continuous function $G$ which is also monotonically increasing in the second argument such that

$$L_m(\theta x; \mu, l) = G[\theta, L_m(x; \mu, l)].$$

Substituting (3.19) into (3.20) and denoting $u = \tilde{L}_r(x; l)$ and $v = L_a(x; l)$, we have

$$f[\mu f^{-1}(u) + (1 - \mu)f^{-1}(\theta v)] = G[\theta, f[\mu f^{-1}(u) + (1 - \mu)f^{-1}(v)]].$$  \hspace{1cm} (3.21)

Since both $\tilde{L}_r(x; l)$ and $L_a(x; l)$ for $x_i \in (0, \infty)$ can vary between $-\infty$ and 0 (when distributions are perfectly equal), the domain for equation (3.21) is $u, v \in (-\infty, 0]$. Now consider all unequal distributions:

$$\Psi^n := \{x \in \Psi^n | x_i \neq x_j \text{ for some } i, j\}.$$

Thus for all $x \in \Psi^n$, neither $u$ nor $v$ can be zero. It follows that equation (3.21) holds for all $u, v$ over interval $(-\infty, 0)$. It is important to note that over $(-\infty, 0)$, $u$ and $v$ are independent variables in the sense that each can vary freely – through an equal-dollar-amount or an equal-proportion change – while holding the other constant.

Rewrite (3.21) as

$$\mu f^{-1}(u) + (1 - \mu) f^{-1}(\theta v) = f^{-1} \circ G(\theta, f[\mu f^{-1}(u) + (1 - \mu)f^{-1}(u)])$$

and denote $\zeta = f^{-1}(u), \xi = f^{-1}(v)$ and $\tilde{G} = f^{-1} \circ G(\theta, \cdot)$, then

$$\mu \zeta + (1 - \mu) f^{-1} \circ \theta \circ f(\xi) = \tilde{G}\{f[\mu \zeta + (1 - \mu)\xi]\}.$$  \hspace{1cm} (3.21a)

Further denoting $\tilde{f} = f^{-1} \circ \theta \circ f$ and $\tilde{G} = \tilde{G} \circ f$, we have

$$\mu \zeta + (1 - \mu) \tilde{f}(\xi) = \tilde{G}[\mu \zeta + (1 - \mu)\xi]$$

for all $\zeta, \xi \in D \equiv \langle f^{-1}(0), f^{-1}(-\infty) \rangle$. Since $f$ is strictly monotonic, so is $f^{-1}$ and $D$ is non-empty.

Setting $\xi = \xi_0 \in D$ in (3.21a), we have

$$\mu \zeta + (1 - \mu) \tilde{f}(\xi_0) = \tilde{G}[\mu \zeta + (1 - \mu)\xi_0]$$

or

$$\tilde{G}(\zeta) = \tilde{\zeta} + (1 - \mu)[\tilde{f}(\xi_0) - \xi_0]$$

where $\tilde{\zeta} = \mu \zeta + (1 - \mu)\xi_0$. That is, $\tilde{G}$ is an affine function: $\tilde{G}(\zeta) = \zeta + a$ for some constant $a$. It follows that

$$\tilde{G}[\mu \zeta + (1 - \mu)\xi] = \mu \zeta + (1 - \mu)\xi + a.$$  \hspace{1cm} (3.22)

Equating (3.22) with (3.21a), we further have

$$\mu \zeta + (1 - \mu)\xi + a = \mu \zeta + (1 - \mu)\tilde{f}(\xi)$$

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or
\[
\tilde{f}(\xi) = \xi + b
\]  
(3.23)

with \( b = \frac{a}{1-\mu} \). By the definitions of \( \tilde{f} \) and \( \xi \), we obtain

\[
f^{-1}(\theta u) = f^{-1}(u) + b(\theta)
\]

which is a Pexider equation. A nontrivial solution to this equation (Aczél, 1966, p. 144, Theorem 4) is:

\[
f^{-1}(u) = \alpha \ln \beta |u| \text{ and } b(\theta) = \alpha \ln \theta
\]  
(3.24)

for some constant \( \alpha \neq 0 \) and some \( \beta > 0 \). This further implies

\[
f(u) = -\frac{1}{\beta} e^{u/\alpha}
\]  
(3.25)

because \( L_m(x; \mu, l) < 0 \). Substituting (3.24) and (3.25) back into (3.13) leads to the Krtscha intermediate Lorenz ordinate \( L_k(x; \mu, l) \) which does not depend on \( \alpha \) and \( \beta \).

The sufficiency of condition (3.25) for \( L_m(x; \mu, l) \) to satisfy the unit-consistency axiom is already known. \( \square \)

IV. Conclusion

Unit-consistency in the sense that the use of different measuring units should not lead to contradictory conclusions is important to all scientific studies. Recognizing this importance, it is surprising that until recently the issue has not been appropriately addressed in the inequality measurement literature.

One might, however, argue that this introduction is not really necessary because a unit-invariance condition has long existed in the literature and a unit-invariant measure is unit-consistent. But requiring unit-invariance in lieu of unit-consistency is unnecessarily strong since unit-invariance accepts only the rightist view of inequality measurement which leads only to the relative class of inequality measures. In this sense, unit-invariance cannot be regarded as an axiom; it only represents one of many possible value judgements in inequality measurement. Compared with unit-invariance, unit-consistency is a much weaker condition that accommodates other views of inequality measurement and can be satisfied by not only relative measures of inequality but also absolute as well as intermediate measures of inequality.

The objective of the paper has been to investigate the implications of unit-consistency for partial inequality orderings. Both relative Lorenz dominance and absolute Lorenz dominance are unit-consistent, but some intermediate Lorenz dominance conditions such as the one introduced by Pfingsten (1986) violate the unit-consistency axiom. The only proposed intermediate Lorenz dominance that is unit-consistent is the one related to Krtscha’s notion of intermediate inequality which has also been further studied by Zoli (1998, 1999) and Yoshida (2003).
To axiomatically characterize unit-consistent intermediate Lorenz orderings, we considered a class of single-parameter intermediate Lorenz dominance and proposed two new axioms on the notion of intermediate inequality – the intermediateness axiom and the $\mu$-continuity axiom. These two axioms jointly imply that a single-parameter intermediate Lorenz ordinate must be a function of a relative Lorenz ordinate and an absolute Lorenz ordinate along with a parameter of intermediateness. By considering additional plausible properties that such an intermediate Lorenz dominance condition may possess, we characterized the quasilinear weighted mean as a necessary realization of the single-parameter intermediate Lorenz orderings. Finally, when the unit-consistency condition is considered, the Kartscha-type intermediate Lorenz dominance emerges as the only member of the single-parameter class that can be unit-consistent.
References


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